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Cutting and Pasting of Families of Submanifolds Modeled on $\mathbf{Z}_2$-Manifolds

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Introduction

All manifolds considered in this paper are in the smooth category, and they are all unoriented, with or without boundary. $\mathbf{Z}_2$ denotes the cyclic group of order 2.

We will consider families of submanifolds of a manifold, and define the SK-group of such families. We will investigate the relationship between the SK-group of families and the SK-group of $\mathbf{Z}_2$-manifolds.

Let $m \geq 0$ be an integer. Let P and Q be m-dimensional compact manifolds with boundary ∂P and ∂Q, respectively, and $\varphi : \partial P \to \partial Q$ be a diffeomorphism. Pasting P and Q along the boundary by φ, we obtain a closed manifold $P \cup_\varphi Q$. For another diffeomorphism $\psi : \partial P \to \partial Q$ we obtain another closed manifold $P \cup_\psi Q$. The two closed manifolds $P \cup_\varphi Q$ and $P \cup_\psi Q$ are said to be *obtained from each other by cutting and pasting* (*S*chneiden und *K*leben in German). Two m-dimensional closed manifolds M and N are said to be SK-*equivalent* to each other, if there is an m-dimensional closed manifold L such that the disjoint union $M + L$ is obtained from $N + L$ by a finite sequence of cuttings and pastings. This is an equivalence relation on $\mathfrak{M}_m$, the set of m-dimensional closed manifolds. Note that if M and N are SK-equivalent then $\chi(M) = \chi(N)$ since

$$\chi(P \cup_\varphi Q) = \chi(P) + \chi(Q) - \chi(\partial P) = \chi(P \cup_\psi Q),$$

where χ denotes the Euler characteristic. Denote by $[M]$ the equivalence class represented by M, and by $\mathfrak{M}_m/SK$ the quotient set of $\mathfrak{M}_m$ by the SK-equivalence. $\mathfrak{M}_m/SK$ becomes a semigroup with the addition induced from the disjoint union of manifolds. The Grothendieck group of $\mathfrak{M}_m/SK$ is called the SK-*group* of m-dimensional closed manifolds and is denoted by SK_m. This group has been introduced and observed by Karras, Kreck, Neumann and Ossa [7]. Note that $[M] = [N]$ in SK_m if and only if M, N are SK-equivalent to each other.

Let $\mathfrak{M}_m^{\mathbf{Z}_2}$ be the set of m-dimensional closed $\mathbf{Z}_2$-manifolds. Taking $\mathbf{Z}_2$-equivariant diffeomorphisms as pasting diffeomorphisms, we can perform $\mathbf{Z}_2$-equivariant cuttings and pastings in $\mathfrak{M}_m^{\mathbf{Z}_2}$ in a similar way as in $\mathfrak{M}_m$, and define an SK-equivalence relation on $\mathfrak{M}_m^{\mathbf{Z}_2}$. Then we

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obtain the SK-group $SK_m^{\mathbf{Z}_2}$ of m-dimensional closed $\mathbf{Z}_2$-manifolds. See for details Karras et al [7] and Kosniowski [14].

The fixed point set $M^{\mathbf{Z}_2}$ of a $\mathbf{Z}_2$-manifold M is a submanifold of M with various dimensions. Let $M_i^{\mathbf{Z}_2}$ be the i-dimensional component of $M^{\mathbf{Z}_2}$ for $0 \leq i \leq m = \dim M$. Then we have a family of submanifolds of M, denoted by $(M; M_m^{\mathbf{Z}_2}, M_{m-1}^{\mathbf{Z}_2}, \cdots, M_0^{\mathbf{Z}_2})$. An equivariant cutting and pasting on M induces a cutting and pasting on each $M_i^{\mathbf{Z}_2}$. Taking this into account, we introduce the following definitions.

Let P be an m-dimensional compact manifold. For any i with $0 \leq i \leq m$ let P_i be an i-dimensional submanifold of P such that $\partial P_i = P_i \cap \partial P$ and $P_i \cap P_j = \emptyset$ if $i \neq j$. We write $\tilde{P} = (P; P_m, P_{m-1}, \cdots, P_0)$ for a family of such submanifolds, and call this an *m-dimensional family*. For another such family $\tilde{Q} = (Q; Q_m, Q_{m-1}, \cdots, Q_0)$, let $\varphi : \partial P \to \partial Q$ be a diffeomorphism which restricts to a diffeomorphism $\varphi_i = \varphi|\partial P_i : \partial P_i \to \partial Q_i$ for any i. Then we obtain a family of submanifolds of a closed manifold,

$$\tilde{P} \cup_{\varphi} \tilde{Q} = (P \cup_{\varphi} Q; P_m \cup_{\varphi_m} Q_m, \cdots, P_0 \cup_{\varphi_0} Q_0).$$

Here $P_0 \cup_{\varphi_0} Q_0$ is a finite set which is the disjoint union of P_0 and Q_0. Let $\psi : \partial P \to \partial Q$ be another diffeomorphism which restricts to a diffeomorphism $\psi_i : \partial P_i \to \partial Q_i$ for any i. We obtain another family

$$\tilde{P} \cup_{\psi} \tilde{Q} = (P \cup_{\psi} Q; P_m \cup_{\psi_m} Q_m, \cdots, P_0 \cup_{\psi_0} Q_0).$$

The two families $\tilde{P} \cup_{\varphi} \tilde{Q}$ and $\tilde{P} \cup_{\psi} \tilde{Q}$ are said to be *obtained from each other by cutting and pasting*. Let $\mathfrak{M}_m^{\mathcal{F}}$ be the set of m-dimensional family of submanifolds of closed manifolds. Two families $\tilde{M}, \tilde{N} \in \mathfrak{M}_m^{\mathcal{F}}$ are said to be *SK-equivalent* to each other, if there is an $\tilde{L} \in \mathfrak{M}_m^{\mathcal{F}}$ such that $\tilde{M} + \tilde{L}$ is obtained from $\tilde{N} + \tilde{L}$ by a finite sequence of cuttings and pastings, where $\tilde{M} + \tilde{L}$ is the disjoint union of $\tilde{M}$ and $\tilde{L}$, i.e.,

$$\tilde{M} + \tilde{L} = (M + L; M_m + L_m, \cdots, M_0 + L_0)$$

for $\tilde{M} = (M; M_m, \cdots, M_0)$ and $\tilde{L} = (L; L_m, \cdots, L_0)$. The quotient set $\mathfrak{M}_m^{\mathcal{F}}/SK$ by this SK-equivalence becomes a semigroup with the addition induced from the disjoint union of families. The SK-group of m-dimensional families of submanifolds is defined as the Grothendieck group of $\mathfrak{M}_m^{\mathcal{F}}/SK$ and is denoted by $SK_m^{\mathcal{F}}$. Any element $x \in SK_m^{\mathcal{F}}$ is written in the form $x = [\tilde{M}] - [\tilde{N}]$ for some $\tilde{M} = (M; M_m, \cdots, M_0)$, $\tilde{N} = (N; N_m, \cdots, N_0) \in \mathfrak{M}_m^{\mathcal{F}}$. Define $\chi(x) = \chi(M) - \chi(N)$ and $\chi_i(x) = \chi(M_i) - \chi(N_i)$ for $0 \leq i \leq m$. This is well-defined since a cutting and pasting operation keeps the Euler characteristic invariant.

We have a natural correspondence $\mathfrak{M}_m^{\mathbf{Z}_2} \to \mathfrak{M}_m^{\mathcal{F}}$ which assigns to a $\mathbf{Z}_2$-manifold $M \in \mathfrak{M}_m^{\mathbf{Z}_2}$ the family $(M; M_m^{\mathbf{Z}_2}, \cdots, M_0^{\mathbf{Z}_2}) \in \mathfrak{M}_m^{\mathcal{F}}$. This induces a homomorphism $\eta : SK_m^{\mathbf{Z}_2} \to SK_m^{\mathcal{F}}$. In this paper we will obtain the following results:

- *Two families* $\tilde{M} = (M; M_m, \cdots, M_0)$ *and* $\tilde{N} = (N; N_m, \cdots, N_0)$ *are* SK*-equivalent in* $\mathfrak{M}_m^{\mathcal{F}}$ *if and only if* $\chi(M) = \chi(N)$ *and* $\chi(M_i) = \chi(N_i)$ *for any* i *with* $0 \le i \le m$.
- *The homomorphism* $\eta : SK_m^{\mathbf{Z}_2} \to SK_m^{\mathcal{F}}$ *is injective.*
- *An element* $x \in SK_m^{\mathcal{F}}$ *is in the image of* η *if and only if* $\chi(x) \equiv \sum_{i=0}^{m} \chi_i(x) \mod 2$.

This implies the following:

- $2x \in \operatorname{Im} \eta$ *for any* $x \in SK_m^{\mathcal{F}}$, and
- η *induces an isomorphism* $SK_m^{\mathbf{Z}_2} \otimes \mathbf{Z}[1/2] \cong SK_m^{\mathcal{F}} \otimes \mathbf{Z}[1/2]$, *where* $\mathbf{Z}[1/2]$ *is the subring of the rationals* $\mathbf{Q}$ *generated by* $1/2$.

NOTE. Not only the case of $\mathbf{Z}_2$, but in a similar way we can also discuss the case of $\mathbf{Z}_p$, the cyclic group of prime order p. But this case is treated in a more general setting, i.e., in the setting of odd order abelian group, in a separate paper (see Komiya [11]). Komiya [9],[10] treated a cutting and pasting for $\mathbf{Z}_2$-manifolds and pairs of manifolds. Hara [1], [2], [3], Hara and Koshikawa [4], [5], [6], Koshikawa [12], [13] are also relevant to our present work.

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1. Homomorphisms

For $0 \le i \le m+1$, let $\mathfrak{M}_{m,i}^{\mathcal{F}}$ be the subset of $\mathfrak{M}_m^{\mathcal{F}}$ consisting of those $\tilde{M} = (M; M_m, \cdots, M_0) \in \mathfrak{M}_m^{\mathcal{F}}$ for which $M_j = \emptyset$ for any $j < i$. For the simplicity we denote $(M; M_m, \cdots, M_i, \emptyset, \cdots, \emptyset)$ by $(M; M_m, \cdots, M_i)$. We have a sequence of subsets of $\mathfrak{M}_m^{\mathcal{F}}$,

$$\mathfrak{M}_m = \mathfrak{M}_{m,m+1}^{\mathcal{F}} \subset \mathfrak{M}_{m,m}^{\mathcal{F}} \subset \cdots \subset \mathfrak{M}_{m,0}^{\mathcal{F}} = \mathfrak{M}_m^{\mathcal{F}} .$$

As in the Introduction we can define the SK-group from $\mathfrak{M}_{m,i}^{\mathcal{F}}$, which is denoted by $SK_{m,i}^{\mathcal{F}}$. Then $SK_{m,0}^{\mathcal{F}} = SK_m^{\mathcal{F}}$ and $SK_{m,m+1}^{\mathcal{F}} = SK_m$. Let $\iota_i^{\mathcal{F}} : SK_{m,i}^{\mathcal{F}} \to SK_{m,i-1}^{\mathcal{F}}$ be the homomorphism induced from the inclusion $\mathfrak{M}_{m,i}^{\mathcal{F}} \subset \mathfrak{M}_{m,i-1}^{\mathcal{F}}$.

PROPOSITION 1.1. *The homomorphism* $\iota_i^{\mathcal{F}} : SK_{m,i}^{\mathcal{F}} \to SK_{m,i-1}^{\mathcal{F}}$ *is injective.*

PROOF. Take an element $[\tilde{M}] - [\tilde{N}] \in SK_{m,i}^{\mathcal{F}}$ where $\tilde{M} = (M; M_m, \cdots, M_i)$, $\tilde{N} = (N; N_m, \cdots, N_i) \in \mathfrak{M}_{m,i}^{\mathcal{F}}$, and assume $\iota_i^{\mathcal{F}}([\tilde{M}] - [\tilde{N}]) = 0$ in $SK_{m,i-1}^{\mathcal{F}}$. Then there is $\tilde{L} = (L; L_m, \cdots, L_{i-1}) \in \mathfrak{M}_{m,i-1}^{\mathcal{F}}$ such that in $\mathfrak{M}_{m,i-1}^{\mathcal{F}}$, $(M; M_m, \cdots, M_i, \emptyset) + (L; L_m, \cdots, L_{i-1})$ is obtained from $(N; N_m, \cdots, N_i, \emptyset) + (L; L_m, \cdots, L_{i-1})$ by a finite sequence of cuttings and pastings. If we forget here the $(i-1)$-dimensional component, we see that $(M; M_m, \cdots, M_i) + (L; L_m, \cdots, L_i)$ is obtained from $(N; N_m, \cdots, N_i)+$

$(L; L_m, \cdots, L_i)$ by a finite sequence of cuttings and pastings in $\mathfrak{M}^{\mathcal{F}}_{m,i}$. This implies $[\tilde{M}] - [\tilde{N}] = 0$ in $SK^{\mathcal{F}}_{m,i}$. Hence $\iota^{\mathcal{F}}_i$ is injective. □

Let $\rho^{\mathcal{F}}_i : SK^{\mathcal{F}}_{m,i} \to SK_i$ be the homomorphism induced from the correspondence $\mathfrak{M}^{\mathcal{F}}_{m,i} \to \mathfrak{M}_i$ which sends $(M; M_m, \cdots, M_i)$ to M_i. Let an $(m-i)$-dimensional closed manifold S with a base point $*$ be arbitrarily fixed, and consider the correspondence $\mathfrak{M}_i \to \mathfrak{M}^{\mathcal{F}}_{m,i}$ which sends $N \in \mathfrak{M}_i$ to $(S \times N; \emptyset, \cdots, \emptyset, \{*\} \times N) \in \mathfrak{M}^{\mathcal{F}}_{m,i}$. This induces a homomorphism $\gamma^{\mathcal{F}}_i : SK_i \to SK^{\mathcal{F}}_{m,i}$. We see $\rho^{\mathcal{F}}_i \circ \gamma^{\mathcal{F}}_i = \mathrm{id}$, and obtain

PROPOSITION 1.2. *The homomorphism $\rho^{\mathcal{F}}_i : SK^{\mathcal{F}}_{m,i} \to SK_i$ is surjective. In fact, $\rho^{\mathcal{F}}_i$ has a right inverse $\gamma^{\mathcal{F}}_i : SK_i \to SK^{\mathcal{F}}_{m,i}$.*

2. Exact sequences

In this section we will prove the following theorem.

THEOREM 2.1. *For any i with $0 \le i \le m$ we have the split short exact sequence*

$$0 \longrightarrow SK^{\mathcal{F}}_{m,i+1} \xrightarrow{\iota^{\mathcal{F}}_{i+1}} SK^{\mathcal{F}}_{m,i} \xrightarrow{\rho^{\mathcal{F}}_i} SK_i \longrightarrow 0\,.$$

We already know from Propositions 1.1 and 1.2 that $\iota^{\mathcal{F}}_{i+1}$ is injective and $\rho^{\mathcal{F}}_i$ has a right inverse. For a proof of Theorem 2.1 it is sufficient to show $\mathrm{Im}\ \iota^{\mathcal{F}}_{i+1} = \mathrm{Ker}\ \rho^{\mathcal{F}}_i$. This will be shown in Proposition 2.3.

Given $(M; M_m, \cdots, M_i) \in \mathfrak{M}^{\mathcal{F}}_{m,i}$, let $\nu(M_i)$ be the normal bundle of M_i in M, and consider the projective space bundle $RP(\nu(M_i) \oplus \mathbf{R})$ associated to $\nu(M_i) \oplus \mathbf{R}$, where $\mathbf{R}$ is the trivial line bundle over M_i. Then $RP(\nu(M_i) \oplus \mathbf{R})$ contains M_i ($\approx RP(\mathbf{R})$) as a submanifold, and we have a family $(RP(\nu(M_i) \oplus \mathbf{R}); \emptyset, \cdots, \emptyset, M_i) \in \mathfrak{M}^{\mathcal{F}}_{m,i}$. Let T and T' be closed tubular neighborhoods of M_i in M and in $RP(\nu(M_i) \oplus \mathbf{R})$, respectively. Here we assume T is sufficiently small so that $T \cap M_j = \emptyset$ for any j with $i < j \le m$. Since T and T' are diffeomorphic to each other, we have a diffeomorphism $\varphi : \partial(M - \mathring{T}) \to \partial(RP(\nu(M_i) \oplus \mathbf{R}) - \mathring{T}')$, and have an m-dimensional closed manifold $(M - \mathring{T}) \cup_\varphi (RP(\nu(M_i) \oplus \mathbf{R}) - \mathring{T}')$, where $\mathring{T}$ and $\mathring{T}'$ are the interiors of T and T', respectively. We have a family

$$((M - \mathring{T}) \cup_\varphi (RP(\nu(M_i) \oplus \mathbf{R}) - \mathring{T}'); M_m, \cdots, M_{i+1}, \emptyset) \in \mathfrak{M}^{\mathcal{F}}_{m,i}\,.$$

Here $M_m, \cdots, M_{i+1}$ are contained in the part $M - \mathring{T}$. We also have a family

$$((M - \mathring{T}) \cup_{\mathrm{id}} (M - \mathring{T}); 2M_m, \cdots, 2M_{i+1}, \emptyset) \in \mathfrak{M}^{\mathcal{F}}_{m,i}\,,$$

where $2M_j = M_j + M_j$. In $SK^{\mathcal{F}}_{m,i}$ we see

$$(*)\quad [M; M_m, \ldots, M_i] + [(M - \mathring{T}) \cup_\varphi (RP(\nu(M_i) \oplus \mathbf{R}) - \mathring{T}'); M_m, \cdots, M_{i+1}, \emptyset]$$
$$= [RP(\nu(M_i) \oplus \mathbf{R}); \emptyset, \cdots, \emptyset, M_i] + [(M - \mathring{T}) \cup_{\mathrm{id}} (M - \mathring{T}); 2M_m, \cdots, 2M_{i+1}, \emptyset],$$

since

$$(M; M_m, \ldots, M_i) = (T'; \emptyset, \cdots, \emptyset, M_i) \cup_\varphi (M - \mathring{T}; M_m, \cdots, M_{i+1}, \emptyset).$$

LEMMA 2.2. *Given* $(M; M_m, \cdots, M_i), (N; N_m, \cdots, N_i) \in \mathfrak{M}^{\mathcal{F}}_{m,i}$, *if* $[M_i] = [N_i]$ *in* SK_i, *then*

$$[RP(\nu(M_i) \oplus \mathbf{R}); \emptyset, \cdots, \emptyset, M_i] = [RP(\nu(N_i) \oplus \mathbf{R}); \emptyset, \cdots, \emptyset, N_i]$$

in $SK^{\mathcal{F}}_{m,i}$.

To prove the lemma we introduce the SK-equivalence for vector bundles and the SK-group of singular manifolds in a space X.

Let E and F be k-dimensional vector bundles over m-dimensional compact manifolds P and Q, respectively, and $\varphi : E|\partial P \to F|\partial Q$ be a bundle isomorphism which induces a diffeomorphism $\overline{\varphi} : \partial P \to \partial Q$. Then we have a k-dimensional vector bundle $E \cup_\varphi F$ over an m-dimensional closed manifold $P \cup_{\overline{\varphi}} Q$. As in the usual way we can define the SK-equivalence for vector bundles.

An *m-dimensional singular manifold* (A, f) in a space X is a continuous map $f : A \to X$ with $A \in \mathfrak{M}_m$. We can define the SK-group $SK_m(X)$ of such singular manifolds (A, f) in X. See for details Karras et al [7] or Kosniowski [14]. The correspondence $(A, f) \mapsto A$ induces a homomorphism $\varepsilon : SK_m(X) \to SK_m$. It is known that ε is an isomorphism for some spaces X.

PROOF OF LEMMA 2.2. Let $f : M_i \to BO(m-i)$ and $g : N_i \to BO(m-i)$ be classifying maps for the $(m-i)$-dimensional vector bundles $\nu(M_i)$ and $\nu(N_i)$, respectively. Since $\varepsilon : SK_i(BO(m-i)) \to SK_i$ is an isomorphism ([7, Theorem 2.11], [14, Theorem 3.5.1]), the assumption $[M_i] = [N_i]$ shows $[M_i, f] = [N_i, g]$ in $SK_i(BO(m-i))$. This implies that $\nu(M_i)$ and $\nu(N_i)$ are SK-equivalent as bundles, and then that $RP(\nu(M_i)\oplus\mathbf{R})$ and $RP(\nu(N_i)\oplus\mathbf{R})$ are SK-equivalent in $\mathfrak{M}_m$. The cutting and pasting operation performing from $RP(\nu(M_i) \oplus \mathbf{R})$ to $RP(\nu(N_i) \oplus \mathbf{R})$ restricts to that from M_i ($\approx RP(\mathbf{R})$) to N_i ($\approx RP(\mathbf{R})$). This implies

$$[RP(\nu(M_i) \oplus \mathbf{R}); \emptyset, \cdots, \emptyset, M_i] = [RP(\nu(N_i) \oplus \mathbf{R}); \emptyset, \cdots, \emptyset, N_i]$$

in $SK^{\mathcal{F}}_{m,i}$. □

PROPOSITION 2.3. *In the sequence in Theorem* 2.1, $\mathrm{Im}\, \iota^{\mathcal{F}}_{i+1} = \mathrm{Ker}\, \rho^{\mathcal{F}}_i$.

PROOF. Im $\iota^{\mathcal{F}}_{i+1} \subset$ Ker $\rho^{\mathcal{F}}_i$ is easily shown. To show the reversed inclusion, assume $\rho^{\mathcal{F}}_i(x) = 0$ for $x = [M; M_m, \cdots, M_i] - [N; N_m, \cdots, N_i] \in SK^{\mathcal{F}}_{m,i}$, i.e., $[M_i] = [N_i]$ in SK_i. Using the equality $(*)$ and Lemma 2.2, we see

$$\begin{aligned} x &= [M; M_m, \cdots, M_i] - [N; N_m, \cdots, N_i] \\ &= [(M - \overset{\circ}{T}) \cup_{\mathrm{id}} (M - \overset{\circ}{T}); 2M_m, \cdots, 2M_{i+1}, \emptyset] \\ &\quad - [(M - \overset{\circ}{T}) \cup_{\varphi} (RP(\nu(M_i) \oplus \mathbf{R}) - \overset{\circ}{T}{}'); M_m, \cdots, M_{i+1}, \emptyset] \\ &\quad - [(N - \overset{\circ}{U}) \cup_{\mathrm{id}} (N - \overset{\circ}{U}); 2N_m, \cdots, 2N_{i+1}, \emptyset] \\ &\quad + [(N - \overset{\circ}{U}) \cup_{\psi} (RP(\nu(N_i) \oplus \mathbf{R}) - \overset{\circ}{U}{}'); N_m, \cdots, N_{i+1}, \emptyset], \end{aligned}$$

where ψ, U and U' are ones obtained in the same way as φ, T and T'. This shows $x \in$ Im $\iota^{\mathcal{F}}_{i+1}$. □

From Propositions 1.1, 1.2 and 2.3 we obtain Theorem 2.1.

COROLLARY 2.4. *$\tilde{M} = (M; M_m, \cdots, M_0)$, $\tilde{N} = (N; N_m, \cdots, N_0) \in \mathfrak{M}^{\mathcal{F}}_m$ are SK-equivalent if and only if $\chi(M) = \chi(N)$ and $\chi(M_i) = \chi(N_i)$ for any i with $0 \leq i \leq m$.*

PROOF. If $\tilde{M}$ and $\tilde{N}$ are SK-equivalent, then it is easily shown $\chi(M) = \chi(N)$ and $\chi(M_i) = \chi(N_i)$ since a cutting and pasting operation keeps the Euler characteristic invariant. The converse is shown by showing the following assertion A(i) for any i with $0 \leq i \leq m+1$ by downward induction.

A(i): *If $\chi(M) = \chi(N)$ and $\chi(M_j) = \chi(N_j)$ for any j with $i \leq j \leq m$, then $[M; M_m, \cdots, M_i] = [N; N_m, \cdots, N_i]$ in $SK^{\mathcal{F}}_{m,i}$.*

Since $SK^{\mathcal{F}}_{m,m+1} = SK_m$, the assertion A$(m+1)$ is already known from Karras et al. [7]. As the induction hypothesis we assume A$(i+1)$ holds, and also assume the assumption in A(i). In the sequence in Theorem 2.1 we see in SK_i,

$$\rho^{\mathcal{F}}_i([M; M_m, \cdots, M_i] - [N; N_m, \cdots, N_i]) = [M_i] - [N_i] = 0\,.$$

Hence we have $(M'; M'_m, \cdots, M'_{i+1}), (N'; N'_m, \cdots, N'_{i+1}) \in \mathfrak{M}^{\mathcal{F}}_{m,i+1}$ such that

$$\iota^{\mathcal{F}}_{i+1}([M'; M'_m, \cdots, M'_{i+1}] - [N'; N'_m, \cdots, N'_{i+1}]) = [M; M_m, \cdots, M_i] - [N; N_m, \cdots, N_i]$$

in $SK^{\mathcal{F}}_{m,i}$, i.e.,

$$[M; M_m, \cdots, M_i] + [N'; N'_m, \cdots, N'_{i+1}, \emptyset] = [N; N_m, \cdots, N_i] + [M'; M'_m, \cdots, M'_{i+1}, \emptyset]\,.$$

From this and the assumption in A(i) we have $\chi(M') = \chi(N')$ and $\chi(M'_j) = \chi(N'_j)$ for $i+1 \leq j \leq m$. A$(i+1)$ says $[M'; M'_m, \cdots, M'_{i+1}] = [N'; N'_m, \cdots, N'_{i+1}]$ in $SK^{\mathcal{F}}_{m,i+1}$. Hence we have $[M; M_m, \cdots, M_i] = [N; N_m, \cdots, N_i]$ in $SK^{\mathcal{F}}_{m,i}$. This proves A(i). □

3. SK-group of $\mathbf{Z}_2$-manifolds

For $0 \le i \le m+1$, let $\mathfrak{M}^{\mathbf{Z}_2}_{m,i}$ be the subset of $\mathfrak{M}^{\mathbf{Z}_2}_m$ consisting of those $M \in \mathfrak{M}^{\mathbf{Z}_2}_m$ for which $M^{\mathbf{Z}_2}_j = \emptyset$ for any $j < i$. Then we have a sequence of subsets of $\mathfrak{M}^{\mathbf{Z}_2}_m$,

$$\mathfrak{M}^{\mathbf{Z}_2}_{m,m+1} \subset \mathfrak{M}^{\mathbf{Z}_2}_{m,m} \subset \cdots \subset \mathfrak{M}^{\mathbf{Z}_2}_{m,0} = \mathfrak{M}^{\mathbf{Z}_2}_m .$$

As in the same way for $\mathfrak{M}^{\mathbf{Z}_2}_m$ we can define the SK-equivalence in $\mathfrak{M}^{\mathbf{Z}_2}_{m,i}$ and then define the SK-group $SK^{\mathbf{Z}_2}_{m,i}$ as the Grothendieck group of the semigroup $\mathfrak{M}^{\mathbf{Z}_2}_{m,i}/SK$. $SK^{\mathbf{Z}_2}_{m,i}$ is the same group as $SK^{\mathbf{Z}_2}_m[\mathcal{F}_{m-i}]$ in his notation in Kosniowski [14, §5.3].

The inclusion $\mathfrak{M}^{\mathbf{Z}_2}_{m,i} \subset \mathfrak{M}^{\mathbf{Z}_2}_{m,i-1}$ induces a homomorphism $\iota^{\mathbf{Z}_2}_i : SK^{\mathbf{Z}_2}_{m,i} \to SK^{\mathbf{Z}_2}_{m,i-1}$.

It is well-known that $\chi(M) \equiv \chi(M^{\mathbf{Z}_2}) \mod 2$ for $M \in \mathfrak{M}^{\mathbf{Z}_2}_m$ (see for example Kawakubo [8, Chapter 5]). For $M \in \mathfrak{M}^{\mathbf{Z}_2}_{m,m-1}$ we see $\chi(M^{\mathbf{Z}_2}_{m-1}) \equiv 0 \mod 2$. This is seen as follows. If m is even, then $\chi(M^{\mathbf{Z}_2}_{m-1}) = 0$. If m is odd, then

$$\begin{aligned} 0 = \chi(M) &\equiv \chi(M^{\mathbf{Z}_2}) \mod 2 \\ &= \chi(M^{\mathbf{Z}_2}_{m-1}) + \chi(M^{\mathbf{Z}_2}_m) = \chi(M^{\mathbf{Z}_2}_{m-1}) . \end{aligned}$$

Hence we have $M' \in \mathfrak{M}_{m-1}$ such that $\chi(M') = \chi(M^{\mathbf{Z}_2}_{m-1})/2$. Hence the correspondence $M \mapsto M'$ induces a homomorphism $\rho^{\mathbf{Z}_2}_{m-1} : SK^{\mathbf{Z}_2}_{m,m-1} \to SK_{m-1}$. For $i \neq m-1$ we define a homomorphism $\rho^{\mathbf{Z}_2}_i : SK^{\mathbf{Z}_2}_{m,i} \to SK_i$ by the correspondence $M \mapsto M^{\mathbf{Z}_2}_i$.

THEOREM 3.1. *For any i with $0 \le i \le m$ we have the split short exact sequence*

$$0 \longrightarrow SK^{\mathbf{Z}_2}_{m,i+1} \xrightarrow{\iota^{\mathbf{Z}_2}_{i+1}} SK^{\mathbf{Z}_2}_{m,i} \xrightarrow{\rho^{\mathbf{Z}_2}_i} SK_i \longrightarrow 0 .$$

PROOF. Let $SK^{\mathbf{Z}_2}_m[\mathcal{F}_j]$ and $SK^{\mathbf{Z}_2}_m[\sigma_j]$ be the ones in Kosniowski [14, §5.3]. We see $SK_i \cong SK^{\mathbf{Z}_2}_m[\sigma_{m-i}]$, and $SK^{\mathbf{Z}_2}_{m,i} = SK^{\mathbf{Z}_2}_m[\mathcal{F}_{m-i}]$ as noted before. Therefore the sequence in the theorem is already shown in Kosniowski [14, §5.3] to be split short exact. □

4. Relations of $SK^{\mathcal{F}}_m$ to $SK^{\mathbf{Z}_2}_m$

In this section we will prove the results stated in the Introduction.

We have a correspondence $\mathfrak{M}^{\mathbf{Z}_2}_{m,i} \to \mathfrak{M}^{\mathcal{F}}_{m,i}$ which assigns to a $\mathbf{Z}_2$-manifold $M \in \mathfrak{M}^{\mathbf{Z}_2}_{m,i}$ the family $(M; M^{\mathbf{Z}_2}_m, \cdots, M^{\mathbf{Z}_2}_i) \in \mathfrak{M}^{\mathcal{F}}_{m,i}$. This induces a homomorphism $\eta_i : SK^{\mathbf{Z}_2}_{m,i} \to SK^{\mathcal{F}}_{m,i}$. Note that $\eta_0 = \eta : SK^{\mathbf{Z}_2}_m \to \mathfrak{M}^{\mathcal{F}}_m$. Define $\theta_i : SK_i \to SK_i$ by $\theta_i = \mathrm{id}$ if $i \neq m-1$, and by $\theta_i = 2$ (the multiplication by 2) if $i = m-1$. Then we have the following commutative diagram for any i with $0 \le i \le m$,

$$
\text{(D)}\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & SK^{\mathbf{Z}_2}_{m,i+1} & \xrightarrow{\iota^{\mathbf{Z}_2}_{i+1}} & SK^{\mathbf{Z}_2}_{m,i} & \xrightarrow{\rho^{\mathbf{Z}_2}_{i}} & SK_i & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \eta_{i+1}} & & \Big\downarrow{\scriptstyle \eta_i} & & \Big\downarrow{\scriptstyle \theta_i} & & \\
0 & \longrightarrow & SK^{\mathcal{F}}_{m,i+1} & \xrightarrow{\iota^{\mathcal{F}}_{i+1}} & SK^{\mathcal{F}}_{m,i} & \xrightarrow{\rho^{\mathcal{F}}_{i}} & SK_i & \longrightarrow & 0\,.
\end{array}
$$

The correspondence $\mathfrak{M}^{\mathbf{Z}_2}_{m,m+1} \to \mathfrak{M}_m$, $M \mapsto M/\mathbf{Z}_2$, induces an isomorphism $SK^{\mathbf{Z}_2}_{m,m+1} \cong SK_m$, since $\mathfrak{M}^{\mathbf{Z}_2}_{m,m+1}$ is the set of m-dimensional closed free $\mathbf{Z}_2$-manifolds, where $M/\mathbf{Z}_2$ denotes the orbit space of $M \in \mathfrak{M}^{\mathbf{Z}_2}_{m,m+1}$. We see $\eta_{m+1} : SK^{\mathbf{Z}_2}_{m,m+1} \to SK^{\mathcal{F}}_{m,m+1}$ is injective, since $\eta_{m+1}([M]) = [M] = 2[M/\mathbf{Z}_2]$ for $M \in \mathfrak{M}^{\mathbf{Z}_2}_{m,m+1}$. θ_i is injective for any i. Therefore, using the diagram (D) we have the following theorem by downward induction for i.

THEOREM 4.1. *The homomorphism $\eta : SK^{\mathbf{Z}_2}_m \to SK^{\mathcal{F}}_m$ is injective.*

For any $x \in \operatorname{Im}\eta$ we see $\chi(x) \equiv \sum_{i=0}^m \chi_i(x) \mod 2$, since $\chi(M) \equiv \chi(M^{\mathbf{Z}_2}) \mod 2$ for $M \in \mathfrak{M}^{\mathbf{Z}_2}_m$. The following theorem shows that this congruence is also sufficient for $x \in SK^{\mathcal{F}}_m$ to be in the image of η.

THEOREM 4.2. *An element $x \in SK^{\mathcal{F}}_m$ to be in the image of η if and only if $\chi(x) \equiv \sum_{i=0}^m \chi_i(x) \mod 2$. In particular, $2x \in \operatorname{Im}\eta$ for any $x \in SK^{\mathcal{F}}_m$.*

PROOF. It is sufficient to prove the following assertion B(i) for any i with $0 \le i \le m+1$:

B(i): *If $\chi(x) \equiv \sum_{j=i}^m \chi_j(x) \mod 2$ for $x \in SK^{\mathcal{F}}_{m,i}$, then $x \in \operatorname{Im}\eta_i$.*

We prove this by downward induction for i. If $i = m+1$, then the hypothesis is $\chi(x) \equiv 0 \mod 2$. Put $x = [M] - [N]$ for $M, N \in \mathfrak{M}_m$, then $\chi(M) - \chi(N) \equiv 0 \mod 2$. If $m > 0$, or if $m = 0$ and $\chi(x) \ge 0$, we have $L \in \mathfrak{M}_m$ such that $\chi(L) = (\chi(M) - \chi(N))/2$. Then, for $\mathbf{Z}_2 \times L \in \mathfrak{M}^{\mathbf{Z}_2}_{m,m+1}$ we have $\eta_{m+1}([\mathbf{Z}_2 \times L]) = [M] - [N] = x$. If $m = 0$ and $\chi(x) < 0$, then consider $-x$. This proves B($m+1$).

As the induction hypothesis, assume B($i+1$) holds. Let $x \in SK^{\mathcal{F}}_{m,i}$ be as in B(i). Then we see $\rho^{\mathcal{F}}_i(x) \in \operatorname{Im}\theta_i$ even if $i = m-1$, since we see $\chi(\rho^{\mathcal{F}}_i(x)) \equiv 0 \mod 2$ as in §3. By a diagram chasing in (D), we have $y \in SK^{\mathbf{Z}_2}_{m,i}$ and $z \in SK^{\mathcal{F}}_{m,i+1}$ such that $\iota^{\mathcal{F}}_{i+1}(z) = x - \eta_i(y)$. Then

$$
\begin{aligned}
\chi(z) &= \chi(x - \eta_i(y)) \\
&\equiv \sum_{j=i}^m \chi_j(x - \eta_i(y)) \mod 2 \quad \text{by the assumption of B}(i)\,,
\end{aligned}
$$

$$= \sum_{j=i}^{m} \chi_j(z) \ = \ \sum_{j=i+1}^{m} \chi_j(z) \quad (\because \ \chi_i(z) = 0) \,.$$

Using B$(i+1)$, we have $w \in SK^{\mathbf{Z}_2}_{m,i+1}$ such that $\eta_{i+1}(w) = z$. Using the diagram (D) again, we have

$$\eta_i(\iota^{\mathbf{Z}_2}_{i+1}(w)) = \iota^{\mathcal{F}}_{i+1}(\eta_{i+1}(w)) = \iota^{\mathcal{F}}_{i+1}(z) = x - \eta_i(y) \,.$$

This shows $x \in \mathrm{Im}\, \eta_i$. □

From Corollary 2.4 we know that $SK^{\mathcal{F}}_m$ has no torsion. Hence, tensoring $\mathbf{Z}[1/2]$ with the monomorphism $\eta : SK^{\mathbf{Z}_2}_m \to SK^{\mathcal{F}}_m$, we have

COROLLARY 4.3. $SK^{\mathbf{Z}_2}_m \otimes \mathbf{Z}[1/2] \cong SK^{\mathcal{F}}_m \otimes \mathbf{Z}[1/2]$.

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