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A Diffusion Process with a Random Potential Consisting of Two Self-Similar Processes with Different Indices

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Abstract. A diffusion process with a random potential consisting of two independent self-similar processes with different indices for the right and the left hand sides of the origin is considered. The limiting behavior of the process as time goes to infinity is investigated.

1. Introduction

Denote by $\mathbf{W}$ the space of real-valued functions w defined on $\mathbf{R}$ and satisfying the following:

(i) $w(0) = 0$,

(ii) w is right-continuous and has left limits on $[0, \infty)$,

(iii) w is left-continuous and has right limits on $(-\infty, 0]$.

For $\alpha, \beta > 0$, let $P_{\alpha,\beta}$ be the probability measure on $\mathbf{W}$ such that $\{w(-x), x \geq 0, P_{\alpha,\beta}\}$ and $\{w(x), x \geq 0, P_{\alpha,\beta}\}$ are, respectively, α^{-1}-self-similar and β^{-1}-self-similar processes with time parameter x, and these two processes are independent. For $w \in \mathbf{W}$ and $\lambda > 0$, define $\tau_\lambda^{\alpha,\beta} w \in \mathbf{W}$ by

$$\left(\tau_\lambda^{\alpha,\beta} w\right)(x) = \begin{cases} \lambda^{-1} w(\lambda^\alpha x) & \text{for } x \leq 0, \\ \lambda^{-1} w(\lambda^\beta x) & \text{for } x > 0. \end{cases}$$

Then we have

$$\{\tau_\lambda^{\alpha,\beta} w, P_{\alpha,\beta}\} \overset{\mathrm{d}}{=} \{w, P_{\alpha,\beta}\}, \tag{1.1}$$

where $\overset{\mathrm{d}}{=}$ means the equality in distribution. Let Ω be the space of real-valued continuous functions defined on $[0, \infty)$, and for $\omega \in \Omega$ write $X(t) = X(t, \omega) = \omega(t)$, where $\omega(t)$ is the value of ω at t. For $w \in \mathbf{W}$ and $x_0 \in \mathbf{R}$, denote by $P_w^{x_0}$ the probability measure on Ω such that $\{X(t), t \geq 0, P_w^{x_0}\}$ is a diffusion process with generator

$$\mathcal{L}_w = \frac{1}{2} e^{w(x)} \frac{d}{dx}\left(e^{-w(x)} \frac{d}{dx}\right)$$

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starting from x_0. Define the probability measure $\mathcal{P}^{x_0}_{\alpha,\beta}$ on $\mathbf{W}\times\Omega$ by

$$\mathcal{P}^{x_0}_{\alpha,\beta}(dwd\omega) = P_{\alpha,\beta}(dw)P^{x_0}_w(d\omega)\,.$$

We study the limiting behavior of the diffusion process $\{X(t), t\geq 0, \mathcal{P}^0_{\alpha,\beta}\}$ (as $t\to\infty$) which is regarded as one defined on the probability space $(\mathbf{W}\times\Omega, \mathcal{P}^0_{\alpha,\beta})$. We set

$$Y_{\alpha,\beta}(t) = \begin{cases} (\log t)^{-\alpha}X(t) & \text{if } X(t)\leq 0\,,\\ (\log t)^{-\beta}X(t) & \text{if } X(t)> 0\,,\end{cases}$$

and show the distributions of $Y_{\alpha,\beta}(t)$ are tight (as $t\to\infty$).

When $P_{\alpha,\beta}$ is the Wiener measure (in this case $\alpha=\beta=2$), the corresponding diffusion process was introduced by Brox([1]) and Schumacher([8]). They showed that $(\log t)^{-2}X(t)$ has a nondegenerate limit distribution as $t\to\infty$. Their result was extended to the case $P_{\alpha,\beta}=P_{\alpha,\alpha}$, $\alpha>0$, by Kawazu, Tamura and Tanaka([5], [6]). They proved that the distributions of $(\log t)^{-\alpha}X(t)$ are tight (as $t\to\infty$). On the other hand, in [3] and [4] a diffusion process with a one-sided Brownian potential starting from the origin was studied. They showed that $t^{-1/2}X(t)$ has a limit distribution as $t\to\infty$ with probability $1/2$ and $(\log t)^{-2}X(t)$ has a limit distribution as $t\to\infty$ with the remaining probability $1/2$.

To state our theorem, we introduce some notation. For $w\in\mathbf{W}$, we set

$$w^*(x) = w(x-)\vee w(x+)\,,\quad w_*(x) = w(x-)\wedge w(x+)\,,\quad x\in\mathbf{R}\,,$$

where $w(x-)=\lim_{\varepsilon\downarrow 0}w(x-\varepsilon)$, $w(x+)=\lim_{\varepsilon\downarrow 0}w(x+\varepsilon)$. We define a subset $\mathbf{W}^\#$ of $\mathbf{W}$ and some functions of $w\in\mathbf{W}^\#$, following [6]. Let $\mathbf{W}^\#$ be the set of $w\in\mathbf{W}$ satisfying

$$\limsup_{x\to\infty}\{w(x)-\inf_{0\leq y\leq x}w(y)\} = \limsup_{x\to-\infty}\{w(x)-\inf_{x\leq y\leq 0}w(y)\}=\infty\,.$$

For $w\in\mathbf{W}^\#$, we define

$$\zeta_1=\zeta_1(w)=\sup\left\{x<0: w^*(x)-\inf_{x<y\leq 0}w(y)\geq 1\right\},$$
$$\zeta_2=\zeta_2(w)=\inf\left\{x>0: w^*(x)-\inf_{0\leq y<x}w(y)\geq 1\right\}.$$

By the definition of $\mathbf{W}^\#$, we notice $-\infty<\zeta_1<0$ and $0<\zeta_2<\infty$. We also set, for $w\in\mathbf{W}^\#$,

$$V_1=V_1(w)=\inf\{w_*(x):\zeta_1\leq x\leq 0\}\,,$$
$$V_2=V_2(w)=\inf\{w_*(x):0\leq x\leq \zeta_2\}\,,$$
$$\mathbf{b}_1=\mathbf{b}_1(w)=\{x\in[\zeta_1,0]:w_*(x)=V_1\}\,,$$
$$\mathbf{b}_2=\mathbf{b}_2(w)=\{x\in[0,\zeta_2]:w_*(x)=V_2\}\,,$$
$$b_i^-=\min\mathbf{b}_i\,,\ \ b_i^+=\max\mathbf{b}_i\,,\ \ i=1,2\,,$$

$$M_1 = M_1(w) = \begin{cases} \sup\left\{w^*(x) : b_1^- < x \le 0\right\} & \text{if } b_1^- < 0\,, \\ 0 & \text{if } b_1^- = 0\,, \end{cases}$$

$$M_2 = M_2(w) = \begin{cases} \sup\left\{w^*(x) : 0 \le x < b_2^+\right\} & \text{if } b_2^+ > 0\,, \\ 0 & \text{if } b_2^+ = 0\,, \end{cases}$$

$$\mathbf{a}_1 = \mathbf{a}_1(w) = \begin{cases} \{x \in [b_1^-, 0] : w^*(x) = M_1\} & \text{if } w(b_1^-) \le w(b_1^- +)\,, \\ \{x \in (b_1^-, 0] : w^*(x) = M_1\} & \text{if } w(b_1^-) > w(b_1^- +)\,, \end{cases}$$

$$\mathbf{a}_2 = \mathbf{a}_2(w) = \begin{cases} \{x \in [0, b_2^+] : w^*(x) = M_2\} & \text{if } w(b_2^+) \le w(b_2^+ -)\,, \\ \{x \in [0, b_2^+) : w^*(x) = M_2\} & \text{if } w(b_2^+) > w(b_2^+ -)\,. \end{cases}$$

We divide $\mathbf{W}^{\#}$ into three subsets $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as follows (cf. [7]):

$$\begin{aligned} \mathbf{A} &= \{w \in \mathbf{W}^{\#} : M_1 \vee (V_1 + 1) < M_2 \vee (V_2 + 1)\}\,, \\ \mathbf{B} &= \{w \in \mathbf{W}^{\#} : M_1 \vee (V_1 + 1) > M_2 \vee (V_2 + 1)\}\,, \\ \mathbf{C} &= \{w \in \mathbf{W}^{\#} : M_1 \vee (V_1 + 1) = M_2 \vee (V_2 + 1)\}\,. \end{aligned}$$

For each $\lambda > 0$, we also divide $\mathbf{W}^{\#}$ into three subsets $\mathbf{A}_\lambda^{\alpha,\beta}$, $\mathbf{B}_\lambda^{\alpha,\beta}$ and $\mathbf{C}_\lambda^{\alpha,\beta}$ (cf. [3]):

$$\begin{aligned} \mathbf{A}_\lambda^{\alpha,\beta} &= \{w \in \mathbf{W}^{\#} : \tau_\lambda^{\alpha,\beta} w \in \mathbf{A}\}\,, \\ \mathbf{B}_\lambda^{\alpha,\beta} &= \{w \in \mathbf{W}^{\#} : \tau_\lambda^{\alpha,\beta} w \in \mathbf{B}\}\,, \\ \mathbf{C}_\lambda^{\alpha,\beta} &= \{w \in \mathbf{W}^{\#} : \tau_\lambda^{\alpha,\beta} w \in \mathbf{C}\}\,. \end{aligned}$$

By (1.1), we have

$$P_{\alpha,\beta}\{\mathbf{A}_\lambda^{\alpha,\beta}\} = P_{\alpha,\beta}\{\mathbf{A}\}\,,\ P_{\alpha,\beta}\{\mathbf{B}_\lambda^{\alpha,\beta}\} = P_{\alpha,\beta}\{\mathbf{B}\}\,,\ P_{\alpha,\beta}\{\mathbf{C}_\lambda^{\alpha,\beta}\} = P_{\alpha,\beta}\{\mathbf{C}\}\,.$$

In the following theorem, $P_{\alpha,\beta}\{\cdots|\cdot\}$ denotes the conditional probability.

THEOREM 1.1. *Let* $P_{\alpha,\beta}\{\mathbf{W}^{\#}\} = 1$. *Then for any* $\varepsilon > 0$ *the following* (i)–(iii) *hold.*

(i) $\lim_{t\to\infty} P_{\alpha,\beta}\{P_w^0\{Y_{\alpha,\beta}(t) \in U_\varepsilon(\mathbf{b}_1(\tau_{\log t}^{\alpha,\beta} w))\} > 1 - \varepsilon \mid \mathbf{A}_{\log t}^{\alpha,\beta}\} = 1.$

(ii) $\lim_{t\to\infty} P_{\alpha,\beta}\{P_w^0\{Y_{\alpha,\beta}(t) \in U_\varepsilon(\mathbf{b}_2(\tau_{\log t}^{\alpha,\beta} w))\} > 1 - \varepsilon \mid \mathbf{B}_{\log t}^{\alpha,\beta}\} = 1.$

(iii) $\lim_{t\to\infty} P_{\alpha,\beta}\{P_w^0\{Y_{\alpha,\beta}(t) \in U_\varepsilon(\mathbf{b}_1(\tau_{\log t}^{\alpha,\beta} w)) \cup U_\varepsilon(\mathbf{b}_2(\tau_{\log t}^{\alpha,\beta} w))\} > 1 - \varepsilon \mid \mathbf{C}_{\log t}^{\alpha,\beta}\} = 1.$

Here $U_\varepsilon(K)$ *denotes the open* ε*-neighborhood of a set* K *in* $\mathbf{R}$.

EXAMPLE 1. For $\alpha, \beta \in (0, 2)$, let $P_{\alpha,\beta}$ be the probability measure on $\mathbf{W}$ such that $\{w(-x), x \ge 0, P_{\alpha,\beta}\}$ and $\{w(x), x \ge 0, P_{\alpha,\beta}\}$ are, respectively, symmetric α-stable and symmetric β-stable Lévy motions with time parameter x, and these two processes are independent. Then $P_{\alpha,\beta}$ satisfies our assumptions and $P_{\alpha,\beta}\{\mathbf{W}^{\#}\} = 1$. In this case $P_{\alpha,\beta}\{\mathbf{C}\} = 0$.

EXAMPLE 2. For $\alpha, \beta \in (0,1)$, let $P_{\alpha,\beta}$ be the probability measure on $\mathbf{W}$ such that $\{w(-x), x \geq 0, P_{\alpha,\beta}\}$ and $\{w(x), x \geq 0, P_{\alpha,\beta}\}$ are, respectively, α-stable and β-stable subordinators with time parameter x, and these two processes are independent. Then $P_{\alpha,\beta}$ satisfies our assumptions and $P_{\alpha,\beta}\{\mathbf{W}^{\#}\} = P_{\alpha,\beta}\{\mathbf{C}\} = 1$. Since in this case $\mathbf{b}_1(w) = \mathbf{b}_2(w) = \{0\}$ for any $w \in \mathbf{W}^{\#}$, Theorem 1.1 (iii) is restated as follows:

$$\lim_{t\to\infty} P_{\alpha,\beta}\{P_w^0\{|Y_{\alpha,\beta}(t)| < \varepsilon\} > 1 - \varepsilon \mid \mathbf{C}_{\log t}^{\alpha,\beta}\} = 1 .$$

2. Preliminaries

We begin by introducing the definition of a valley by Kawazu, Tamura and Tanaka ([6]). Denote by $\mathbf{K}$ the space of nonempty compact subsets of $\mathbf{R}$. For $\mathbf{a} \in \mathbf{K}$, we write $a^- = \min \mathbf{a}$ and $a^+ = \max \mathbf{a}$. Let $w \in \mathbf{W}^{\#}$ and $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{K}$. A triplet $\mathbf{V} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ is called a valley of w if the following (i)–(vi) hold.

(i) $-\infty < a^- \leq a^+ \leq b^- \leq b^+ \leq c^- \leq c^+ < \infty$ and $a^+ < c^-$.

(ii) $w_{\mathbf{a}} > w_{\mathbf{b}}$ and $w_{\mathbf{c}} > w_{\mathbf{b}}$,
where

$$w_{\mathbf{a}} = \max_{a^- \leq x \leq a^+} w^*(x),\ w_{\mathbf{b}} = \min_{b^- \leq x \leq b^+} w_*(x),\ w_{\mathbf{c}} = \max_{c^- \leq x \leq c^+} w^*(x) .$$

(iii)

$$\begin{aligned} \mathbf{a} &= \{x \in [a^-, a^+] : w^*(x) = w_{\mathbf{a}}\} , \\ \mathbf{b} &= \{x \in [b^-, b^+] : w_*(x) = w_{\mathbf{b}}\} , \\ \mathbf{c} &= \{x \in [c^-, c^+] : w^*(x) = w_{\mathbf{c}}\} . \end{aligned}$$

(iv) If $a^+ < b^-$, then

$$\begin{aligned} & w_{\mathbf{b}} < w_*(x) \leq w^*(x) < w_{\mathbf{a}} \quad \text{for all } x \in (a^+, b^-) , \\ & w^*(b^-) < w_{\mathbf{a}} \quad \text{in the case } w(b^- -) > w(b^- +) , \\ & w_{\mathbf{b}} < w_*(a^+) \quad \text{in the case } w(a^+ -) > w(a^+ +) . \end{aligned}$$

If $b^+ < c^-$, then

$$\begin{aligned} & w_{\mathbf{b}} < w_*(x) \leq w^*(x) < w_{\mathbf{c}} \quad \text{for all } x \in (b^+, c^-) , \\ & w^*(b^+) < w_{\mathbf{c}} \quad \text{in the case } w(b^+ +) > w(b^+ -) , \\ & w_{\mathbf{b}} < w_*(c^-) \quad \text{in the case } w(c^- +) > w(c^- -) . \end{aligned}$$

(v) If $a^+ = b^-$, then $w(a^+ -) = w_{\mathbf{a}}$ and $w(b^- +) = w_{\mathbf{b}}$.
If $b^+ = c^-$, then $w(b^+ -) = w_{\mathbf{b}}$ and $w(c^- +) = w_{\mathbf{c}}$.

(vi) $H(a^-, b^+) \vee H(c^+, b^-) < (w_{\mathbf{a}} - w_{\mathbf{b}}) \wedge (w_{\mathbf{c}} - w_{\mathbf{b}})$, where

$$H(x,y)=\begin{cases}\sup\limits_{x<x'\le y'<y}\{w(y')-w(x')\} & \text{if } x<y\,,\\ \sup\limits_{y<y'\le x'<x}\{w(y')-w(x')\} & \text{if } x>y\,,\\ 0 & \text{if } x=y\,.\end{cases}$$

For a valley $\mathbf{V}=(\mathbf{a},\mathbf{b},\mathbf{c})$, $D(\mathbf{V})=(w_{\mathbf{a}}-w_{\mathbf{b}})\wedge(w_{\mathbf{c}}-w_{\mathbf{b}})$ is called the depth of $\mathbf{V}$ and $A(\mathbf{V})=H(a^-,b^+)\vee H(c^+,b^-)$ the inner directed ascent of $\mathbf{V}$. Two valleys $\mathbf{V}=(\mathbf{a},\mathbf{b},\mathbf{c})$ and $\mathbf{V}'=(\mathbf{a}',\mathbf{b}',\mathbf{c}')$ are said to be connected at 0 if $\mathbf{c}=\mathbf{a}'$ and $c^-\le 0\le c^+$.

We can show the following proposition in the same way as in [6].

PROPOSITION 2.1. *Let $P_{\alpha,\beta}\{\mathbf{W}^\#\}=1$. Then there exists a subset $\widetilde{\mathbf{W}}^\#$ of $\mathbf{W}^\#$ with $P_{\alpha,\beta}\{\widetilde{\mathbf{W}}^\#\}=1$ such that the following* (i)–(iii) *hold.*

(i) *If $w\in\widetilde{\mathbf{W}}^\#\cap\mathbf{A}$, then for some $\mathbf{a},\mathbf{c}\in\mathbf{K}$, $\mathbf{V}=(\mathbf{a},\mathbf{b}_1,\mathbf{c})$ is a valley of w with $A(\mathbf{V})<1<D(\mathbf{V})$ and $b_1^+<0<c^-$.*

(ii) *If $w\in\widetilde{\mathbf{W}}^\#\cap\mathbf{B}$, then for some $\mathbf{a},\mathbf{c}\in\mathbf{K}$, $\mathbf{V}=(\mathbf{a},\mathbf{b}_2,\mathbf{c})$ is a valley of w with $A(\mathbf{V})<1<D(\mathbf{V})$ and $a^+<0<b_2^-$.*

(iii) *If $w\in\widetilde{\mathbf{W}}^\#\cap\mathbf{C}$, then for some $\mathbf{a},\mathbf{c}\in\mathbf{K}$, either the following* (a) *or* (b) *holds*:

(a) $\mathbf{V}=(\mathbf{a},\mathbf{b}_1\cup\mathbf{b}_2,\mathbf{c})$ *is a valley of w with $A(\mathbf{V})<1<D(\mathbf{V})$,*

(b) $\mathbf{V}=(\mathbf{a},\mathbf{b}_1,\mathbf{a}_1\cup\mathbf{a}_2)$ *and* $\mathbf{V}'=(\mathbf{a}_1\cup\mathbf{a}_2,\mathbf{b}_2,\mathbf{c})$ *are valleys of w connected at* 0 *with $A(\mathbf{V})\vee A(\mathbf{V}')<1<D(\mathbf{V})\wedge D(\mathbf{V}')$.*

For $w\in\mathbf{W}$ and $\lambda>0$, we define $w_{\lambda,+}, w_{\lambda,-}\in\mathbf{W}$ by

$$w_{\lambda,+}(x)=\begin{cases}w(x) & \text{for } x\le 0\,,\\ w(\lambda^{\alpha-\beta}x) & \text{for } x>0\,,\end{cases}$$

$$w_{\lambda,-}(x)=\begin{cases}w(\lambda^{\beta-\alpha}x) & \text{for } x\le 0\,,\\ w(x) & \text{for } x>0\,.\end{cases}$$

Given $w\in\mathbf{W}$, $\lambda>0$ and $x_0\in\mathbf{R}$, we denote by $P^{x_0}_{\lambda w_{\lambda,+}}$ and $P^{x_0}_{\lambda w_{\lambda,-}}$ the probability measures on Ω such that $\{X(t), t\ge 0, P^{x_0}_{\lambda w_{\lambda,+}}\}$ and $\{X(t), t\ge 0, P^{x_0}_{\lambda w_{\lambda,-}}\}$ are diffusion processes with generators $\mathcal{L}_{\lambda w_{\lambda,+}}$ and $\mathcal{L}_{\lambda w_{\lambda,-}}$ starting from x_0, respectively. We can construct such diffusion processes as follows ([2], see also [4]). Let $(\widetilde{\Omega},\widetilde{P})$ be a probability space and $\{B(t), t\ge 0\}$ be a one-dimensional Brownian motion starting from 0 defined on $(\widetilde{\Omega},\widetilde{P})$. We set

$$L(t,x)=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \mathbf{1}_{[x,x+\varepsilon)}(B(s))ds \quad \text{(local time)}\,,$$

$$S_{\lambda,+}(x)=\int_0^x e^{\lambda w_{\lambda,+}(y)}dy\,,\quad x\in\mathbf{R}\,,$$

$$A_{\lambda,+}(t) = \int_0^t e^{-2\lambda w_{\lambda,+}(S_{\lambda,+}{}^{-1}(B(s)))} ds$$
$$= \int_{-\infty}^{\infty} e^{-2\lambda w_{\lambda,+}(S_{\lambda,+}{}^{-1}(x))} L(t,x)dx\,, \quad t \geq 0\,,$$
$$X(t;0,\lambda w_{\lambda,+}) = S_{\lambda,+}{}^{-1}(B(A_{\lambda,+}{}^{-1}(t)))\,, \quad t \geq 0\,, \tag{2.1}$$

where $S_{\lambda,+}{}^{-1}$ and $A_{\lambda,+}{}^{-1}$ denote the inverse functions of $S_{\lambda,+}$ and $A_{\lambda,+}$, respectively. Then $\{X(t;0,\lambda w_{\lambda,+}),\, t \geq 0\}$ defined on $(\widetilde{\Omega}, \widetilde{P})$ is a diffusion process with generator $\mathcal{L}_{\lambda w_{\lambda,+}}$ starting from 0. We also set

$$X(t;x_0,\lambda w_{\lambda,+}) = x_0 + X(t;0,\lambda(w_{\lambda,+})^{x_0})\,, \quad t \geq 0\,,$$

where $(w_{\lambda,+})^{x_0}(\cdot) = w_{\lambda,+}(\cdot + x_0)$. Then $\{X(t;x_0,\lambda w_{\lambda,+}),\, t \geq 0\}$ is a diffusion process with generator $\mathcal{L}_{\lambda w_{\lambda,+}}$ starting from x_0. We can construct a diffusion process with generator $\mathcal{L}_{\lambda w_{\lambda,-}}$ starting from x_0 on $(\widetilde{\Omega}, \widetilde{P})$ in the similar manner.

LEMMA 2.2. *For any $w \in \mathbf{W}$ and $\lambda > 0$,*

$$\{X(t), t \geq 0, P^0_{\lambda(\tau_\lambda^{\alpha,\beta} w)_{\lambda,+}}\} \overset{\mathrm{d}}{=} \{\lambda^{-\alpha} X(\lambda^{2\alpha} t), t \geq 0, P^0_w\}\,,$$
$$\{X(t), t \geq 0, P^0_{\lambda(\tau_\lambda^{\alpha,\beta} w)_{\lambda,-}}\} \overset{\mathrm{d}}{=} \{\lambda^{-\beta} X(\lambda^{2\beta} t), t \geq 0, P^0_w\}\,.$$

PROOF. For $w \in \mathbf{W}$, $\lambda > 0$ and $\alpha > 0$, we define $\tau_\lambda^\alpha w \in \mathbf{W}$ by

$$(\tau_\lambda^\alpha w)(x) = \lambda^{-1} w(\lambda^\alpha x)\,, \quad x \in \mathbf{R}\,.$$

Then we have

$$(\tau_\lambda^{\alpha,\beta} w)_{\lambda,+} = \tau_\lambda^\alpha w\,, \quad (\tau_\lambda^{\alpha,\beta} w)_{\lambda,-} = \tau_\lambda^\beta w\,.$$

Since it was shown in [6] that

$$\{X(t), t \geq 0, P^0_{\lambda \tau_\lambda^\alpha w}\} \overset{\mathrm{d}}{=} \{\lambda^{-\alpha} X(\lambda^{2\alpha} t), t \geq 0, P^0_w\}\,,$$

we obtain the lemma. □

In preparation for the proof of Theorem 1.1, we present the following theorem.

THEOREM 2.3. *Let $w \in \mathbf{W}^\#$ and let r be a real-valued function of $\lambda > 0$ such that $r(\lambda) \to 1$ as $\lambda \to \infty$.*

(i) *If $\mathbf{V} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a valley of w with $A(\mathbf{V}) < 1 < D(\mathbf{V})$ and $b^+ < 0 < c^-$, then for any $\varepsilon > 0$*

$$\lim_{\lambda \to \infty} P^0_{\lambda w_{\lambda,+}} \left\{ X(e^{\lambda r(\lambda)}) \in U_\varepsilon(\mathbf{b}) \right\} = 1\,. \tag{2.2}$$

(ii) *If* $\mathbf{V} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ *is a valley of* w *with* $A(\mathbf{V}) < 1 < D(\mathbf{V})$ *and* $a^+ < 0 < b^-$, *then for any* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^0_{\lambda w_{\lambda,-}} \left\{X(e^{\lambda r(\lambda)}) \in U_\varepsilon(\mathbf{b})\right\} = 1\,.$$

(iii) *If* $\mathbf{V} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ *is a valley of* w *with* $A(\mathbf{V}) < 1 < D(\mathbf{V})$ *and* $b^- \leq 0 \leq b^+$, *then for any* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^0_{\lambda w_{\lambda,+}} \left\{X(e^{\lambda r(\lambda)}) \in (-\infty, 0] \cap U_\varepsilon(\mathbf{b})^c\right\} = 0\,, \tag{2.3}$$

$$\lim_{\lambda\to\infty} P^0_{\lambda w_{\lambda,-}} \left\{X(e^{\lambda r(\lambda)}) \in [0, \infty) \cap U_\varepsilon(\mathbf{b})^c\right\} = 0\,. \tag{2.4}$$

(iv) *If* $\mathbf{V} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ *and* $\mathbf{V}' = (\mathbf{a}', \mathbf{b}', \mathbf{c}')$ *are valleys of* w *connected at* 0 *with* $A(\mathbf{V}) \vee A(\mathbf{V}') < 1 < D(\mathbf{V}) \wedge D(\mathbf{V}')$, *then for any* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^0_{\lambda w_{\lambda,+}} \left\{X(e^{\lambda r(\lambda)}) \in (-\infty, 0] \cap U_\varepsilon(\mathbf{b})^c\right\} = 0\,, \tag{2.5}$$

$$\lim_{\lambda\to\infty} P^0_{\lambda w_{\lambda,-}} \{X(e^{\lambda r(\lambda)}) \in [0, \infty) \cap U_\varepsilon(\mathbf{b}')^c\} = 0\,.$$

In Section 3 we prepare lemmas needed for the proof of Theorem 2.3. In Section 4 we prove Theorem 2.3 (i). We can prove Theorem 2.3 (ii) in the same way as (i). In Section 5 we prove Theorem 2.3 (iii), (iv), and in Section 6 we prove Theorem 1.1.

3. Lemmas on hitting times

In this section we present some lemmas on hitting times of the diffusion process $\{X(t), t \geq 0, P^{x_0}_{\lambda w_{\lambda,+}}\}$. We prove them by employing the method of [1] (see also [4, Lemma 5.1]).

LEMMA 3.1. *Let* $w \in \mathbf{W}$ *and* $p < x_0 \leq 0$. *Assume* $w(p+) \geq w^*(x)$ *for all* $x \in (p, x_0)$, *and assume* $q \equiv \inf\{x > x_0 : w(x) > w(p+)\} < \infty$. *Then for any* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^{x_0}_{\lambda w_{\lambda,+}} \left\{\tau(p) < e^{\lambda(J_1+\varepsilon)}\right\} = 1\,, \tag{3.1}$$

where

$$\tau(p) = \tau(p, \omega) = \inf\{t > 0 : X(t) = p\}\,,$$
$$J_1 = w(p+) - \inf_{p<x<q} w(x)\,.$$

PROOF. Let

$$\tau(a; x_0, \lambda w_{\lambda,+}) = \inf\{t > 0 : X(t; x_0, \lambda w_{\lambda,+}) = a\}\,, \quad a \in \mathbf{R}\,,$$

which is defined on the probability space $(\widetilde{\Omega}, \widetilde{P})$. The assertion (3.1) is equivalent to

$$\lim_{\lambda\to\infty} \widetilde{P} \left\{\tau(p; x_0, \lambda w_{\lambda,+}) < e^{\lambda(J_1+\varepsilon)}\right\} = 1\,. \tag{3.2}$$

We prove (3.2) just in the case $x_0 = 0$. Choose $q' > q$ satisfying $\inf_{q<x<q'} w(x) > \inf_{p<x<q} w(x)$, and set

$$E_\lambda = \{\tau(p; 0, \lambda w_{\lambda,+}) < \tau(\lambda^{\beta-\alpha} q'; 0, \lambda w_{\lambda,+})\}.$$

We have

$$\lim_{\lambda\to\infty} \widetilde{P}\{E_\lambda\} = \lim_{\lambda\to\infty} \frac{\int_0^{q'} e^{\lambda w(y)} dy}{\lambda^{\alpha-\beta} \int_p^0 e^{\lambda w(y)} dy + \int_0^{q'} e^{\lambda w(y)} dy} = 1\,, \tag{3.3}$$

since

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log\left(\lambda^{\alpha-\beta} \int_p^0 e^{\lambda w(y)} dy\right) = w(p+)\,,$$

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \int_0^{q'} e^{\lambda w(y)} dy = \sup_{q<y<q'} w(y) > w(p+)\,.$$

Setting

$$T(a) = \inf\{t > 0 : B(t) = a\}\,, \quad a \in \mathbf{R}\,,$$

we get, from (2.1),

$$\begin{aligned} \tau(p; 0, \lambda w_{\lambda,+}) &= A_{\lambda,+}(T(S_{\lambda,+}(p))) \\ &= \int_p^\infty e^{-\lambda w_{\lambda,+}(y)} L(T(S_{\lambda,+}(p)), S_{\lambda,+}(y)) dy\,. \end{aligned} \tag{3.4}$$

On E_λ, the right-hand side of (3.4) is equal to

$$\int_p^{\lambda^{\beta-\alpha} q'} e^{-\lambda w_{\lambda,+}(y)} L(T(S_{\lambda,+}(p)), S_{\lambda,+}(y)) dy \equiv I_\lambda\,.$$

Since

$$\begin{aligned} &\{L(T(S_{\lambda,+}(p)), S_{\lambda,+}(y)), y \in \mathbf{R}\} \\ &\quad \stackrel{\mathrm{d}}{=} \{|S_{\lambda,+}(p)| L(T(-1), S_{\lambda,+}(y)/|S_{\lambda,+}(p)|), y \in \mathbf{R}\}\,, \end{aligned}$$

we have

$$\begin{aligned} I_\lambda &\stackrel{\mathrm{d}}{=} |S_{\lambda,+}(p)| \int_p^{\lambda^{\beta-\alpha} q'} e^{-\lambda w_{\lambda,+}(y)} L\left(T(-1), \frac{S_{\lambda,+}(y)}{|S_{\lambda,+}(p)|}\right) dy \\ &= \int_p^0 e^{\lambda w(x)} dx \int_p^0 e^{-\lambda w(y)} L\left(T(-1), \frac{S_\lambda(y)}{|S_\lambda(p)|}\right) dy \\ &\quad + \int_p^0 e^{\lambda w(x)} dx \int_0^{q'} e^{-\lambda w(z)} L\left(T(-1), \frac{\lambda^{\beta-\alpha} S_\lambda(z)}{|S_\lambda(p)|}\right) \lambda^{\beta-\alpha} dz \end{aligned}$$

$$\equiv I\!I_\lambda + I\!I\!I_\lambda\,, \tag{3.5}$$

where $S_\lambda(x) = \int_0^x e^{\lambda w(y)} dy$. Observing that $I\!I_\lambda \le p^2 e^{\lambda J_1} T(-1)$ ($\widetilde{P}$-a.s.) and $I\!I\!I_\lambda \le |p| q' e^{\lambda J_1} T(-1) \lambda^{\beta-\alpha}$ ($\widetilde{P}$-a.s.), we get

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log I\!I_\lambda \le J_1\,, \quad \widetilde{P}\text{-a.s.}\,, \tag{3.6}$$

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log I\!I\!I_\lambda \le J_1\,, \quad \widetilde{P}\text{-a.s.} \tag{3.7}$$

By (3.5), (3.6) and (3.7), we obtain for any $\varepsilon > 0$

$$\lim_{\lambda\to\infty} \widetilde{P}\left\{I_\lambda < e^{\lambda(J_1+\varepsilon)}\right\} = 1\,. \tag{3.8}$$

Since $\tau(p; 0, \lambda w_{\lambda,+})$ is equal to I_λ on E_λ, it follows that

$$\begin{aligned}\widetilde{P}\{\tau(p; 0, \lambda w_{\lambda,+}) < e^{\lambda(J_1+\varepsilon)}\} &\ge \widetilde{P}\{I_\lambda < e^{\lambda(J_1+\varepsilon)}, E_\lambda\}\\ &\ge \widetilde{P}\{I_\lambda < e^{\lambda(J_1+\varepsilon)}\} - \widetilde{P}\{E_\lambda{}^c\}\,.\end{aligned} \tag{3.9}$$

The right-hand side of (3.9) converges to 1 as $\lambda \to \infty$ by (3.3) and (3.8). Therefore we obtain (3.2) in the case $x_0 = 0$. □

The following lemma can be proved in the same way as above.

LEMMA 3.2. *Let $w \in \mathbf{W}$ and $q > 0$. Assume $w(q-) \ge w^*(x)$ for all $x \in (0, q)$, and assume $p \equiv \sup\{x < 0 : w(x) > w(q-)\} > -\infty$. Then for any $\varepsilon > 0$*

$$\lim_{\lambda\to\infty} P^0_{\lambda w_{\lambda,+}} \left\{\tau(\lambda^{\beta-\alpha} q) < e^{\lambda(J_2+\varepsilon)}\right\} = 1\,,$$

where

$$J_2 = w(q-) - \inf_{p<x<q} w(x)\,.$$

LEMMA 3.3. *Let $w \in \mathbf{W}$, $p < x_0 \le 0$ and $x_0 \le q$.*

(i) *Assume $w(p) > w^*(x)$ for all $x \in (p, q)$ and $w(p) > w(x_0)$ (in the case $q = x_0$). Then for any $\varepsilon > 0$ and $\varepsilon' > 0$*

$$\lim_{\lambda\to\infty} P^{x_0}_{\lambda w_{\lambda,+}} \left\{\tau(p - \varepsilon') > e^{\lambda(J_3-\varepsilon)}\right\} = 1\,, \tag{3.10}$$

where

$$J_3 = \sup_{p-\varepsilon'<x<p} w(x) - \inf_{p<x<q} w(x)\,.$$

(ii) *Assume $w(p+) > w^*(x)$ for all $x \in (p, q)$ and $w(p+) > w(x_0)$ (in the case $q = x_0$). Then for any $\varepsilon > 0$*

$$\lim_{\lambda\to\infty} P^{x_0}_{\lambda w_{\lambda,+}} \left\{\tau(p) > e^{\lambda(J_4-\varepsilon)}\right\} = 1\,,$$

where

$$J_4 = w(p+) - \inf_{p<x<q} w(x)\,.$$

PROOF. We just prove (i) in the case $x_0 = 0$. In this case the assertion (3.10) is equivalent to

$$\lim_{\lambda\to\infty} \widetilde{P}\left\{\tau(p-\varepsilon'; 0, \lambda w_{\lambda,+}) > e^{\lambda(J_3-\varepsilon)}\right\} = 1\,. \tag{3.11}$$

By the same argument as in the proof of Lemma 3.1, we have

$$\begin{aligned}
&\tau(p-\varepsilon'; 0, \lambda w_{\lambda,+})\\
&\overset{\mathrm{d}}{=} |S_{\lambda,+}(p-\varepsilon')| \int_{p-\varepsilon'}^{\infty} e^{-\lambda w_{\lambda,+}(y)} L\left(T(-1), \frac{S_{\lambda,+}(y)}{|S_{\lambda,+}(p-\varepsilon')|}\right) dy\\
&\geq \int_{p-\varepsilon'}^{0} e^{\lambda w(x)} dx \int_{p}^{\lambda^{\beta-\alpha}q} e^{-\lambda w_{\lambda,+}(y)} L\left(T(-1), \frac{S_{\lambda,+}(y)}{|S_{\lambda}(p-\varepsilon')|}\right) dy\\
&= \int_{p-\varepsilon'}^{0} e^{\lambda w(x)} dx \int_{p}^{0} e^{-\lambda w(y)} L\left(T(-1), \frac{S_{\lambda}(y)}{|S_{\lambda}(p-\varepsilon')|}\right) dy\\
&\quad + \int_{p-\varepsilon'}^{0} e^{\lambda w(x)} dx \int_{0}^{q} e^{-\lambda w(z)} L\left(T(-1), \frac{\lambda^{\beta-\alpha} S_{\lambda}(z)}{|S_{\lambda}(p-\varepsilon')|}\right) \lambda^{\beta-\alpha} dz\\
&\equiv IV_\lambda + V_\lambda\,.
\end{aligned} \tag{3.12}$$

Let us estimate IV_λ first. We notice that $S_\lambda(y)/|S_\lambda(p-\varepsilon')|$ in IV_λ tends to 0 as $\lambda \to \infty$ uniformly on any closed interval contained in $(p, 0]$. This implies

$$L\left(T(-1), \frac{S_\lambda(y)}{|S_\lambda(p-\varepsilon')|}\right) \to L(T(-1), 0) > 0 \quad (\widetilde{P}\text{-a.s.})$$

as $\lambda \to \infty$ uniformly on any closed interval contained in $(p, 0]$. Therefore, by the classical Laplace method, we get

$$\begin{aligned}
\lim_{\lambda\to\infty} \frac{1}{\lambda} \log IV_\lambda &= \sup_{p-\varepsilon'<x<0,\ p<y<0} \{w(x) - w(y)\}\\
&= \sup_{p-\varepsilon'<x<p} w(x) - \inf_{p<y<0} w(y) \equiv J_{IV}, \qquad \widetilde{P}\text{-a.s.}
\end{aligned} \tag{3.13}$$

As for V_λ, we observe that $\lambda^{\beta-\alpha} S_\lambda(z)/|S_\lambda(p-\varepsilon')|$ tends to 0 as $\lambda \to \infty$ uniformly on any closed interval contained in $(0, q)$. Therefore, in the same way as above, we get

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log V_\lambda = \sup_{p-\varepsilon'<x<p} w(x) - \inf_{0<z<q} w(z) \equiv J_V, \qquad \widetilde{P}\text{-a.s.} \tag{3.14}$$

By (3.13) and (3.14), we obtain

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log(IV_\lambda + V_\lambda) = \max\{J_{IV}, J_V\} = J_3 \tag{3.15}$$

in probability with respect to $\widetilde{P}$. Combining (3.12) and (3.15), we arrive at (3.11). □

The following three lemmas can be shown in the same way as Lemma 3.3.

LEMMA 3.4. *Let $w \in \mathbf{W}$ and $p \le 0 < q$.*

(i) *Assume $w(q) > w^*(x)$ for all $x \in (p, q)$ and $w(q) > 0$ (in the case $p = 0$). Then for any $\varepsilon > 0$ and $\varepsilon' > 0$*

$$\lim_{\lambda\to\infty} P^0_{\lambda w_{\lambda,+}}\left\{\tau(\lambda^{\beta-\alpha}(q+\varepsilon')) > e^{\lambda(J_5-\varepsilon)}\right\} = 1,$$

where

$$J_5 = \sup_{q<x<q+\varepsilon'} w(x) - \inf_{p<x<q} w(x).$$

(ii) *Assume $w(q-) > w^*(x)$ for all $x \in (p, q)$ and $w(q-) > 0$ (in the case $p = 0$). Then for any $\varepsilon > 0$*

$$\lim_{\lambda\to\infty} P^0_{\lambda w_{\lambda,+}}\left\{\tau(\lambda^{\beta-\alpha} q) > e^{\lambda(J_6-\varepsilon)}\right\} = 1,$$

where

$$J_6 = w(q-) - \inf_{p<x<q} w(x).$$

LEMMA 3.5. (i) *Let $w \in \mathbf{W}$ and $p < x_0 \le q < 0$. Assume $w(q+) > w^*(x)$ for all $x \in (p, q)$ and $w(q+) > w(x_0)$ (in the case $q = x_0$). Then for any $\varepsilon > 0$ and $\varepsilon' > 0$ satisfying $q + \varepsilon' < 0$*

$$\lim_{\lambda\to\infty} P^{x_0}_{\lambda w_{\lambda,+}}\left\{\tau(q+\varepsilon') > e^{\lambda(J_7-\varepsilon)}\right\} = 1,$$

where

$$J_7 = \sup_{q<x<q+\varepsilon'} w(x) - \inf_{p<x<q} w(x).$$

(ii) *Let $w \in \mathbf{W}$ and $p < x_0 < q \le 0$. Assume $w(q) > w^*(x)$ for all $x \in (p, q)$. Then for any $\varepsilon > 0$*

$$\lim_{\lambda\to\infty} P^{x_0}_{\lambda w_{\lambda,+}}\left\{\tau(q) > e^{\lambda(J_8-\varepsilon)}\right\} = 1,$$

where

$$J_8 = w(q) - \inf_{p<x<q} w(x).$$

LEMMA 3.6. (i) *Let* $w \in \mathbf{W}$ *and* $0 < p \le x_0 < q$. *Assume* $w(p-) > w^*(x)$ *for all* $x \in (p,q)$ *and* $w(p-) > w(x_0)$ (*in the case* $p = x_0$). *Then for any* $\varepsilon > 0$ *and* $\varepsilon' > 0$ *satisfying* $p - \varepsilon' > 0$

$$\lim_{\lambda\to\infty} P^{\lambda^{\beta-\alpha}x_0}_{\lambda w_{\lambda,+}} \left\{ \tau(\lambda^{\beta-\alpha}(p-\varepsilon')) > e^{\lambda(J_9-\varepsilon)} \right\} = 1\,,$$

where

$$J_9 = \sup_{p-\varepsilon'<x<p} w(x) - \inf_{p<x<q} w(x)\,.$$

(ii) *Let* $w \in \mathbf{W}$ *and* $0 \le p < x_0 < q$. *Assume* $w(p) > w^*(x)$ *for all* $x \in (p,q)$. *Then for any* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^{\lambda^{\beta-\alpha}x_0}_{\lambda w_{\lambda,+}} \left\{ \tau(\lambda^{\beta-\alpha}p) > e^{\lambda(J_{10}-\varepsilon)} \right\} = 1\,,$$

where

$$J_{10} = w(p) - \inf_{p<x<q} w(x)\,.$$

4. Proof of Theorem 2.3 (i)

We prepare three lemmas for the proof of Theorem 2.3 (i).

LEMMA 4.1. *Under the hypotheses of Theorem* 2.3 (i), *for any sufficiently small* $\varepsilon > 0$ *there exists* $\theta \in (0,1)$ *such that*

$$\lim_{\lambda\to\infty} P^{0}_{\lambda w_{\lambda,+}} \left\{ \tau(b^-(\varepsilon)) < e^{\lambda\theta} \right\} = 1\,, \tag{4.1}$$

where

$$b^-(\varepsilon) = \begin{cases} b^- & \text{if } w(b^-) > w(b^-+)\,, \\ b^- - \varepsilon & \text{if } w(b^-) \le w(b^-+)\,. \end{cases} \tag{4.2}$$

PROOF. We set

$$M = \sup\{w^*(x) : b^- < x \le 0\}\,. \tag{4.3}$$

First we show (4.1) in the case $M < w_{\mathbf{b}} + 1$. We define $w_0 \in \mathbf{W}^\#$ by

$$w_0(x) = \begin{cases} w(x) & \text{for } x > b^-(\varepsilon)\,, \\ w(b^-(\varepsilon)+) & \text{for } x = b^-(\varepsilon)\,, \\ -x + w(b^-(\varepsilon)+) + b^-(\varepsilon) & \text{for } x < b^-(\varepsilon)\,, \end{cases}$$

where $\varepsilon > 0$ is chosen to be sufficiently small (cf. [4]). We choose $p < b^-(\varepsilon)$ satisfying $M < w_0(p) < w_{\mathbf{b}} + 1$. Then $w_0(p) > (w_0)^*(x)$ for all $x \in (p, 0)$. Setting $q = \inf\{x > 0 :$

$w_0(x) > w_0(p)\}$, we have $q \le c^-$ and therefore $\inf_{p<x<q} w_0(x) = w_{\mathbf{b}}$. By virtue of Lemma 3.1, we get, for any $\varepsilon' > 0$

$$\lim_{\lambda\to\infty} P^0_{\lambda(w_0)_{\lambda,+}} \left\{\tau(p) < e^{\lambda(J_1'+\varepsilon')}\right\} = 1\,, \tag{4.4}$$

where $J_1' = w_0(p) - w_{\mathbf{b}} < 1$. Since

$$\begin{aligned} P^0_{\lambda(w_0)_{\lambda,+}}\{\tau(p) < e^{\lambda(J_1'+\varepsilon')}\} &\le P^0_{\lambda(w_0)_{\lambda,+}}\{\tau(b^-(\varepsilon)) < e^{\lambda(J_1'+\varepsilon')}\} \\ &= P^0_{\lambda w_{\lambda,+}}\{\tau(b^-(\varepsilon)) < e^{\lambda(J_1'+\varepsilon')}\}\,, \end{aligned}$$

we obtain (4.1) from (4.4) in this case.

Next we prove (4.1) in the case $M \ge w_{\mathbf{b}} + 1$. Since $H(c^+, b^-) < 1$, we can choose $c_k \in \mathbf{R}$, $k = 0, 1, \ldots, n$ for some $n \in \mathbf{N}$ satisfying $0 = c_0 > c_1 > c_2 > \cdots > c_{n-1} > c_n = b^-(\varepsilon)$ and the following: for any $k \in \{1, 2, \ldots, n\}$ there exists $p_k \in \mathbf{R}$ such that

$$\begin{cases} p_k < c_k\,, \\ w_{\mathbf{c}} > w_k(p_k) \ge (w_k)^*(x) \quad \text{for all} \quad x \in (p_k, c_{k-1})\,, \\ H_k \equiv w_k(p_k) - \inf_{c_k<x<q_k} w(x) < 1\,, \\ q_k = \inf\{x > c_{k-1} : w(x) > w_k(p_k)\}\,. \end{cases} \tag{4.5}$$

Here $w_k \in \mathbf{W}^\#$ is defined by

$$w_k(x) = \begin{cases} w(x) & \text{for } x > c_k, \\ w(c_k+) & \text{for } x = c_k, \\ -x + w(c_k+) + c_k & \text{for } x < c_k. \end{cases}$$

For any $k \in \{1, 2, \ldots, n\}$ and $\varepsilon_k > 0$, we have, by (4.5) and Lemma 3.1,

$$\lim_{\lambda\to\infty} P^{c_{k-1}}_{\lambda(w_k)_{\lambda,+}} \left\{\tau(p_k) < e^{\lambda(H_k+\varepsilon_k)}\right\} = 1\,. \tag{4.6}$$

In the same way as obtaining (4.1) from (4.4), we get from (4.6)

$$\lim_{\lambda\to\infty} P^{c_{k-1}}_{\lambda w_{\lambda,+}} \left\{\tau(c_k) < e^{\lambda(H_k+\varepsilon_k)}\right\} = 1\,.$$

Therefore, by using the strong Markov property of $\{X(t), t \ge 0, P^{\cdot}_{\lambda w_{\lambda,+}}\}$, we obtain (4.1) in this case, too. □

LEMMA 4.2. *Under the hypotheses of Theorem* 2.3 (i), *for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\lim_{\lambda\to\infty} P^0_{\lambda w_{\lambda,+}} \left\{\tau(a^+(\varepsilon)) > e^{\lambda(1+\delta)}\right\} = 1\,, \tag{4.7}$$

where

$$a^+(\varepsilon) = \begin{cases} a^+ - \varepsilon & \text{if } w(a^+) > w(a^+ +)\,, \\ a^+ & \text{if } w(a^+) \le w(a^+ +)\,. \end{cases} \tag{4.8}$$

PROOF. First we prove (4.7) in the case $M < w_{\mathbf{b}} + 1$, where M is defined in (4.3). In this case $w_{\mathbf{a}} > w^*(x)$ for all $x \in (a^+, 0)$, $w_{\mathbf{a}} > 0$ and $\inf_{a^+ < x < 0} w(x) = w_{\mathbf{b}}$. Therefore, by Lemma 3.3, we get, for any $\varepsilon > 0$ and $\varepsilon' > 0$

$$\lim_{\lambda \to \infty} P^0_{\lambda w_{\lambda,+}} \left\{ \tau(a^+(\varepsilon)) > e^{\lambda(J - \varepsilon')} \right\} = 1 ,$$

where

$$J = \begin{cases} \sup_{a^+ - \varepsilon < x < a^+} w(x) - w_{\mathbf{b}} & \text{if } w(a^+) > w(a^+ +) , \\ w(a^+ +) - w_{\mathbf{b}} & \text{if } w(a^+) \le w(a^+ +) . \end{cases} \tag{4.9}$$

Since $J \ge w_{\mathbf{a}} - w_{\mathbf{b}} > 1$, we obtain (4.7) in this case.

Next we show (4.7) in the case $M \ge w_{\mathbf{b}} + 1$. We use the same notation as in the proof of Lemma 4.1. We notice that

$$\lim_{\lambda \to \infty} P^0_{\lambda w_{\lambda,+}} \left\{ \tau(c_{n-1}) < e^{\lambda \theta} \right\} = 1 \quad \text{for some } \theta \in (0, 1) . \tag{4.10}$$

Moreover, we observe that $w_{\mathbf{a}} > w^*(x)$ for all $x \in (c_n, c_{n-1})$. Combining this with the definition of a valley, we get $w_{\mathbf{a}} > w^*(x)$ for all $x \in (a^+, c_{n-1})$. We also have $w_{\mathbf{a}} > w(c_{n-1})$ and $\inf_{a^+ < x < c_{n-1}} w(x) = w_{\mathbf{b}}$. Therefore, by Lemma 3.3, we get, for any $\varepsilon > 0$ and $\varepsilon' > 0$

$$\lim_{\lambda \to \infty} P^{c_{n-1}}_{\lambda w_{\lambda,+}} \left\{ \tau(a^+(\varepsilon)) > e^{\lambda(J - \varepsilon')} \right\} = 1 , \tag{4.11}$$

where $J(> 1)$ is defined in (4.9). By (4.10), (4.11) and the strong Markov property of $\{X(t), t \ge 0, P^{\cdot}_{\lambda w_{\lambda,+}}\}$, we obtain (4.7) in this case, too. □

LEMMA 4.3. *Under the hypotheses of Theorem* 2.3 (i), *for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\lim_{\lambda \to \infty} P^0_{\lambda w_{\lambda,+}} \left\{ \tau(\lambda^{\beta - \alpha} c^-(\varepsilon)) > e^{\lambda(1 + \delta)} \right\} = 1 , \tag{4.12}$$

where

$$c^-(\varepsilon) = \begin{cases} c^- + \varepsilon & \text{if } w(c^-) > w(c^- -) , \\ c^- & \text{if } w(c^-) \le w(c^- -) . \end{cases}$$

PROOF. We set

$$p = \begin{cases} b^+ & \text{if } w(b^+) > w(b^+ +) , \\ b^+ - \varepsilon_0 & \text{if } w(b^+) \le w(b^+ +) , \end{cases}$$

where $\varepsilon_0 > 0$ is chosen to be sufficiently small. Then $w_{\mathbf{c}} > w^*(x)$ for all $x \in (p, c^-)$ and $\inf_{p < x < c^-} w(x) = w_{\mathbf{b}}$. By virtue of Lemma 3.4, we get, for any $\varepsilon > 0$ and $\varepsilon' > 0$

$$\lim_{\lambda \to \infty} P^0_{\lambda w_{\lambda,+}} \left\{ \tau(\lambda^{\beta - \alpha} c^-(\varepsilon)) > e^{\lambda(J' - \varepsilon')} \right\} = 1 ,$$

where

$$J' = \begin{cases} \sup_{c^- < x < c^- + \varepsilon} w(x) - w_{\mathbf{b}} & \text{if } w(c^-) > w(c^- -)\,, \\ w(c^- -) - w_{\mathbf{b}} & \text{if } w(c^-) \le w(c^- -)\,. \end{cases}$$

Since $J' \ge w_{\mathbf{c}} - w_{\mathbf{b}} > 1$, we obtain (4.12). □

Let us now prove Theorem 2.3 (i) by using the coupling method in [6].

PROOF OF THEOREM 2.3 (i). Consider the interval $K_\lambda = [a^+(\varepsilon_1), \lambda^{\beta-\alpha} c^-(\varepsilon_2)]$, where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are chosen to be small enough that $\inf_{a^+(\varepsilon_1) < x < a^+} w(x) > w_{\mathbf{b}}$ in the case $w(a^+) > w(a^+ +)$ and $\inf_{c^- < x < c^-(\varepsilon_2)} w(x) > w_{\mathbf{b}}$ in the case $w(c^-) > w(c^- -)$. Define $m_{\lambda w_{\lambda,+}}$, a probability measure on K_λ, by

$$m_{\lambda w_{\lambda,+}}\{E\} = \frac{\int_E e^{-\lambda w_{\lambda,+}(x)} dx}{\int_{a^+(\varepsilon_1)}^{\lambda^{\beta-\alpha} c^-(\varepsilon_2)} e^{-\lambda w_{\lambda,+}(x)} dx}$$

for any Borel set E in K_λ. This is the invariant probability measure for the reflecting $\mathcal{L}_{\lambda w_{\lambda,+}}$-diffusion process on K_λ. We notice, for any $\varepsilon > 0$ satisfying $U_\varepsilon(\mathbf{b}) \cap [b^-(\varepsilon), 0) \subset (a^+(\varepsilon_1), 0)$,

$$\begin{aligned} &\lim_{\lambda\to\infty} m_{\lambda w_{\lambda,+}}\{U_\varepsilon(\mathbf{b}) \cap [b^-(\varepsilon), 0)\} \\ &= \lim_{\lambda\to\infty} \frac{\int_{U_\varepsilon(\mathbf{b})\cap[b^-(\varepsilon),0)} e^{-\lambda w(x)} dx}{\int_{a^+(\varepsilon_1)}^0 e^{-\lambda w(x)} dx + \lambda^{\beta-\alpha} \int_0^{c^-(\varepsilon_2)} e^{-\lambda w(x)} dx} \\ &= 1\,, \end{aligned} \tag{4.13}$$

since

$$\begin{aligned} &\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \int_{U_\varepsilon(\mathbf{b})\cap[b^-(\varepsilon),0)} e^{-\lambda w(x)} dx = -w_{\mathbf{b}}\,, \\ &\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \int_{(a^+(\varepsilon_1),0)\setminus(U_\varepsilon(\mathbf{b})\cap[b^-(\varepsilon),0))} e^{-\lambda w(x)} dx < -w_{\mathbf{b}}\,, \\ &\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \left(\lambda^{\beta-\alpha} \int_0^{c^-(\varepsilon_2)} e^{-\lambda w(x)} dx \right) < -w_{\mathbf{b}}\,. \end{aligned}$$

Let $\{X_\lambda^{(R)}(t), t \ge 0\}$ be the reflecting $\mathcal{L}_{\lambda w_{\lambda,+}}$-diffusion process on K_λ with initial distribution $m_{\lambda w_{\lambda,+}}$ defined on the probability space $(\widetilde{\Omega}, \widetilde{P})$. This is a stationary process. By (4.13), we have, for any $t \ge 0$

$$\lim_{\lambda\to\infty} \widetilde{P}\left\{ X_\lambda^{(R)}(t) \in U_\varepsilon(\mathbf{b}) \cap [b^-(\varepsilon), 0) \right\} = 1\,. \tag{4.14}$$

We couple the processes $\{X(t; 0, \lambda w_{\lambda,+}), t \ge 0\}$ and $\{X_\lambda^{(R)}(t), t \ge 0\}$ as follows: Two processes move independently until they meet each other for the first time; then they move together until they go out from the open interval $(a^+(\varepsilon_1), \lambda^{\beta-\alpha}c^-(\varepsilon_2))$; after that they again move independently. We set

$$\sigma_\lambda = \inf\{t \geq 0 : X(t; 0, \lambda w_{\lambda,+}) = X_\lambda^{(R)}(t)\},$$
$$T_\lambda = \inf\{t \geq \sigma_\lambda : X(t; 0, \lambda w_{\lambda,+}) \notin (a^+(\varepsilon_1), \lambda^{\beta-\alpha}c^-(\varepsilon_2))\}.$$

By (4.14), we notice

$$\lim_{\lambda\to\infty} \widetilde{P}\Big\{\sigma_\lambda \leq \tau(b^-(\varepsilon); 0, \lambda w_{\lambda,+})\Big\} = 1. \tag{4.15}$$

Therefore, by Lemma 4.1, we have

$$\lim_{\lambda\to\infty} \widetilde{P}\Big\{\sigma_\lambda < e^{\lambda\theta_0}\Big\} = 1 \quad \text{for some} \quad \theta_0 \in (0, 1). \tag{4.16}$$

Moreover, by Lemmas 4.2 and 4.3, we have

$$\lim_{\lambda\to\infty} \widetilde{P}\Big\{T_\lambda > e^{\lambda(1+\delta_0)}\Big\} = 1 \quad \text{for some} \quad \delta_0 > 0. \tag{4.17}$$

Choose any $\delta \in (0, (1-\theta_0) \wedge \delta_0)$ and any r_1 and r_2 satisfying $1-\delta < r_1 < r_2 < 1+\delta$. For any sufficiently small $\varepsilon > 0$ and any $r \in [r_1, r_2]$, we observe that

$$\begin{aligned}
&\widetilde{P}\{X(e^{\lambda r}; 0, \lambda w_{\lambda,+}) \in U_\varepsilon(\mathbf{b})\}\\
&\quad \geq \widetilde{P}\{\sigma_\lambda \leq e^{\lambda r_1}, X(e^{\lambda r}; 0, \lambda w_{\lambda,+}) \in U_\varepsilon(\mathbf{b}), e^{\lambda r_2} \leq T_\lambda\}\\
&\quad = \widetilde{P}\{\sigma_\lambda \leq e^{\lambda r_1}, X_\lambda^{(R)}(e^{\lambda r}) \in U_\varepsilon(\mathbf{b}), e^{\lambda r_2} \leq T_\lambda\}\\
&\quad \geq \widetilde{P}\{\sigma_\lambda \leq e^{\lambda r_1}\} + \widetilde{P}\{X_\lambda^{(R)}(e^{\lambda r}) \in U_\varepsilon(\mathbf{b})\} + \widetilde{P}\left\{e^{\lambda r_2} \leq T_\lambda\right\} - 2\\
&\quad = \widetilde{P}\{\sigma_\lambda \leq e^{\lambda r_1}\} + m_{\lambda w_{\lambda,+}}\{U_\varepsilon(\mathbf{b})\} + \widetilde{P}\{e^{\lambda r_2} \leq T_\lambda\} - 2.
\end{aligned}$$

Therefore, by (4.13), (4.16) and (4.17), we get

$$\lim_{\lambda\to\infty} \inf_{r\in[r_1,r_2]} \widetilde{P}\Big\{X(e^{\lambda r}; 0, \lambda w_{\lambda,+}) \in U_\varepsilon(\mathbf{b})\Big\} = 1. \tag{4.18}$$

Hence we obtain (2.2). □

5. Proof of Theorem 2.3 (iii), (iv)

We first present a lemma for the proof of Theorem 2.3 (iii).

LEMMA 5.1. *Under the hypotheses of Theorem* 2.3 (iii), *for any sufficiently small* $\varepsilon >$ 0 *there exists* $\theta \in (0, 1)$ *such that*

$$\lim_{\lambda\to\infty} P^0_{\lambda w_{\lambda,+}}\Big\{\tau(b^-(\varepsilon)) < e^{\lambda\theta}\Big\} = 1, \tag{5.1}$$

$$\lim_{\lambda\to\infty} P^0_{\lambda w_{\lambda,+}}\Big\{\tau(\lambda^{\beta-\alpha}b^+(\varepsilon)) < e^{\lambda\theta}\Big\} = 1, \tag{5.2}$$

where $b^-(\varepsilon)$ *is defined in* (4.2) *and*

$$b^+(\varepsilon) = \begin{cases} b^+ & \text{if } w(b^+) > w(b^+-)\,, \\ b^+ + \varepsilon & \text{if } w(b^+) \le w(b^+-)\,. \end{cases}$$

PROOF. We can prove (5.1) in the same way as (4.1). Thus we just show (5.2). Setting

$$M' = \begin{cases} 0 & \text{if } b^- = b^+ = 0\,, \\ \sup\{w^*(x) : b^- < x < b^+\} & \text{otherwise}\,, \end{cases}$$

we have $M' < w_{\mathbf{b}} + 1$. We define $w_0 \in \mathbf{W}^\#$ by

$$w_0(x) = \begin{cases} w(x) & \text{for } x < b^+(\varepsilon)\,, \\ w(b^+(\varepsilon)-) & \text{for } x = b^+(\varepsilon)\,, \\ x + w(b^+(\varepsilon)-) - b^+(\varepsilon) & \text{for } x > b^+(\varepsilon)\,, \end{cases}$$

where $\varepsilon > 0$ is chosen to be sufficiently small. We can choose $q > b^+(\varepsilon)$ satisfying $M' < w_0(q) < w_{\mathbf{b}} + 1$. Setting $p = \sup\{x < 0 : w_0(x) > w_0(q)\}$, we have $p \ge a^-$ and therefore $\inf_{p<x<q} w_0(x) = w_{\mathbf{b}}$. By virtue of Lemma 3.2, we get, for any $\varepsilon' > 0$

$$\lim_{\lambda\to\infty} P^0_{\lambda(w_0)_{\lambda,+}}\Big\{\tau(\lambda^{\beta-\alpha}q) < e^{\lambda(J_2'+\varepsilon')}\Big\} = 1\,, \tag{5.3}$$

where $J_2' = w_0(q) - w_{\mathbf{b}} < 1$. Since

$$\begin{aligned} P^0_{\lambda(w_0)_{\lambda,+}}\{\tau(\lambda^{\beta-\alpha}q) < e^{\lambda(J_2'+\varepsilon')}\} &\le P^0_{\lambda(w_0)_{\lambda,+}}\{\tau(\lambda^{\beta-\alpha}b^+(\varepsilon)) < e^{\lambda(J_2'+\varepsilon')}\} \\ &= P^0_{\lambda w_{\lambda,+}}\{\tau(\lambda^{\beta-\alpha}b^+(\varepsilon)) < e^{\lambda(J_2'+\varepsilon')}\}\,, \end{aligned}$$

we obtain (5.2) from (5.3). □

PROOF OF THEOREM 2.3 (iii). We use the same notation as in the proof of Theorem 2.3 (i). We couple the processes $\{X(t; 0, \lambda w_{\lambda,+}), t \ge 0\}$ and $\{X_\lambda^{(R)}(t), t \ge 0\}$ in the same way as there. In this case, instead of (4.15), we have

$$\lim_{\lambda\to\infty} \widetilde{P}\Big\{\sigma_\lambda \le \tau(b^-(\varepsilon); 0, \lambda w_{\lambda,+}) \vee \tau(\lambda^{\beta-\alpha}b^+(\varepsilon); 0, \lambda w_{\lambda,+})\Big\} = 1\,.$$

Therefore, by Lemma 5.1, we get (4.16) in this case, too. Hence we obtain (2.3) in the same way as obtaining (2.2). We can prove (2.4) in the same manner. □

Next we prepare lemmas for the proof of Theorem 2.3 (iv). Under the hypotheses of Theorem 2.3 (iv), we set

$$x_1 = \begin{cases} c^- & \text{if } w(c^-) < w(c^-+)\,, \\ c^- - \varepsilon_1 & \text{if } w(c^-) \ge w(c^-+)\,, \end{cases}$$

$$x_2 = \begin{cases} c^+ & \text{if } w(c^+) < w(c^+-)\,, \\ c^+ + \varepsilon_2 & \text{if } w(c^+) \geq w(c^+-)\,, \end{cases}$$

where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are chosen to be small enough that $w_{\mathbf{c}} - \inf_{x_1<x<x_2} w(x) < 1$. We note $x_1 < 0 < x_2$.

LEMMA 5.2. *Under the hypotheses of Theorem* 2.3 (iv), *for any* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^0_{\lambda w_{\lambda,+}} \left\{ \tau(x_1, \lambda^{\beta-\alpha} x_2) < e^{\lambda(J_{11}+\varepsilon)} \right\} = 1\,,$$

where

$$\tau(x_1, \lambda^{\beta-\alpha}x_2) = \inf\{t > 0 : X(t) \notin (x_1, \lambda^{\beta-\alpha}x_2)\}\,,$$
$$J_{11} = w_{\mathbf{c}} - \inf_{x_1<x<x_2} w(x) (< 1)\,.$$

PROOF. We set

$$F_\lambda = \{\tau(x_1; 0, \lambda w_{\lambda,+}) < \tau(\lambda^{\beta-\alpha}x_2; 0, \lambda w_{\lambda,+})\}\,.$$

As in the proof of Lemma 3.1, we have, on F_λ

$$\tau(x_1; 0, \lambda w_{\lambda,+}) = \int_{x_1}^{\lambda^{\beta-\alpha}x_2} e^{-\lambda w_{\lambda,+}(y)} L(T(S_{\lambda,+}(x_1)), S_{\lambda,+}(y))dy$$
$$\equiv VI_\lambda\,,$$

and for any $\varepsilon > 0$

$$\lim_{\lambda\to\infty} \widetilde{P}\left\{ VI_\lambda < e^{\lambda(J_{11}+\varepsilon)} \right\} = 1\,. \tag{5.4}$$

On the other hand, we have, on ${F_\lambda}^c$

$$\tau(\lambda^{\beta-\alpha}x_2; 0, \lambda w_{\lambda,+}) = \int_{x_1}^{\lambda^{\beta-\alpha}x_2} e^{-\lambda w_{\lambda,+}(y)} L(T(S_{\lambda,+}(\lambda^{\beta-\alpha}x_2)), S_{\lambda,+}(y))dy$$
$$\equiv VII_\lambda\,,$$

and for any $\varepsilon > 0$

$$\lim_{\lambda\to\infty} \widetilde{P}\left\{ VII_\lambda < e^{\lambda(J_{11}+\varepsilon)} \right\} = 1\,. \tag{5.5}$$

Setting

$$T^0_\lambda(x_1, \lambda^{\beta-\alpha}x_2) = \inf\{t > 0 : X(t; 0, \lambda w_{\lambda,+}) \notin (x_1, \lambda^{\beta-\alpha}x_2)\}\,,$$

we observe that, for any $\varepsilon > 0$

$$\widetilde{P}\{T^0_\lambda(x_1, \lambda^{\beta-\alpha}x_2) < e^{\lambda(J_{11}+\varepsilon)}\}$$
$$= \widetilde{P}\{VI_\lambda < e^{\lambda(J_{11}+\varepsilon)},\ F_\lambda\} + \widetilde{P}\{VII_\lambda < e^{\lambda(J_{11}+\varepsilon)},\ {F_\lambda}^c\}$$

$$\geq 1 - \widetilde{P}\{VI_\lambda \geq e^{\lambda(J_{11}+\varepsilon)}\} - \widetilde{P}\{VII_\lambda \geq e^{\lambda(J_{11}+\varepsilon)}\}. \tag{5.6}$$

Since the right-hand side of (5.6) converges to 1 as $\lambda \to \infty$ by (5.4) and (5.5), we obtain the lemma. □

LEMMA 5.3. *Under the hypotheses of Theorem* 2.3 (iv), *the following* (i)–(iii) *hold.*

(i) *For any sufficiently small $\varepsilon > 0$ there exists $\theta \in (0,1)$ such that*

$$\lim_{\lambda\to\infty} P^{x_1}_{\lambda w_{\lambda,+}} \left\{\tau(b^-(\varepsilon)) < e^{\lambda\theta}\right\} = 1,$$

where $b^-(\varepsilon)$ is defined in (4.2).

(ii) *For any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\lim_{\lambda\to\infty} P^{x_1}_{\lambda w_{\lambda,+}} \left\{\tau(a^+(\varepsilon)) > e^{\lambda(1+\delta)}\right\} = 1,$$

where $a^+(\varepsilon)$ is defined in (4.8).

(iii) *For any $\varepsilon > 0$ satisfying $c^- + \varepsilon < 0$ in the case $w(c^-) < w(c^-+)$ there exists $\delta > 0$ such that*

$$\lim_{\lambda\to\infty} P^{x_1}_{\lambda w_{\lambda,+}} \left\{\tau(c^-(\varepsilon)') > e^{\lambda(1+\delta)}\right\} = 1,$$

where

$$c^-(\varepsilon)' = \begin{cases} c^- + \varepsilon & \text{if } w(c^-) < w(c^-+), \\ c^- & \text{if } w(c^-) \geq w(c^-+). \end{cases}$$

PROOF. We can prove (i) and (ii) in the same way as Lemmas 4.1 and 4.2, respectively. We just show (iii). We observe that $w_{\mathbf{c}} > w^*(x)$ for all $x \in (b^-(\varepsilon_0), c^-)$ and $\inf_{b^-(\varepsilon_0)<x<c^-} w(x) = w_{\mathbf{b}}$, where $\varepsilon_0 > 0$ is chosen to be sufficiently small. By virtue of Lemma 3.5, we get, for any $\varepsilon > 0$ satisfying the assumption of (iii) and $\varepsilon' > 0$

$$\lim_{\lambda\to\infty} P^{x_1}_{\lambda w_{\lambda,+}} \left\{\tau(c^-(\varepsilon)') > e^{\lambda(J''-\varepsilon')}\right\} = 1,$$

where $J'' = w_{\mathbf{c}} - w_{\mathbf{b}} > 1$. Therefore we obtain (iii). □

Let us now prove Theorem 2.3 (iv) by employing the method of [6].

PROOF OF THEOREM 2.3 (iv). We just prove (2.5). By Lemma 5.2,

$$\lim_{\lambda\to\infty} \widetilde{P}\left\{T^0_\lambda(x_1, \lambda^{\beta-\alpha}x_2) < e^{\lambda\theta_0}\right\} = 1 \quad \text{for some } \theta_0 \in (0,1). \tag{5.7}$$

Using Lemma 5.3, we can show that there exists $\delta_0 > 0$ such that for any r_1 and r_2 satisfying $1 - \delta_0 < r_1 < r_2 < 1 + \delta_0$ and any $\varepsilon > 0$

$$\lim_{\lambda\to\infty} \sup_{r\in[r_1,r_2]} \widetilde{P}\left\{X(e^{\lambda r}; x_1, \lambda w_{\lambda,+}) \in (-\infty, 0] \cap U_\varepsilon(\mathbf{b})^c\right\} = 0 \tag{5.8}$$

in the same way as proving (4.18). Moreover, by using Lemma 3.6, we can show that for any $\varepsilon_0 > 0$ satisfying $a'^+ - \varepsilon_0 > 0$ in the case $w(a'^+) < w(a'^+-)$ there exists $\delta_1 > 0$ such that

$$\lim_{\lambda\to\infty} \widetilde{P}\Big\{\tau(\lambda^{\beta-\alpha}a'^+(\varepsilon_0); \lambda^{\beta-\alpha}x_2, \lambda w_{\lambda,+}) > e^{\lambda(1+\delta_1)}\Big\} = 1\,, \tag{5.9}$$

where

$$a'^+(\varepsilon_0) = \begin{cases} a'^+ - \varepsilon_0 & \text{if } w(a'^+) < w(a'^+-)\,, \\ a'^+ & \text{if } w(a'^+) \geq w(a'^+-)\,. \end{cases}$$

Choose any $\delta \in (0, (1-\theta_0)\wedge\delta_0\wedge\delta_1)$ and any r_1 and r_2 satisfying $1-\delta < r_1 < r_2 < 1+\delta$. For any $r \in [r_1, r_2]$ and $\varepsilon > 0$, we get, by the strong Markov property of $\{X(t; \cdot, \lambda w_{\lambda,+}), t \geq 0\}$,

$$\begin{aligned}
&\widetilde{P}\{X(e^{\lambda r}; 0, \lambda w_{\lambda,+}) \in (-\infty, 0] \cap U_\varepsilon(\mathbf{b})^c\} \\
&\quad \leq \int_0^{e^{\lambda\theta_0}} \widetilde{P}\{T_\lambda^0(x_1, \lambda^{\beta-\alpha}x_2) = \tau(x_1; 0, \lambda w_{\lambda,+}) \in du\} \\
&\qquad\qquad \times \widetilde{P}\{X(e^{\lambda r} - u; x_1, \lambda w_{\lambda,+}) \in (-\infty, 0] \cap U_\varepsilon(\mathbf{b})^c\} \\
&\quad + \int_0^{e^{\lambda\theta_0}} \widetilde{P}\{T_\lambda^0(x_1, \lambda^{\beta-\alpha}x_2) = \tau(\lambda^{\beta-\alpha}x_2; 0, \lambda w_{\lambda,+}) \in du\} \\
&\qquad\qquad \times \widetilde{P}\{X(e^{\lambda r} - u; \lambda^{\beta-\alpha}x_2, \lambda w_{\lambda,+}) \in (-\infty, 0] \cap U_\varepsilon(\mathbf{b})^c\} \\
&\quad + \widetilde{P}\{T_\lambda^0(x_1, \lambda^{\beta-\alpha}x_2) \geq e^{\lambda\theta_0}\}\,.
\end{aligned} \tag{5.10}$$

As for the second term in the right-hand side of (5.10), we observe that

$$\begin{aligned}
&\widetilde{P}\{X(e^{\lambda r} - u; \lambda^{\beta-\alpha}x_2, \lambda w_{\lambda,+}) \in (-\infty, 0] \cap U_\varepsilon(\mathbf{b})^c\} \\
&\quad \leq \widetilde{P}\{\tau(\lambda^{\beta-\alpha}a'^+(\varepsilon_0); \lambda^{\beta-\alpha}x_2, \lambda w_{\lambda,+}) \leq e^{\lambda(1+\delta_1)}\}
\end{aligned}$$

because of $a'^+(\varepsilon_0) \geq 0$ and $r < 1 + \delta_1$. Therefore, for sufficiently large λ, we get

$$\begin{aligned}
&\sup_{r\in[r_1,r_2]} \widetilde{P}\{X(e^{\lambda r}; 0, \lambda w_{\lambda,+}) \in (-\infty, 0] \cap U_\varepsilon(\mathbf{b})^c\} \\
&\quad \leq \widetilde{P}\{T_\lambda^0(x_1, \lambda^{\beta-\alpha}x_2) = \tau(x_1; 0, \lambda w_{\lambda,+}) < e^{\lambda\theta_0}\} \\
&\qquad\qquad \times \sup_{r'\in[1-\delta,1+\delta]} \widetilde{P}\{X(e^{\lambda r'}; x_1, \lambda w_{\lambda,+}) \in (-\infty, 0] \cap U_\varepsilon(\mathbf{b})^c\} \\
&\quad + \widetilde{P}\{T_\lambda^0(x_1, \lambda^{\beta-\alpha}x_2) = \tau(\lambda^{\beta-\alpha}x_2; 0, \lambda w_{\lambda,+}) < e^{\lambda\theta_0}\} \\
&\qquad\qquad \times \widetilde{P}\{\tau(\lambda^{\beta-\alpha}a'^+(\varepsilon_0); \lambda^{\beta-\alpha}x_2, \lambda w_{\lambda,+}) \leq e^{\lambda(1+\delta_1)}\} \\
&\quad + \widetilde{P}\{T_\lambda^0(x_1, \lambda^{\beta-\alpha}x_2) \geq e^{\lambda\theta_0}\}\,.
\end{aligned} \tag{5.11}$$

By (5.7), (5.8) and (5.9), the right-hand side of (5.11) converges to 0 as $\lambda \to \infty$. Hence we obtain (2.5). □

6. Proof of Theorem 1.1

Let us now prove Theorem 1.1.

PROOF OF THEOREM 1.1. First we prove (i). By Lemma 2.2, we have

$$\begin{aligned} &P_{\alpha,\beta}\{P_w^0\{\lambda^{-\alpha}X(e^\lambda) \in U_\varepsilon(\mathbf{b}_1(\tau_\lambda^{\alpha,\beta}w))\} > 1-\varepsilon,\ \mathbf{A}_\lambda^{\alpha,\beta}\} \\ &\quad = P_{\alpha,\beta}\{P^0_{\lambda(\tau_\lambda^{\alpha,\beta}w)_{\lambda,+}}\{X(e^{\lambda r(\lambda)}) \in U_\varepsilon(\mathbf{b}_1(\tau_\lambda^{\alpha,\beta}w))\} > 1-\varepsilon, \mathbf{A}_\lambda^{\alpha,\beta}\}\,, \end{aligned} \tag{6.1}$$

where $r(\lambda) = 1 - 2\alpha\lambda^{-1}\log\lambda$. By (1.1), the right-hand side of (6.1) is equal to

$$P_{\alpha,\beta}\{P^0_{\lambda w_{\lambda,+}}\{X(e^{\lambda r(\lambda)}) \in U_\varepsilon(\mathbf{b}_1)\} > 1-\varepsilon, \mathbf{A}\}\,,$$

which converges to $P_{\alpha,\beta}\{\mathbf{A}\}$ as $\lambda \to \infty$ by Proposition 2.1 (i) and Theorem 2.3 (i). Therefore we obtain (i). We can prove (ii) in the same manner by using Proposition 2.1 (ii) and Theorem 2.3 (ii).

Next we show (iii). In the same way as above, we observe that

$$\begin{aligned} &P_{\alpha,\beta}\{P_w^0\{Y_{\alpha,\beta}(e^\lambda) \in U_\varepsilon(\mathbf{b}_1(\tau_\lambda^{\alpha,\beta}w)) \cup U_\varepsilon(\mathbf{b}_2(\tau_\lambda^{\alpha,\beta}w))\} > 1-\varepsilon,\ \mathbf{C}_\lambda^{\alpha,\beta}\} \\ &\quad = P_{\alpha,\beta}\{P_w^0\{\lambda^{-\alpha}X(e^\lambda) \in (-\infty,0] \cap U_\varepsilon(\mathbf{b}_1(\tau_\lambda^{\alpha,\beta}w))^c\} \\ &\qquad + P_w^0\{\lambda^{-\beta}X(e^\lambda) \in [0,\infty) \cap U_\varepsilon(\mathbf{b}_2(\tau_\lambda^{\alpha,\beta}w))^c\} < \varepsilon,\ \mathbf{C}_\lambda^{\alpha,\beta}\} \\ &\quad = P_{\alpha,\beta}\{P^0_{\lambda w_{\lambda,+}}\{X(e^{\lambda r_1(\lambda)}) \in (-\infty,0] \cap U_\varepsilon(\mathbf{b}_1)^c\} \\ &\qquad + P^0_{\lambda w_{\lambda,-}}\{X(e^{\lambda r_2(\lambda)}) \in [0,\infty) \cap U_\varepsilon(\mathbf{b}_2)^c\} < \varepsilon, \mathbf{C}\}\,, \end{aligned} \tag{6.2}$$

where $r_1(\lambda) = 1 - 2\alpha\lambda^{-1}\log\lambda$, $r_2(\lambda) = 1 - 2\beta\lambda^{-1}\log\lambda$. The right-hand side of (6.2) converges to $P_{\alpha,\beta}\{\mathbf{C}\}$ as $\lambda \to \infty$ by Proposition 2.1 (iii) and Theorem 2.3 (iii), (iv). Hence we obtain (iii). □

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