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A Landau-Kolmogorov Inequality for Lorentz Spaces

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Dedicated to Professor Mitsuo Morimoto on the occasion of his sixtieth birthday

Abstract. In this paper we prove that the Landau-Kolmogorov inequality for functions on the half line holds for Lorentz spaces with the constants, which are best possible for L_∞-space.

1. Introduction

The Landau-Kolmogorov inequality

$$\|f^{(k)}\|_\infty^n \leq K(k,n)\|f\|_\infty^{n-k}\|f^{(n)}\|_\infty^k\,, \tag{1}$$

where $0 < k < n$, is well known and has many interesting applications and generalizations (see [1–7, 13, 16, 17, 20–21]). Its study was initiated by Landau [11] and Hadamard [8] (the case $n = 2$). For functions on the whole real line $\mathbf{R}$, Kolmogorov [10] succeeded in finding in explicit form the best possible constants $K(k,n) = C_{k,n}$ in (1), and Stein proved in [20] that inequality (1) still holds for L_p-norm, $1 \leq p < \infty$, with these constants (the same situation also happens for an arbitrary Orlicz norm [1]). The best constants $C^+_{k,n}$ for the half line $\mathbf{R}_+ = [0,\infty)$ are not known in explicit form except for $n = 2,\ 3,\ 4$ (see [11, 12]), but an algorithm exists for their computation (Schoenberg and Cavaretta [15]). In this paper, essentially developing the Stein method [20], we prove that, for the half line, inequality (1) still holds for Lorentz spaces with the constants $C^+_{k,n}$. Note that a similar result for Orlicz spaces was proved in [2] by the techniques which cannot be used for Lorentz spaces.

2. Results

Let $\Phi : [0,\infty) \to [0,\infty)$ be a non-zero concave function, which is non-decreasing and $\Phi(0+) = \Phi(0) = 0$. We put $\Phi(\infty) = \lim_{t\to\infty}\Phi(t)$. Let S be an interval of $\mathbf{R}$. For an

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arbitrary measurable function f we define

$$\|f\|_{N_\Phi(S)} = \int_0^\infty \Phi(\lambda_f(y))dy\,,$$

where $\lambda_f(y) = \mathrm{mes}\{x \in S : |f(x)| > y\}$, $(y \ge 0)$. If the space $N_\Phi(S)$ consists of measurable functions f such that $\|f\|_{N_\Phi(S)} < \infty$ then $N_\Phi(S)$ is a Banach space. Denote by $M_\Phi(S)$, the space of measurable functions g such that

$$\|g\|_{M_\Phi(S)} = \sup\Big\{\frac{1}{\Phi(\mathrm{mes}\ \Delta)}\int_\Delta |g(x)|dx :\ \Delta \subset S,\ 0 < \mathrm{mes}\ \Delta < \infty\Big\} < \infty\,.$$

Then $M_\Phi(S)$ is a Banach space, too [19], [18], [14]. The $N_\Phi(S)$ and $M_\Phi(S)$ are called Lorentz spaces.

We have the following results [19], [18], [6]:

LEMMA 1. *If $f \in N_\Phi(S)$, $g \in M_\Phi(S)$ then $fg \in L_1(S)$ and*

$$\int_S |f(x)g(x)|dx \le \|f\|_{N_\Phi(S)}\|g\|_{M_\Phi(S)}\,.$$

LEMMA 2. *If $f \in N_\Phi(S)$ then*

$$\|f\|_{N_\Phi(S)} = \sup_{\|g\|_{M_\Phi(S)}\le 1}\Big|\int_S f(x)g(x)dx\Big|\,.$$

LEMMA 3. *Let $n \ge 1$. If $f \in L_{1,loc}(\mathbf{R}_+)$ has a generalized n-th derivative $g \in L_{1,loc}(\mathbf{R}_+)$, then f can be redefined on a set of measure zero so that $f^{(n-1)}$ is absolutely continuous and $f^{(n)} = g$ a.e. on $\mathbf{R}_+$.*

THEOREM 1. *Let f and its generalized derivative $f^{(n)}$ be in $N_\Phi(\mathbf{R}_+)$. Then $f^{(k)} \in N_\Phi(\mathbf{R}_+)$ for all $0 < k < n$ and*

$$\|f^{(k)}\|^n_{N_\Phi(\mathbf{R}_+)} \le C^+_{k,n}\|f\|^{n-k}_{N_\Phi(\mathbf{R}_+)}\|f^{(n)}\|^k_{N_\Phi(\mathbf{R}_+)}\,. \tag{1}$$

PROOF. We begin to prove (1) with the assumption that $f^{(k)} \in N_\Phi(\mathbf{R}_+)$, $0 \le k \le n$. Fixed $0 < k < n$. By Lemma 2 we see that for any $\varepsilon > 0$ there exists a function $v_\varepsilon \in M_\Phi(\mathbf{R}_+)$ such that $\|v_\varepsilon\|_{M_\Phi(\mathbf{R}_+)} \le 1$ and

$$\Big|\int_0^\infty f^{(k)}(x)v_\varepsilon(x)dx\Big| \ge \|f^{(k)}\|_{N_\Phi(\mathbf{R}_+)} - \varepsilon/2\,.$$

By Lemma 1, there is an interval $\mathcal{H} := [c,d],\ c,d \in (0,\infty)$ such that

$$\Big|\int_0^\infty f^{(k)}(x)v(x)dx\Big| \ge \|f^{(k)}\|_{N_\Phi(\mathbf{R}_+)} - \varepsilon\,, \tag{2}$$

where $v = v(\mathcal{H},\varepsilon) := \chi_{\mathcal{H}}v_\varepsilon$. Put

$$F_\varepsilon(x) = \int_0^\infty f(x+y)v(y)dy\,.$$

Then $F_\varepsilon \in L_\infty(\mathbf{R}_+)$ by virtue of Lemma 1, and it is easy to check that

$$F_\varepsilon^{(r)}(x) = \int_0^\infty f^{(r)}(x+y)v(y)dy, \quad 0 \le r \le n \tag{3}$$

in the distribution sense.

For all $x \in \mathbf{R}_+$, clearly,

$$|F_\varepsilon^{(r)}(x)| \le \|f^{(r)}(x+\cdot)\|_{N_\Phi(\mathbf{R}_+)}\|v\|_{M_\Phi(\mathbf{R}_+)} \le \|f^{(r)}\|_{N_\Phi(\mathbf{R}_+)} .$$

Now we prove the continuity of $F_\varepsilon^{(r)}$ on $\mathbf{R}_+$ $(0 \le r \le n)$. We show this for $r = 0$. Clearly, it suffices to prove that for any $x \in \mathbf{R}_+$,

$$\lim_{t\to 0} \|\chi_{\mathcal{H}}(\cdot)\big(f(x+t+\cdot) - f(x+\cdot)\big)\|_{N_\Phi(\mathbf{R}_+)} = 0 .$$

Assume the contrary that for some $\delta > 0$, point x^0 and sequence $\{t_m\}$ with $t_m \to 0$,

$$\|\chi_{\mathcal{H}}(\cdot)\big(f(x^0+t_m+\cdot) - f(x^0+\cdot)\big)\|_{N_\Phi(\mathbf{R}_+)} \ge \delta , \quad m \ge 1 . \tag{4}$$

For simplicity of notation we suppose $x^0 = 0$. Since $f \in N_\Phi(\mathbf{R}_+)$, $f \in L^1_{\ell oc}(\mathbf{R}_+)$. So, it is known that

$$\int_c^d |f(x+t_m) - f(x)|dx \to 0 \qquad \text{as} \quad m \to \infty .$$

Therefore, there exists a subsequence $\{t_{m_j}\}$, we still denote by $\{t_m\}$ such that $f(\cdot + t_m) \to f$ a.e. on $\mathcal{H}$. Define

$$g_n(x) = \inf_{m \ge n} |f(x+t_m)| , \quad x \in \mathcal{H} ,$$

then $\{g_n\}$ is a non-decreasing sequence and $g_n \to |f|$ a.e. on $\mathcal{H}$. It is easy to see that

$$\lambda_{\chi_{\mathcal{H}} g_n}(t) \to \lambda_{\chi_{\mathcal{H}}|f|}(t) \quad \text{as } n \to \infty , \quad \text{for every} \quad t > 0 .$$

We have

$$\Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) = \lim_{m\to\infty} \Phi(\lambda_{\chi_{\mathcal{H}} g_m}(t)) \le \varliminf_{m\to\infty} \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)), \ t > 0 . \tag{5}$$

It follows from the definition of Φ that $\Phi(a+b) \le \Phi(a) + \Phi(b)$ for $a, b \ge 0$. Observing that, for any $f, g \in N_\Phi(\mathbf{R}_+)$ and $t > 0$ we have $\lambda_{\chi_{\mathcal{H}}(f+g)}(2t) \le \lambda_{\chi_{\mathcal{H}} f}(t) + \lambda_{\chi_{\mathcal{H}} g}(t)$, then

$$\Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t)) \le \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)) + \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t)), m \ge 1 .$$

It is easy to check that

$$\lim_{m\to\infty} \|\chi_{\mathcal{H}} f(\cdot + t_m)\|_{N_\Phi(\mathbf{R}_+)} = \|\chi_{\mathcal{H}} f\|_{N_\Phi(\mathbf{R}_+)} .$$

Applying Fatou's lemma, we obtain

$$\int_0^\infty \underline{\lim}_{m\to\infty} [\Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)) + \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) - \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t))]dt$$
$$\leq \underline{\lim}_{m\to\infty} \int_0^\infty [\Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)) + \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) - \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t)]dt$$
$$= 2\int_0^\infty \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t))dt - \frac{1}{2}\overline{\lim}_{m\to\infty}\int_0^\infty \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t))dt\,. \tag{6}$$

On the other hand,

$$\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t) = \text{mes}\{x \in \mathcal{H} : |f(x+t_m) - f(x)| > t\}.$$

Therefore, taking account of $f(\cdot + t_m) \to f$ a.e. on $\mathcal{H}$, we have

$$\lim_{m\to\infty} \lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t) = 0$$

and then

$$\lim_{m\to\infty} \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t)) = 0\,.$$

So, by (5) we get for any $t > 0$

$$2\Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) = \lim_{m\to\infty} \Phi(\lambda_{\chi_{\mathcal{H}} g_m}(t)) + \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) - \lim_{m\to\infty} \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t))$$
$$\leq \underline{\lim}_{m\to\infty} [\Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)) + \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) - \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t))]\,.$$

So, since (6), we have

$$2\int_0^\infty \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t))dt \leq 2\int_0^\infty \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t))dt - \frac{1}{2}\overline{\lim}_{m\to\infty}\int_0^\infty \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t))dt\,.$$

Hence

$$\int_0^\infty \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t))dt \to 0 \ \text{ as } m \to \infty\,,$$

i.e., $\lim_{m\to\infty} \|\chi_{\mathcal{H}}\big(f(\cdot + t_m) - f\big)\|_{N_\Phi(\mathbf{R}_+)} = 0$, which contradicts (4).

The cases $1 \leq r \leq n$ are proved similarly. The continuity of $F_\varepsilon^{(r)}$ has been proved.

Thus by the classical Landau-Kolmogorov inequality we have

$$|F_\varepsilon^{(k)}(0)|^n \leq C_{k,n}^+ \|F_\varepsilon\|_\infty^{n-k} \|F_\varepsilon^{(n)}\|_\infty^k\,,$$

which shows, with the help of (2) and the fact that $|F_\varepsilon^{(r)}(x)| \leq \|f^{(r)}\|_{N_\Phi(\mathbf{R}_+)}$ $(0 \leq r \leq n)$, the inequality

$$\{\|f^{(k)}\|_{N_\Phi(\mathbf{R}_+)} - \varepsilon\}^n \leq C_{k,n}^+ \|f\|_{N_\Phi(\mathbf{R}_+)}^{n-k} \|f^{(n)}\|_{N_\Phi(\mathbf{R}_+)}^k\,.$$

Therefore, by letting $\varepsilon \to 0$ we have (1) under the additional assumption that $f^{(r)} \in N_\Phi(\mathbf{R}_+)$ for $r = 1, 2, \cdots, n-1$.

To complete the proof, it remains to show that $f^{(k)} \in N_\Phi(\mathbf{R}_+), \forall k = 1, \cdots, n-1$ if $f, f^{(n)} \in N_\Phi(\mathbf{R}_+)$. By Lemma 3 we can assume that $f, f', \cdots, f^{(n-1)}$ are continuous on $[0, \infty)$ and $f^{(n-1)}$ is absolutely continuous on $[0, \infty)$.

We put for $k = 0, \cdots, n$,

$$f_{(k)}(x) = \begin{cases} f^{(k)}(x), & x \in [0, \infty) \\ 0, & x \in (-\infty, 0)\,. \end{cases}$$

Let $\psi \in C_0^\infty(0, \infty), \psi \geq 0, \psi(x) = 0$ for $x \geq 1$ and $\int_{\mathbf{R}_+} \psi(x)dx = 1$. We put $\psi_\lambda(x) = \frac{1}{\lambda}\psi(\frac{x}{\lambda}),\ \lambda > 0$ and $f_\lambda = f_{(0)} * \psi_\lambda$.

Fix $b > 0$. Then $\forall \varphi \in C_0^\infty(b, \infty)$ we have for $0 < \lambda < b, k = 1, \cdots, n$:

$$\begin{aligned}
\langle f_\lambda^{(k)}, \varphi > &= (-1)^k < f_\lambda, \varphi^{(k)}\rangle \\
&= (-1)^k \int_0^\infty \Big(\int_0^\infty f_{(0)}(x-y)\psi_\lambda(y)dy\Big)\varphi^{(k)}(x)dx \\
&= \int_0^\lambda \Big((-1)^k \int_b^\infty f_{(0)}(x-y)\varphi^{(k)}(x)dx\Big)\psi_\lambda(y)dy \\
&= \int_0^\lambda \Big(\int_b^\infty f^{(k)}(x-y)\varphi(x)dx\Big)\psi_\lambda(y)dy \\
&= \int_b^\infty \Big(\int_0^\lambda f^{(k)}(x-y)\psi_\lambda(y)dy\Big)\varphi(x)dx \\
&= \int_b^\infty (f_{(k)} * \psi_\lambda)(x)\varphi(x)dx \\
&= \langle f_{(k)} * \psi_\lambda,\ \varphi\rangle\,.
\end{aligned}$$

So, we have proved for $0 < \lambda < b$,

$$f_\lambda^{(k)} = (f_{(0)} * \psi_\lambda)^{(k)} = f_{(k)} * \psi_\lambda \tag{7}$$

in the $\mathcal{D}'(b, \infty)$ sense. Therefore, for $0 < \lambda < b$ we have

$$\begin{aligned}
\|(f_{(0)} * \psi_\lambda)^{(n)}\|_{N_\Phi[b,\infty)} &= \|f_{(n)} * \psi_\lambda\|_{N_\Phi[b,\infty)} \\
&\leq \|f_{(n)} * \psi_\lambda\|_{N_\Phi(\mathbf{R})} \leq \|f_{(n)}\|_{N_\Phi(\mathbf{R})} \qquad (8) \\
&= \|f_{(n)}\|_{N_\Phi(\mathbf{R}_+)} = \|f^{(n)}\|_{N_\Phi(\mathbf{R}_+)}\,.
\end{aligned}$$

On the other hand, using $(f_{(0)}*\psi_\lambda)^{(k)} = f_{(0)}*\psi_\lambda^{(k)} \in N_\Phi(\mathbf{R}), \forall k = 0, 1, \cdots, n$ and the above proved Landau-Kolmogorov inequality for functions on $[b, \infty)$, we get for $k = 1, \cdots, n-1$,

$$\|f_\lambda^{(k)}\|^n_{N_\Phi[b,\infty)} \le C^+_{k,n}\|f_\lambda\|^{n-k}_{N_\Phi[b,\infty)}\|f_\lambda^{(n)}\|^k_{N_\Phi[b,\infty)}.$$

Hence, combining (7), (8) we get for all $0 < \lambda < b$, $k = 1, \cdots, n-1$,

$$\begin{aligned}\|f_{(k)} * \psi_\lambda\|^n_{N_\Phi[b,\infty)} &\le C^+_{k,n}\|f_{(0)} * \psi_\lambda\|^{n-k}_{N_\Phi[b,\infty)}\|f_{(n)} * \psi_\lambda\|^k_{N_\Phi[b,\infty)}\\ &\le C^+_{k,n}\|f\|^{n-k}_{N_\Phi[0,\infty)}\|f^{(n)}\|^k_{N_\Phi[0,\infty)}. \end{aligned} \tag{9}$$

On the other hand, because $f_{(k)}$ is continuous on $\mathbf{R}_+$, we get easily

$$\lim_{\lambda\to 0} f_{(k)} * \psi_\lambda(x) = f_{(k)}(x) = f^{(k)}(x), \forall x > 0. \tag{10}$$

For each function $v \in M_\Phi[b, \infty)$, $\|v\|_{M_\Phi[b,\infty)} \le 1$ and $0 < \lambda < b$, by (9) and the definition of the $N_\Phi[b, \infty)$-norm we get

$$\int_b^\infty |(f_{(k)} * \psi_\lambda)(x)v(x)|dx \le \{C^+_{k,n}\|f\|^{n-k}_{N_\Phi[0,\infty)}\|f^{(n)}\|^k_{N_\Phi[0,\infty)}\}^{1/n}.$$

Therefore, using the Fatou lemma, (9) and (10) we have

$$\begin{aligned}\Big|\int_b^\infty (f^{(k)}(x)v(x)dx\Big| &= \Big|\int_b^\infty \lim_{\lambda\to 0}(f_{(k)} * \psi_\lambda)(x)v(x)dx\Big|\\ &\le \int_b^\infty (\lim_{\lambda\to 0}|(f_{(k)} * \psi_\lambda)(x)v(x)|)dx\\ &\le \underline{\lim}_{\lambda\to 0}\int_b^\infty |(f_{(k)} * \psi_\lambda)(x)v(x)|dx\\ &\le \underline{\lim}_{\lambda\to 0}\|f_{(k)} * \psi_\lambda\|_{N_\Phi[b,\infty)}\\ &\le \big\{C^+_{k,n}\|f\|^{(n-k)}_{N_\Phi[0,\infty)}\|f^{(n)}\|^k_{N_\Phi[0,\infty)}\big\}^{1/n}.\end{aligned}$$

So, by the definition,

$$\|f^{(k)}\|^n_{N_\Phi[b,\infty)} \le C^+_{k,n}\|f\|^{(n-k)}_{N_\Phi[0,\infty)}\|f^{(n)}\|^k_{N_\Phi[0,\infty)} < \infty.$$

On the other hand, it follows from the continuity of $f^{(k)}$ on $[0, \infty]$ that $f^{(k)} \in N_\Phi[0, b)$ for any $b > 0$. Therefore,

$$\|f^{(k)}\|_{N_\Phi(\mathbf{R}_+)} \le \|f^{(k)}\|_{N_\Phi[0,b]} + \|f^{(k)}\|_{N_\Phi[b,\infty)} < \infty.$$

The proof is complete.

Finally, it is known that there is a smallest constant C^+ depending only on n such that

$$\delta^k||f^{(k)}||_\infty \le C^+(||f||_\infty + \delta^n||f^{(n)}||_\infty), \tag{11}$$

where $\delta > 0$ is arbitrary (see [6]). Modifying the above proof, we can get the following result.

THEOREM 2. *Let f and its generalized derivative $f^{(n)}$ be in $N_\Phi(\mathbf{R}_+)$. Then $f^{(k)} \in N_\Phi(\mathbf{R}_+)$ for all $0 < k < n$ and*

$$\delta^k ||f^{(k)}||_{N_\Phi(\mathbf{R}_+)} \le C^+(||f||_{N_\Phi(\mathbf{R}_+)} + \delta^n ||f^{(n)}||_{N_\Phi(\mathbf{R}_+)}) ,$$

where $\delta > 0$ is arbitrary and C^+ is defined in (11).

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