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Boussinesq Equations in Three Dimensional Thin Domains and the Corresponding Two Dimensional Limit

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1. Introduction

In this article we study about the solution of three-dimensional Boussinesq equations. The Boussinesq equations is studied in the field of the fluid dynamics of Earth and Planets fluids systems. The fluid on the planet is laid in a thin layer, so that we consider the Boussinesq equations in thin domains.

Let Ω_ε be a thin domain defined by

$$\Omega_\varepsilon = \{(x_1, x_2, x_3) \in \mathbf{R}^3\ ;\ (x_1, x_2) \in \omega,\ 0 < x_3 < \varepsilon\}\,, \tag{1.1}$$

where ω is an C^2 bounded domain in $\mathbf{R}^2$ and $0 < \varepsilon < 1$. We denote the boundary of Ω_ε by $\partial\Omega_\varepsilon = \Gamma_t \cup \Gamma_b \cup \Gamma_l$, where

$$\Gamma_t = \bar{\omega} \times \{\varepsilon\},\ \Gamma_b = \bar{\omega} \times \{0\} \text{ and } \Gamma_l = \partial\omega \times (0, \varepsilon)\,. \tag{1.2}$$

We are concerned in this article with the following initial boundary value problem in a thin domain:

$$\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p + 2fk \times u = g\theta \qquad \text{in} \quad \Omega_\varepsilon \times (0, T)\,, \tag{1.3}$$

$$\operatorname{div} u = 0 \qquad \text{in} \quad \Omega_\varepsilon \times (0, T)\,, \tag{1.4}$$

$$\partial_t\theta + (u \cdot \nabla)\theta - \kappa\Delta\theta = Q \qquad \text{in} \quad \Omega_\varepsilon \times (0, T)\,, \tag{1.5}$$

$$u_3 = 0,\ \partial_3 u_\alpha = 0,\ \alpha = 1, 2 \ \text{ and } \ \partial_3\theta = 0 \qquad \text{on} \quad (\Gamma_t \cup \Gamma_b) \times (0, T)\,, \tag{1.6}$$

$$u = 0,\ \theta = 0 \qquad \text{on} \quad \Gamma_l \times (0, T)\,, \tag{1.7}$$

$$u(\cdot, 0) = u_0,\ \theta(\cdot, 0) = \theta_0 \qquad \text{in} \quad \Omega_\varepsilon\,, \tag{1.8}$$

where $u = (u_1, u_2, u_3)$ is the fluid velocity, p is the pressure, θ is the temperature, $g = (0, 0, g_0)$ is the gravitational vector (g_0 is a constant), Q is the heat source function, ν (kinematic viscosity) and κ (thermal diffusivity) are positive constants, $fk =$

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$f(0, \cos l(x_2), \sin l(x_2))$ is the earth rotation angular speed (f a constant and $l(x_2)$ the latitude) and $2fk \times u$ represents the Coriolis acceleration. u_0 (resp. θ_0) is a vector (resp. scalar) function defined on Ω_ε.

The system of equations (1.3)–(1.5) describes the large scale motion in the ocean (the Boussinesq approximation). For not only a three-dimensional (3D) case but an arbitrary dimension, the existence and uniqueness of the weak solution of the Boussinesq equations (1.3)–(1.8) has been studied (see [4], [7]). Moreover, the planetary geostrophic (PG) equations are derived from the Boussinesq equations using standard scale analysis and the existance of the weak solutions of the PG equations also has been studied (see [12]).

The solutions of the Navier-Stokes equations (NSE) in thin domains (flat, curved, and with various boundary conditions) has been extensively studied; see [1, 2, 8, 10, 11]. Azérad and Guillen derived, by making use of anisotropic eddy viscosities, a 3D limit nonlinear model in [1]. While, by averaging along the vertical direction and using the uniqueness of solutions of two-dimensional (2D) NSE, Temam and Ziane obtained 2D limit models, together with existence and global regularity results in [10, 11].

Our purpose in this paper is to prove that the averages in the vertical direction of the weak solution of the 3D Boussinesq equations (1.3)–(1.8) converge as the thickness $\varepsilon \to 0$ to the weak solution of the following 2D initial boundary value problem:

$$\partial_t \tilde{u} + (\tilde{u} \cdot \nabla')\tilde{u} - \nu \Delta' \tilde{u} + \nabla' \tilde{p} + b(\tilde{u}) = 0 \quad \text{in } \omega \times (0, T), \tag{1.9}$$

$$\operatorname{div}' \tilde{u} = 0 \quad \text{in } \omega \times (0, T), \tag{1.10}$$

$$\partial_t \tilde{\theta} + (\tilde{u} \cdot \nabla')\tilde{\theta} - \kappa \Delta' \tilde{\theta} = 0 \quad \text{in } \omega \times (0, T), \tag{1.11}$$

$$\tilde{u} = 0, \ \tilde{\theta} = 0 \quad \text{on } \partial\omega \times (0, T), \tag{1.12}$$

$$\tilde{u}(x', 0) = \tilde{u}_0, \ \tilde{\theta}(x', 0) = \tilde{\theta}_0 \quad \text{in } \omega, \tag{1.13}$$

where $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, 0)$ and $b(\tilde{u}) = 2f \sin l(x_2)\,(-\tilde{u}_2, \tilde{u}_1, 0)$. We denote, here and henceforth, the 2D operators with a prime, for example

$$\nabla' = (\partial_1, \partial_2, 0) \ \text{ and } \ x' = (x_1, x_2).$$

In order to explain $\tilde{u}_0$ and $\tilde{\theta}_0$, we introduce some Hilbert spaces.

For $\Omega = \Omega_\varepsilon$ or ω, We denote by $H^s(\Omega)$, $s \in \mathbf{R}$, the Sobolev space constructed as subspace in $L^2(\Omega)$ and define

$$H_0^1(\Omega) = \overline{C_0^\infty(\Omega)}^{H^1(\Omega)} = \{v \in H^1(\Omega);\ v = 0 \text{ on } \partial\Omega\},$$

where $C_0^\infty(\Omega) = \{\varphi \in C^\infty(\bar{\Omega});\ \varphi = 0 \text{ in some neighborhood of } \partial\Omega\}$. We also define the following:

$$H_l^1(\Omega_\varepsilon) = \overline{C_l^\infty(\Omega_\varepsilon)}^{H^1(\Omega_\varepsilon)} = \{v \in H^1(\Omega_\varepsilon);\ v = 0 \text{ on } \Gamma_l\}$$

(where $C_l^\infty(\Omega_\varepsilon) = \{\varphi \in C^\infty(\bar{\Omega}_\varepsilon);\ \varphi = 0 \text{ in some neighborhood of } \Gamma_l\}$),

$$\mathbf{V}_\varepsilon = \{v \in H_l^1(\Omega_\varepsilon) \times H_l^1(\Omega_\varepsilon) \times H_0^1(\Omega_\varepsilon);\ \operatorname{div} v = 0 \text{ in } \Omega_\varepsilon\},$$

$$\mathbf{H}_\varepsilon = \overline{\{\varphi \in C_l^\infty(\Omega_\varepsilon) \times C_l^\infty(\Omega_\varepsilon) \times C_0^\infty(\Omega_\varepsilon);\ \operatorname{div}\varphi = 0 \text{ in } \Omega_\varepsilon\}}^{(L^2(\Omega_\varepsilon))^3},$$

$$\tilde{\mathbf{V}} = \{v \in (H_0^1(\omega))^2;\ \operatorname{div}' v = 0 \text{ in } \omega\}, \text{ and}$$

$$\tilde{\mathbf{H}} = \overline{\{\varphi \in (C_0^\infty(\omega))^2;\ \operatorname{div}'\varphi = 0 \text{ in } \omega\}}^{(L^2(\omega))^2}.$$

Here and henceforth, $\mathbf{X}^*$ denotes the dual space of $\mathbf{X}$. The scalar product in $L^2(\Omega_\varepsilon)^d$ is denoted by $(\cdot,\cdot)$ and in $H^1(\Omega_\varepsilon)^d$ is denoted by $((\cdot,\cdot))$, $d \in \mathbf{N}$, and the associated norms are denoted by $|\cdot|$ and $\|\cdot\|$ respectively.

From the definition mentioned above, we assume that $\tilde{u}_0$ and $\tilde{\theta}_0$ are the functions such that the following equations are satisfied:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon u_0(\cdot, x_3)\,dx_3 = \tilde{u}_0 \text{ weakly in } \tilde{\mathbf{H}}, \tag{1.14}$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon \theta_0(\cdot, x_3)\,dx_3 = \tilde{\theta}_0 \text{ weakly in } L^2(\omega). \tag{1.15}$$

Our main result is the following theorem.

THEOREM 1.1. *Let T be a finite positive constant, and let $\{u,\theta\}$ be the weak solution of Boussinesq equations* (1.3)–(1.8). *We assume that $Q \in L^2(0,T;\ H_0^1(\Omega_\varepsilon))$, $u_0 \in \mathbf{V}_\varepsilon$ and $\theta_0 \in H_l^1(\Omega_\varepsilon)$ and set $q_\varepsilon = \|Q\|_{L^2(0,T;\ H_0^1(\Omega_\varepsilon))}$, $\alpha_\varepsilon = \|u_0\|_{\mathbf{V}_\varepsilon}$, $\beta_\varepsilon = \|\theta_0\|_{H^1(\Omega_\varepsilon)}$. If $(q_\varepsilon)_{\varepsilon>0}$, $(\alpha_\varepsilon)_{\varepsilon>0}$ and $(\beta_\varepsilon)_{\varepsilon>0}$ are bounded and there exist $\tilde{u}_0 \in \tilde{\mathbf{H}}$ and $\tilde{\theta}_0 \in L^2(\omega)$ such that*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon u_0(x', x_3)\,dx_3 = \tilde{u}_0 \textit{ weakly in } \tilde{\mathbf{H}}, \tag{1.16}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon \theta_0(x', x_3)\,dx_3 = \tilde{\theta}_0 \textit{ weakly in } L^2(\omega), \tag{1.17}$$

then there exist $\tilde{u}$ and $\tilde{\theta}$ such that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon u(\cdot, x_3)\,dx_3 = \tilde{u} \textit{ strongly in } \mathcal{C}([0,T];\ \tilde{\mathbf{V}}^*) \cap L^2(0,T;\ \tilde{\mathbf{H}}), \tag{1.18}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon \theta(\cdot, x_3)\,dx_3 = \tilde{\theta} \textit{ strongly in } \mathcal{C}([0,T];\ H^{-1}(\omega)) \cap L^2(0,T;\ L^2(\omega)), \tag{1.19}$$

where $\{\tilde{u},\tilde{\theta}\}$ are the weak solution of (1.9)–(1.13).

2. Preliminaries

For a scalar function $\varphi \in L^2(\Omega_\varepsilon)$, we define its average in the thin direction as follows:

$$(M_\varepsilon \varphi)(x_1, x_2) = \frac{1}{\varepsilon} \int_0^\varepsilon \varphi(x_1, x_2, x_3)\, dx_3 \tag{2.1}$$

and set

$$N_\varepsilon \varphi = \varphi - M_\varepsilon \varphi\, . \tag{2.2}$$

We also define the average operator $\tilde{M}_\varepsilon$ as follows:

$$\tilde{M}_\varepsilon u = (M_\varepsilon u_1, M_\varepsilon u_2, 0) \quad \text{for} \quad u = (u_1, u_2, u_3) \in L^2(\Omega_\varepsilon)^3 \tag{2.3}$$

and set

$$\tilde{N}_\varepsilon u = u - \tilde{M}_\varepsilon u\, . \tag{2.4}$$

For the boundary conditions under consideration, $M_\varepsilon u_i$, $N_\varepsilon u_i$, $i = 1, 2, 3$, $M_\varepsilon \theta$ and $N_\varepsilon \theta$ satisfy following conditions:

$$M_\varepsilon u_1 = M_\varepsilon u_2 = M_\varepsilon \theta = 0 \ \text{ on } \partial\omega\, , \tag{2.5}$$

$$\tilde{N}_\varepsilon u \cdot \vec{n} = 0 \text{ on } \Gamma_t \cup \Gamma_b,\ \int_0^\varepsilon N_\varepsilon u_i(x_1, x_2, x_3, t)\, dx_3 = 0 \text{ for } i = 1, 2 \tag{2.6}$$

and

$$\int_0^\varepsilon N_\varepsilon \theta(x_1, x_2, x_3, t)\, dx_3 = 0\, , \tag{2.7}$$

where $\vec{n}$ is the outward unit normal vector to $\partial\Omega_\varepsilon$. All these operators are projectors; i.e.,

$$M_\varepsilon^2 = M_\varepsilon,\ N_\varepsilon^2 = N_\varepsilon,\ \tilde{M}_\varepsilon^2 = \tilde{M}_\varepsilon,\ \tilde{N}_\varepsilon^2 = \tilde{N}_\varepsilon\, . \tag{2.8}$$

Furthermore, we have the following properties which are obvious:

(i) M_ε is an orthogonal projector from $L^2(\Omega_\varepsilon)$ onto $L^2(\omega)$,

(ii) $M_\varepsilon N_\varepsilon = 0$, and $\tilde{M}_\varepsilon \tilde{N}_\varepsilon = 0$,

(iii) $M_\varepsilon \nabla' = \nabla' M_\varepsilon$, $N_\varepsilon \nabla' = \nabla' N_\varepsilon$ and $\tilde{M}_\varepsilon \nabla' = \nabla' \tilde{M}_\varepsilon$, $\tilde{N}_\varepsilon \nabla' = \nabla' \tilde{N}_\varepsilon$,

(iv) $\varphi \in H^k(\Omega_\varepsilon) \Rightarrow M_\varepsilon \varphi \in H^k(\omega)$ and $N_\varepsilon \varphi \in H^k(\Omega_\varepsilon),\ k \geq 0$.

In the following lemma, we give the basic properties of the operators M_ε and $\tilde{M}_\varepsilon$.

LEMMA 2.1. *For all $u, v \in H^1(\Omega_\varepsilon)$, we have*

$$\int_{\Omega_\varepsilon} \nabla N_\varepsilon u \cdot \nabla M_\varepsilon v\, dx = 0\, , \tag{2.9}$$

$$|u|^2 = |M_\varepsilon u|^2 + |N_\varepsilon u|^2 \quad \textit{and} \quad \|u\|^2 = \|M_\varepsilon u\|^2 + \|N_\varepsilon u\|^2\, . \tag{2.10}$$

$$\textit{If } v \in \mathbf{V}_\varepsilon,\ \textit{then } \tilde{M}_\varepsilon v \in \tilde{\mathbf{V}} \textit{ and } \tilde{N}_\varepsilon v \in \mathbf{V}_\varepsilon\, . \tag{2.11}$$

(2.9) and (2.10) are proved from calculating directly. For (2.11), we have

$$\operatorname{div}' \tilde{M}_\varepsilon v = 0\,,$$

by (iii) of the properities and $\int_0^\varepsilon \partial_3 v_3 \, dx_3 = 0$.

Now, we prepare some propositions and its corollaries.

PROPOSITION 2.1 (Poincaré 's inequality). *For $u \in H^1(\Omega_\varepsilon)$ satisfying one of the following conditions*:

$$\begin{cases} \text{(i)} \quad u = 0 \quad \text{on } \Gamma_t\,, \\ \text{(ii)} \quad u = 0 \quad \text{on } \Gamma_b\,, \\ \text{(iii)} \quad \displaystyle\int_0^\varepsilon u(x_1, x_2, x_3)\, dx_3 = 0 \quad \text{a.e. in } \omega\,, \end{cases}$$

we have

$$|u| \le \varepsilon\, |\partial_3 u|\ . \tag{2.12}$$

For the proof, see [10], Proposition 2.1. Thanks to Proposition 2.1 and (2.6), we have the following:

COROLLARY 2.1. *For $u \in H^1(\Omega_\varepsilon)$,*

$$|N_\varepsilon u| \le \varepsilon\, |\partial_3 N_\varepsilon u|\ . \tag{2.13}$$

PROPOSITION 2.2 (Anisotropic Ladyzhenskaya's inequality). *For $u \in H^1(\Omega_\varepsilon)$, there exists a constant $c_0(\omega)$, depending on ω, such that*

$$|u|_{L^6(\Omega_\varepsilon)} \le c_0(\omega) \left(\frac{1}{\varepsilon}|u| + |\partial_3 u|\right)^{\frac{1}{3}} (|u| + |\partial_1 u| + |\partial_2 u|)^{\frac{2}{3}}\ . \tag{2.14}$$

For the proof, see [10], Remark 2.1. Thanks to Proposition 2.2 and Corollary 2.1, we obtain

COROLLARY 2.2. *There exists a positive constant c_0, independent of ε, such that*

$$|N_\varepsilon u|^2_{L^6(\Omega_\varepsilon)} \le c_0 \|N_\varepsilon u\|^2\,. \tag{2.15}$$

By Corollary 2.1, 2.2 and Hölder's inequality, we get

$$|N_\varepsilon u|^2_{L^3(\Omega_\varepsilon)} \le c_0 \varepsilon \|N_\varepsilon u\|^2 \tag{2.16}$$

for $\forall u \in H^1(\Omega_\varepsilon)$.

Finally, we quote the following theorem.

THEOREM 2.1 ([1]). *Let $T > 0$, and let the Banach spaces $\mathbf{X} \overset{compact}{\hookrightarrow} \mathbf{B} \hookrightarrow \mathbf{Y}$. Let $(f_\varepsilon)_{\varepsilon>0}$ be a family of functions of $L^p(0, T;\ \mathbf{X})$, $1 \le p \le \infty$, with the extra condition $(f_\varepsilon)_{\varepsilon>0} \subset \mathcal{C}(0, T;\ \mathbf{Y})$ if $p = \infty$, such that*

(H1) $(f_\varepsilon)_{\varepsilon>0}$ *is bounded in* $L^p(0,T;\ \mathbf{X})$,

(H2) $|f_\varepsilon(x,t+h) - f_\varepsilon(x,t)|_{L^p(0,T-h;\ \mathbf{Y})} \le \varphi(h) + \psi(\varepsilon)$ *with*

$$\begin{cases} \lim_{h\to 0} \varphi(h) = 0\,, \\ \lim_{\varepsilon\to 0} \psi(\varepsilon) = 0\,. \end{cases}$$

Then the family $(f_\varepsilon)_{\varepsilon>0}$ *possesses a cluster point in* $L^p(0,T;\ \mathbf{B})$ *and also in* $\mathcal{C}(0,T;\ \mathbf{B})$ *if* $p=\infty$, *as* $\varepsilon \to 0$.

For the proof, see [1], Theorem 5.1.

3. Weak formulation and a priori estimates

In this section we derive some a priori estimates for $\tilde{M}_\varepsilon u$, $\tilde{N}_\varepsilon u$, $M_\varepsilon\theta$ and $N_\varepsilon\theta$. The weak formulations of (1.3)–(1.7) are then as follows.

$$\frac{d}{dt}(u,v) + ((u\cdot\nabla)u, v) + \nu(\nabla u, \nabla v) + 2f(k\times u, v) = (g\theta, v)\,, \tag{3.1}$$

$$\frac{d}{dt}(\theta, w) + ((u\cdot\nabla)\theta, w) + \kappa(\nabla\theta, \nabla w) = (Q, w) \tag{3.2}$$

for all $v = (v_1, v_2, v_3) \in \mathbf{V}_\varepsilon$ and $w \in H^1_l(\Omega_\varepsilon)$.

Now we define the weak solution of (1.3)–(1.8).

DEFINITION 3.1. A pair of functions $\{u,\theta\}$ is called a weak solution of (1.3)–(1.8) if

1. $\{u,\theta\}$ satisfies (3.1) and (3.2) for any $v\in\mathbf{V}_\varepsilon$ and $w\in H^1_l(\Omega_\varepsilon)$,
2. $\{u,\theta\}$ also satisfies energy inequalities

$$|\theta(t)|^2 + \kappa\int_0^t |\nabla\theta|^2\,ds \le c\,(|\theta_0|^2 + \|Q\|^2_{L^2(0,T;\ L^2(\Omega_\varepsilon))}) \tag{3.3}$$

where c is a constant, independent of ε, and

$$|u(t)|^2 + \nu\int_0^t |\nabla u|^2\,dt \le |u_0|^2 + \frac{1}{\nu}\int_0^t |g\theta|^2\,ds\,. \tag{3.4}$$

From (3.3), we have

$$\theta \in L^\infty(0,T;\ L^2(\Omega_\varepsilon)) \cap L^2(0,T;\ H^1(\Omega_\varepsilon)) \tag{3.5}$$

because of the assumption of Theorem 1.1. We also obtain

$$g\theta \in L^\infty(0,T;\ L^2(\Omega_\varepsilon)^3) \cap L^2(0,T;\ H^1(\Omega_\varepsilon)^3)\,. \tag{3.6}$$

Furthermore, from (2.10), (3.3), Proposition 2.1 and the assumption of Theorem 1.1, we have

$$\varepsilon\biggl(|M_\varepsilon\theta(t)|^2_{L^2(\omega)} + \kappa\int_0^t |\nabla' M_\varepsilon\theta|^2_{L^2(\omega)}\,ds\biggr)$$

$$\leq c\,\varepsilon(|M_\varepsilon\theta_0|^2_{L^2(\omega)} + \|N_\varepsilon\theta_0\|^2 + \|Q\|_{L^2(0,T;\,H^1_0(\Omega_\varepsilon))})$$
$$\leq C_0\varepsilon\,, \tag{3.7}$$

where C_0 is independent of ε. Then (3.7) amounts to saying that

$$\{M_\varepsilon\theta\}_{\varepsilon>0} \text{ is a bounded sequence in } L^\infty(0,T;\ L^2(\omega)) \cap L^2(0,T;\ H^1_0(\omega))\,. \tag{3.8}$$

Similarly, from (2.10) and (3.3), we have

$$|N_\varepsilon\theta(t)|^2 + \kappa\int_0^t |\nabla N_\varepsilon\theta|^2\,ds \leq C_0'\varepsilon\,, \tag{3.9}$$

where C_0' is independent of ε. Therefore we see that

$$\{\|N_\varepsilon\theta\|_{L^\infty(0,T;\,L^2(\Omega_\varepsilon))}\}_{\varepsilon>0} \quad\text{and}\quad \left\{\|N_\varepsilon\theta\|_{L^2(0,T;\,H^1(\Omega_\varepsilon))}\right\}_{\varepsilon>0} \text{ are bounded}\,. \tag{3.10}$$

Next, from (3.4), we obtain

$$u \in L^\infty(0,T;\ \mathbf{H}_\varepsilon) \cap L^2(0,T;\ \mathbf{V}_\varepsilon)\,. \tag{3.11}$$

because of $u_0 \in \mathbf{V}_\varepsilon$ and (3.6).

Moreover, from (2.10), (3.4), (3.6), Proposition 2.1 and the assumption of Theorem 1.1, we also obtain

$$\varepsilon\left(|\tilde{M}_\varepsilon u(t)|^2_{L^2(\omega)} + \nu\int_0^t |\nabla'\tilde{M}_\varepsilon u|^2_{L^2(\omega)}\,ds\right)$$
$$\leq \varepsilon(|\tilde{M}_\varepsilon u_0|^2_{L^2(\omega)} + \|\tilde{N}_\varepsilon u_0\|^2 + c\|g\theta\|_{L^2(0,T;\,H^1(\Omega_\varepsilon)^3)})$$
$$\leq C_1\varepsilon\,, \tag{3.12}$$

where C_1 is independent of ε. Then (3.12) implies that

$$\{\tilde{M}_\varepsilon u\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;\ \tilde{\mathbf{H}}) \cap L^2(0,T;\ \tilde{\mathbf{V}})\,. \tag{3.13}$$

Similarly, from (2.10) and (3.4), we obtain

$$|\tilde{N}_\varepsilon u(t)|^2 + \nu\int_0^t |\nabla\tilde{N}_\varepsilon u|^2\,ds \leq C_1'\varepsilon\,, \tag{3.14}$$

where C_1' is independent of ε. Then we see that

$$\{\|\tilde{N}_\varepsilon u\|_{L^\infty(0,T;\,\mathbf{H}_\varepsilon)}\}_{\varepsilon>0} \quad\text{and}\quad \{\|\tilde{N}_\varepsilon u\|_{L^2(0,T;\,\mathbf{V}_\varepsilon)}\}_{\varepsilon>0} \text{ are bounded}\,. \tag{3.15}$$

By the above estimates and interpolation between $L^\infty(0,T;\ L^2(\Omega_\varepsilon))$ and $L^2(0,T;\ L^6(\Omega_\varepsilon))$, we have

$$\tilde{M}_\varepsilon u \in L^4(0,T;\ L^3(\omega)^3),\ \ \tilde{N}_\varepsilon u \in L^4(0,T;\ L^3(\Omega_\varepsilon)^3)\,. \tag{3.16}$$

for $0 < \forall\varepsilon < 1$. we obtain similar

$$M_\varepsilon\theta \in L^4(0, T;\ L^3(\omega)),\ \ N_\varepsilon\theta \in L^4(0, T;\ L^3(\Omega_\varepsilon))\,, \tag{3.17}$$

for $0 < \forall\varepsilon < 1$.

REMARK. Because of (3.9), we can see that the estimate of the norm of $L^\infty(0, T;\ L^2(\Omega_\varepsilon))$ and $L^2(0, T;\ H^1(\Omega_\varepsilon))$ for the difference between the temperature in the thin domain Ω_ε and the average temperature in the vertical direction is less than $C_0'\varepsilon$. For the velocity, we can obtain the similar estimate from (3.14).

Let

$$\tilde{\mathbf{W}} = \overline{\{\varphi \in (C_0^\infty(\omega))^2;\ \mathrm{div}'\,\varphi = 0 \text{ in } \omega\}}^{H^2(\omega)^2}.$$

Then, from the Sobolev-Rellich embeddings, one deduces easily that

$$\tilde{\mathbf{W}} \hookrightarrow \tilde{\mathbf{V}} \hookrightarrow \tilde{\mathbf{H}} \equiv \tilde{\mathbf{H}}^* \hookrightarrow \tilde{\mathbf{V}}^* \hookrightarrow \tilde{\mathbf{W}}^*\,, \tag{3.18}$$

where all are dence and compact embeddings. We also define

$$H_0^2(\omega) = \overline{C_0^\infty(\omega)}^{H^2(\omega)} = \left\{ v \in H^2(\omega);\ v = 0 \text{ and } \frac{\partial v}{\partial n} = 0 \text{ on } \partial\omega \right\}$$

and denote

$$H^{-s}(\omega) = (H_0^s(\omega))^*$$

for $s = 1, 2$.

Now, we have the following lemma.

LEMMA 3.1. *For $0 < \forall h < T$, there exist positive constants c_0 and c_1, independent of h and ε, such that*

$$\|M_\varepsilon\theta(x', t+h) - M_\varepsilon\theta(x', t)\|_{L^\infty(0,T-h;\ H^{-2}(\omega))} \le c_0(h^{\frac{1}{2}} + \varepsilon^{\frac{2}{3}}) \tag{3.19}$$

and

$$\|\tilde{M}_\varepsilon u(x', t+h) - \tilde{M}_\varepsilon u(x', t)\|_{L^\infty(0,T-h;\ \tilde{\mathbf{W}}^*)} \le c_1(h^{\frac{1}{2}} + \varepsilon^{\frac{2}{3}})\,. \tag{3.20}$$

PROOF. From (2.2) and (3.2), we obtain

$$\frac{d}{dt}(M_\varepsilon\theta, w) + \frac{d}{dt}(N_\varepsilon\theta, w) + ((u\cdot\nabla)\theta, w) + \kappa(\nabla\theta, \nabla w) = (Q, w) \tag{3.21}$$

for $\forall w \in H_0^2(\omega)$. Integrate (3.21) from t to $t+h$, $t \in [0, T-h]$, we have

$$(M_\varepsilon\theta(x, t+h) - M_\varepsilon\theta(x, t), w) + (N_\varepsilon\theta(x, t+h) - N_\varepsilon\theta(x, t), w) = \int_t^{t+h} g_0^\varepsilon(t)\,ds\,, \tag{3.22}$$

where

$$g_0^\varepsilon(t) = -((u\cdot\nabla)\theta, w) - \kappa(\nabla\theta, \nabla w) + (Q, w)\,. \tag{3.23}$$

For the first term of g_0^ε, since $u = \tilde{M}_\varepsilon u + \tilde{N}_\varepsilon u$, we have

$$((u \cdot \nabla)\theta, w) = ((\tilde{M}_\varepsilon u \cdot \nabla) M_\varepsilon \theta, w) + ((\tilde{M}_\varepsilon u \cdot \nabla) N_\varepsilon \theta, w) \\ + ((\tilde{N}_\varepsilon u \cdot \nabla) M_\varepsilon \theta, w) + ((\tilde{N}_\varepsilon u \cdot \nabla) N_\varepsilon \theta, w)\,.$$

Thanks to (2.6) and (2.7), we obtain $((\tilde{M}_\varepsilon u \cdot \nabla) N_\varepsilon \theta, w) = ((\tilde{N}_\varepsilon u \cdot \nabla) M_\varepsilon \theta, w) = 0\,.$ Moreover, we have

$$\begin{aligned} |((\tilde{M}_\varepsilon u \cdot \nabla) M_\varepsilon \theta, w)| &= |(M_\varepsilon \theta, (\tilde{M}_\varepsilon u \cdot \nabla) w)| \\ &\le |M_\varepsilon \theta|_{L^3(\Omega_\varepsilon)} |\tilde{M}_\varepsilon u| \, |\nabla' w|_{L^6(\Omega_\varepsilon)} \\ &\le c\, \varepsilon^{\frac{1}{3}} |M_\varepsilon \theta|_{L^3(\omega)} |\tilde{M}_\varepsilon u| \, \|w\|_{H^2(\Omega_\varepsilon)} \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} |((\tilde{N}_\varepsilon u \cdot \nabla) N_\varepsilon \theta, w)| &= |(N_\varepsilon \theta, (\tilde{N}_\varepsilon u \cdot \nabla) w)| \\ &\le |N_\varepsilon \theta|_{L^3(\Omega_\varepsilon)} |\tilde{N}_\varepsilon u| \, |\nabla' w|_{L^6(\Omega_\varepsilon)} \\ &\le c\, \varepsilon^{\frac{1}{2}} \, \|N_\varepsilon \theta\| \, |\tilde{N}_\varepsilon u| \, \|w\|_{H^2(\Omega_\varepsilon)}\,, \end{aligned} \tag{3.25}$$

because of (2.16). Thus, since $|\tilde{M}_\varepsilon u| = \varepsilon^{1/2} |\tilde{M}_\varepsilon u|_{L^2(\omega)}$ and $\|w\|_{H^2(\Omega_\varepsilon)} = \varepsilon^{1/2} \|w\|_{H^2(\omega)}$, we obtain

$$\begin{aligned} |((u \cdot \nabla)\theta, w)| &\le c(\varepsilon^{\frac{1}{3}} |M_\varepsilon \theta|_{L^3(\omega)} |\tilde{M}_\varepsilon u| + \varepsilon^{\frac{1}{2}} \|N_\varepsilon \theta\| \, |\tilde{N}_\varepsilon u|) \|w\|_{H^2(\Omega_\varepsilon)} \\ &\le c\, \varepsilon (\varepsilon^{\frac{1}{3}} |M_\varepsilon \theta|_{L^3(\omega)} |\tilde{M}_\varepsilon u|_{L^2(\omega)} + \|N_\varepsilon \theta\| \, |\tilde{N}_\varepsilon u|) \|w\|_{H^2(\omega)}\,. \end{aligned} \tag{3.26}$$

Next, we consider the second term of g_0^ε. Thanks to the properties of N_ε, we have

$$(\nabla N_\varepsilon \theta, \nabla w) = \int_\omega \left(\int_0^\varepsilon N_\varepsilon \nabla' \theta \, dx_3 \right) \nabla' w \, dx' = 0\,. \tag{3.27}$$

Therefore, we obtain the following estimate:

$$\begin{aligned} |(\nabla \theta, \nabla w)| &\le \varepsilon \, |(\nabla' M_\varepsilon \theta, \nabla' w)_{L^2(\omega)}| + |(\nabla N_\varepsilon \theta, \nabla w)| \\ &\le \varepsilon \, \|M_\varepsilon \theta\|_{H^1(\omega)} \|w\|_{H^2(\omega)}\,. \end{aligned} \tag{3.28}$$

Finally, the estimate for the third term of g_0^ε is

$$|(Q, w)| \le \varepsilon \, \|Q\| \, \|w\|_{H^2(\omega)}\,. \tag{3.29}$$

by Proposition 2.1.

On the other hand,

$$(N_\varepsilon \theta(x, t+h) - N_\varepsilon \theta(x, t), w) = \int_\omega \left(\int_0^\varepsilon N_\varepsilon (\theta(x, t+h) - \theta(x, t)) \, dx_3 \right) w \, dx' = 0$$

by the properties of N_ε.

From the estimations mentioned above, we have

$$
\begin{aligned}
|(M_\varepsilon\theta(x,t+h) - M_\varepsilon\theta(x,t), w)| &\le \int_t^{t+h} |g_0^\varepsilon(s)|\,ds \\
&\le \int_t^{t+h} c\,\varepsilon(\varepsilon^{\frac{1}{3}}|M_\varepsilon\theta|_{L^3(\omega)}|\tilde{M}_\varepsilon u|_{L^2(\omega)} + \|N_\varepsilon\theta\|\,|\tilde{N}_\varepsilon u| \\
&\quad + \|M_\varepsilon\theta\|_{H^1(\omega)} + \|Q\|)\,ds\cdot\|w\|_{H^2(\omega)} \\
&\le \text{(with Hölder's inequality)} \\
&\le c\,\varepsilon(h^{\frac{3}{4}}\varepsilon^{\frac{1}{3}}\|M_\varepsilon\theta\|_{L^4(0,T;\,L^3(\omega))}\|\tilde{M}_\varepsilon u\|_{L^\infty(0,T;\,L^2(\omega))} \\
&\quad + h^{\frac{1}{2}}\|N_\varepsilon\theta\|_{L^2(0,T;\,H^1(\Omega_\varepsilon))}\|\tilde{N}_\varepsilon u\|_{L^\infty(0,T;\,L^2(\Omega_\varepsilon))} \\
&\quad + h^{\frac{1}{2}}\|M_\varepsilon\theta\|_{L^2(0,T;\,H^1(\omega))} + h^{\frac{1}{2}}\,\|Q\|_{L^2(0,T;\,H_0^1(\Omega_\varepsilon))})\|w\|_{H^2(\omega)}\,,
\end{aligned}
$$

and, taking into account $Q \in L^2(0,T;\,H_0^1(\omega))$ and $0 < h < T$, according to (3.7), (3.8), (3.13), (3.15) and (3.17), we obtain

$$
|(M_\varepsilon\theta(x,t+h) - M_\varepsilon\theta(x,t), w)| \le c_0\,\varepsilon\,(h^{\frac{1}{2}} + \varepsilon^{\frac{2}{3}})\|w\|_{H^2(\omega)}\,.
$$

Hence, since $(M_\varepsilon\theta(x,t+h) - M_\varepsilon\theta(x,t), w) = \varepsilon\,(M_\varepsilon\theta(x',t+h) - M_\varepsilon\theta(x',t), w)_{L^2(\omega)}$, we obtain

$$
|(M_\varepsilon\theta(x',t+h) - M_\varepsilon\theta(x',t), w)_{L^2(\omega)}| \le c_0(h^{\frac{1}{2}} + \varepsilon^{\frac{2}{3}})\|w\|_{H^2(\omega)} \tag{3.30}
$$

for $\forall w \in H_0^2(\omega)$. Hence we have the estimation of (3.19).

Similarly, from (2.4) and (3.1), we have

$$
\frac{d}{dt}(\tilde{M}_\varepsilon u, v) + \frac{d}{dt}(\tilde{N}_\varepsilon u, v) + ((u\cdot\nabla)u, v) + \nu(\nabla u, \nabla v) = (g\theta, v)\,, \tag{3.31}
$$

for $\forall v \in \tilde{\mathbf{W}}$. Hence, integrating (3.31) from t to $t+h$, $t \in [0, T-h]$, we obtain

$$
(\tilde{M}_\varepsilon u(x,t+h) - \tilde{M}_\varepsilon u(x,t), v) + (\tilde{N}_\varepsilon u(x,t+h) - \tilde{N}_\varepsilon u(x,t), v) = \int_t^{t+h} g_1^\varepsilon(t)\,dt\,, \tag{3.32}
$$

where

$$
g_1^\varepsilon(t) = -((u\cdot\nabla)u, v) - \nu(\nabla u, \nabla v) + 2f(k\times u, v) + (g\theta, v)\,.
$$

The following estimations is obtained in the same way as (3.26)–(3.28):

$$
\begin{aligned}
|((u\cdot\nabla)u, v)| &\le c\,\varepsilon^{\frac{1}{2}}(|\tilde{M}_\varepsilon u|_{L^3(\Omega_\varepsilon)}|\tilde{M}_\varepsilon u| + \varepsilon^{\frac{1}{2}}\|\tilde{N}_\varepsilon u\|\,|\tilde{N}_\varepsilon u|)\|v\|_{\tilde{\mathbf{W}}} \\
&\le c\,\varepsilon(\varepsilon^{\frac{1}{3}}|\tilde{M}_\varepsilon u|_{L^3(\omega)}|\tilde{M}_\varepsilon u|_{L^2(\omega)} + \|\tilde{N}_\varepsilon u\|\,|\tilde{N}_\varepsilon u|)\|v\|_{\tilde{\mathbf{W}}}\,,
\end{aligned} \tag{3.33}
$$

$$
|(\nabla u, \nabla v)| \le \varepsilon\,\|\tilde{M}_\varepsilon u\|_{H^1(\omega)}\|v\|_{\tilde{\mathbf{W}}}\,. \tag{3.34}
$$

For the Coriolis term, we obtain

$$|(k \times u, v)| \leq |\tilde{M}_\varepsilon u|\,|v| + |\tilde{N}_\varepsilon u|\,|v| \leq c\,\varepsilon(|\tilde{M}_\varepsilon u|_{L^2(\omega)} + \varepsilon^{\frac{1}{2}}\|\tilde{N}_\varepsilon u\|)\|v\|_{\tilde{\mathbf{W}}}\,. \tag{3.35}$$

And, since $\tilde{M}_\varepsilon g\theta = 0$, we have

$$|(g\theta, v)| = 0\,. \tag{3.36}$$

Hence, since $(\tilde{N}_\varepsilon u(x, t+h) - \tilde{N}_\varepsilon u(x,t), v) = \sum_{i=1}^{2}\int_\omega(\int_0^\varepsilon \tilde{N}_\varepsilon(u_i(x, t+h) - u_i(x,t))\,dx_3)v_i\,dx' = 0$, we have

$$\begin{aligned}
&|(\tilde{M}_\varepsilon u(x, t+h) - \tilde{M}_\varepsilon u(x,t), v)|\\
&\quad\leq \int_t^{t+h} |g_1^\varepsilon(s)|\,ds\\
&\quad\leq \int_t^{t+h} c\,\varepsilon(\varepsilon^{\frac{1}{3}}|\tilde{M}_\varepsilon u|_{L^3(\omega)}|\tilde{M}_\varepsilon u|_{L^2(\omega)} + \|\tilde{N}_\varepsilon u\|\,|\tilde{N}_\varepsilon u|\\
&\qquad\quad + |\tilde{M}_\varepsilon u|_{L^2(\omega)} + \varepsilon^{\frac{1}{2}}\|\tilde{N}_\varepsilon u\| + \|\tilde{M}_\varepsilon u\|_{H^1(\omega)})\,ds \cdot \|v\|_{\tilde{\mathbf{W}}}\\
&\quad\leq \text{(with Hölder's inequality)}\\
&\quad\leq c\,\varepsilon(h^{\frac{3}{4}}\varepsilon^{\frac{1}{3}}\|\tilde{M}_\varepsilon u\|_{L^4(0,T;\,L^3(\omega)^3)}\|\tilde{M}_\varepsilon u\|_{L^\infty(0,T;\,L^2(\omega))}\\
&\qquad\quad + h^{\frac{1}{2}}\|\tilde{N}_\varepsilon u\|_{L^2(0,T;\,H^1(\Omega_\varepsilon)^3)}\|\tilde{N}_\varepsilon u\|_{L^\infty(0,T;\,L^2(\Omega_\varepsilon))}\\
&\qquad\quad + h\|\tilde{M}_\varepsilon u\|_{L^\infty(0,T;\,L^2(\omega))} + \varepsilon^{\frac{1}{2}}h^{\frac{1}{2}}\|\tilde{N}_\varepsilon u\|_{L^2(0,T;\,H^1(\Omega_\varepsilon))}\\
&\qquad\quad + h^{\frac{1}{2}}\|\tilde{M}_\varepsilon u\|_{L^2(0,T;\,H^1(\omega)^3)})\|v\|_{\tilde{\mathbf{W}}},
\end{aligned}$$

and, according to (3.13), (3.15) and (3.16), we obtain

$$|(\tilde{M}_\varepsilon u(x, t+h) - \tilde{M}_\varepsilon u(x,t), v)| \leq c_1\,\varepsilon(h^{\frac{1}{2}} + \varepsilon^{\frac{2}{3}})\|v\|_{\tilde{\mathbf{W}}}\,. \tag{3.37}$$

Therefore, since $(\tilde{M}_\varepsilon u(x, t+h) - \tilde{M}_\varepsilon u(x,t), v) = \varepsilon\,(\tilde{M}_\varepsilon u(x', t+h) - \tilde{M}_\varepsilon u(x',t), v)_{L^2(\omega)}$, we obtain

$$|(\tilde{M}_\varepsilon u(x', t+h) - \tilde{M}_\varepsilon u(x',t), v)_{L^2(\omega)}| \leq c_1(h^{\frac{1}{2}} + \varepsilon^{\frac{2}{3}})\|v\|_{\tilde{\mathbf{W}}} \tag{3.38}$$

for $\forall v \in \tilde{\mathbf{W}}$. The proof is complete. □

4. Proof of Theorem

The space-time weak formulations of (1.9)–(1.13) is the following:

$$(\tilde{u}, v) = (\tilde{u}_0, v) - \int_0^t [((\tilde{u}\cdot\nabla')\tilde{u}, v) + \nu(\nabla'\tilde{u}, \nabla' v) + (b(\tilde{u}), v)]\,ds\,, \tag{4.1}$$

$$(\tilde{\theta}, w) = (\tilde{\theta}_0, w) - \int_0^t [((\tilde{u}\cdot\nabla')\tilde{\theta}, w) + \kappa(\nabla'\tilde{\theta}, \nabla' w)]\,ds\,, \tag{4.2}$$

for all $v \in \tilde{\mathbf{V}}$ and $w \in H_0^1(\omega)$.

DEFINITION 4.1. A pair of functions $\{\tilde{u}, \tilde{\theta}\}$ is called a weak solution of (1.9)–(1.13) if $\{\tilde{u}, \tilde{\theta}\}$ satisfies (4.1) and (4.2) for any $v \in \tilde{\mathbf{V}}$ and $w \in H_0^1(\omega)$.

The purpose of the following is that $\tilde{M}_\varepsilon u$ and $M_\varepsilon \theta$ converge, as $\varepsilon \to 0$, to the weak solution of $\{\tilde{u}, \tilde{\theta}\}$ in $\mathcal{C}(0, T;\ \tilde{\mathbf{V}}^*) \cap L^2(0, T;\ \tilde{\mathbf{H}})$ and $\mathcal{C}(0, T;\ H^{-1}(\omega)) \cap L^2(0, T;\ L^2(\omega))$ respectively.

Thanks to Lemma 3.1 and (3.8), we can apply Theorem 2.1 for $p = \infty$ and $L^2(\omega) \overset{compact}{\hookrightarrow} H^{-1}(\omega) \hookrightarrow H^{-2}(\omega)$. Therefore, there exists a subsequence, still denoted by $M_\varepsilon \theta$, and a funchtion $\hat{\theta}$ such that

$$M_\varepsilon \theta \to \hat{\theta} \ \text{ strongly in } \mathcal{C}(0, T;\ H^{-1}(\omega))\,. \tag{4.3}$$

Similarly, combining Lemma 3.1 and (3.13), we can apply Theorem 2.1 for $p = \infty$ and $\tilde{\mathbf{H}} \overset{compact}{\hookrightarrow} \tilde{\mathbf{V}}^* \hookrightarrow \tilde{\mathbf{W}}^*$. Therefore, there exists a subsequence, still denoted by $\tilde{M}_\varepsilon u$, and a function $\hat{u}$ such that

$$\tilde{M}_\varepsilon u \to \hat{u} \ \text{ strongly in } \mathcal{C}(0, T;\ \tilde{\mathbf{V}}^*)\,. \tag{4.4}$$

Moreover, we also apply Theorem 2.1 for $p = 2$ and $H_0^1(\omega) \overset{compact}{\hookrightarrow} L^2(\omega) \hookrightarrow H^{-2}(\omega)$. then, since Lemma 3.1, (3.8) and $L^\infty(0, T-h;\ H^{-2}(\omega)) \subset L^2(0, T-h;\ H^{-2}(\omega))$, there exists a subsequence, still denoted by $M_\varepsilon \theta$, such that

$$M_\varepsilon \theta \to \hat{\theta} \ \text{ strongly in } L^2(0, T;\ L^2(\omega))\,. \tag{4.5}$$

Similarly, we apply Theorem 2.1 for $p = 2$ and $\tilde{\mathbf{V}} \overset{compact}{\hookrightarrow} \tilde{\mathbf{H}} \hookrightarrow \tilde{\mathbf{W}}^*$. By Lemma 3.1, (3.13) and $L^\infty(0, T-h;\ \tilde{\mathbf{W}}^*) \subset L^2(0, T-h;\ \tilde{\mathbf{W}}^*)$, there exists a subsequence, still denoted by $\tilde{M}_\varepsilon u$, such that

$$\tilde{M}_\varepsilon u \to \hat{u} \ \text{ strongly in } L^2(0, T;\ \tilde{\mathbf{H}})\,. \tag{4.6}$$

Now we will prove that $\{\hat{u}, \hat{\theta}\}$ is the weak solution of (1.9)–(1.13).

First, because of (3.8) and (3.13), there exists the subsequences, still denoted by $\tilde{M}_\varepsilon u$ and $M_\varepsilon \theta$, such that

$$\tilde{M}_\varepsilon u \to \hat{u} \ \text{ weakly in } L^\infty(0, T;\ \tilde{\mathbf{H}})\,, \tag{4.7}$$

and

$$M_\varepsilon \theta \to \hat{\theta} \ \text{ weakly in } L^\infty(0, T;\ L^2(\omega))\,. \tag{4.8}$$

Moreover, by the assumption of Theorem 1.1, we have

$$\hat{u}(x', 0) = \tilde{u}_0 \ \text{ and } \ \hat{\theta}(x', 0) = \tilde{\theta}_0\,.$$

Integrating (3.2) between 0 and t, we have

$$\begin{aligned}(M_\varepsilon\theta, w)_{L^2(\omega)} &\\ &= (M_\varepsilon\theta_0, w)_{L^2(\omega)} + \int_0^t \bigg[\frac{1}{\varepsilon}(Q, w) - ((\tilde{M}_\varepsilon u\cdot\nabla')M_\varepsilon\theta, w)_{L^2(\omega)}\\ &\quad - \frac{1}{\varepsilon}((\tilde{N}_\varepsilon u\cdot\nabla)N_\varepsilon\theta, w) - \kappa(\nabla' M_\varepsilon\theta, \nabla' w)\bigg]\,ds\,, \end{aligned}\tag{4.9}$$

for all $w \in C_0^\infty(\omega)$ and $t \in [0, T]$. For the second term of the right-hand side in (4.9), Proposition 2.1 yields

$$\left|\int_0^t \frac{1}{\varepsilon}(Q, w)\,ds\right| \le \int_0^T \varepsilon^{-\frac{1}{2}}|Q|\,|w|_{L^2(\omega)}ds \le c\,\varepsilon^{\frac{1}{2}}\|Q\|_{L^2(0,T;\,H_0^1(\Omega_\varepsilon))}|w|_{\mathcal{C}(\bar{\omega})}\,. \tag{4.10}$$

Then we obtain

$$\left|\int_0^t \frac{1}{\varepsilon}(Q, w)\,ds\right| \to 0$$

as $\varepsilon \to 0$. For the third term, we have the following:

$$\begin{aligned}&\left|\int_0^t ((\tilde{M}_\varepsilon u\cdot\nabla')M_\varepsilon\theta, w)_{L^2(\omega)}\,ds - \int_0^t ((\hat{u}\cdot\nabla')\hat{\theta}, w)_{L^2(\omega)}\,ds\right|\\ &\qquad \le \int_0^T |((\tilde{M}_\varepsilon u - \hat{u})\cdot\nabla' M_\varepsilon\theta, w)_{L^2(\omega)}|\,ds\\ &\qquad\quad + \int_0^T ((\hat{u}\cdot\nabla')(M_\varepsilon\theta - \hat{\theta}), w)_{L^2(\omega)}\,ds\\ &\qquad \le \|\tilde{M}_\varepsilon u - \hat{u}\|_{L^2(0,T;\,\tilde{\mathbf{H}})}\|M_\varepsilon\theta\|_{L^2(0,T;\,H_0^1(\omega))}|w|_{\mathcal{C}(\bar{\omega})}\\ &\qquad\quad + \|M_\varepsilon\theta - \hat{\theta}\|_{L^2(0,T;\,L^2(\omega))}\|\hat{u}\|_{L^2(0,T;\,\tilde{\mathbf{V}})}|w|_{\mathcal{C}(\bar{\omega})}\\ &\qquad \to 0\,,\end{aligned}$$

as $\varepsilon \to 0$, because of (3.8), (4.5), (4.6) and $\hat{u} \in L^2(0, T;\, \tilde{\mathbf{V}})$. For the fourth term, from the Cauchy-Schwarz inequality and Corollary 2.1, we obtain

$$\begin{aligned}\left|\frac{1}{\varepsilon}((\tilde{N}_\varepsilon u\cdot\nabla)N_\varepsilon\theta, w)\right| &\le \frac{1}{\varepsilon}|\tilde{N}_\varepsilon u|\,|N_\varepsilon\theta|\,|\nabla' w|_{\mathcal{C}(\bar{\omega})}\\ &\le \varepsilon\|\tilde{N}_\varepsilon u\|\,\|N_\varepsilon\theta\|\,|\nabla' w|_{\mathcal{C}(\bar{\omega})}\,.\end{aligned}$$

Then, as $\varepsilon \to 0$, we have

$$\left|\int_0^t \frac{1}{\varepsilon}((\tilde{N}_\varepsilon u\cdot\nabla)N_\varepsilon\theta, w)\,ds\right| \to 0\,,$$

since (3.10) and (3.15). And, from (4.8), we have

$$\int_0^t (M_\varepsilon\theta, \Delta' w)\, ds \to \int_0^t (\hat{\theta}, \Delta' w)\, ds$$

as $\varepsilon \to 0$. Hence, for the last term, we get

$$\int_0^t (\nabla' M_\varepsilon\theta, \nabla' w)\, ds \to \int_0^t (\nabla'\hat{\theta}, \nabla' w)\, ds\,,$$

as ε tends to zero.

Therefore the right-hand side in (4.9) converges, as $\varepsilon \to 0$ (t is fixed), to

$$(\tilde{\theta}_0, w)_{L^2(\omega)} - \int_0^t [((\hat{u}\cdot\nabla')\hat{\theta}, w)_{L^2(\omega)} + \kappa(\nabla'\hat{\theta}, \nabla' w)_{L^2(\omega)}]\, ds,$$

which is equal to $(\hat{\theta}(t), w)_{L^2(\omega)}$ by (4.8). Hence, since $H_0^1(\omega)$ is the completion of $C_0^\infty(\omega)$ under the $H_0^1(\omega)$ norm, we obtain

$$(\hat{\theta}, w) = (\tilde{\theta}_0, w) - \int_0^t [((\hat{u}\cdot\nabla')\hat{\theta}, w) + \kappa(\nabla'\hat{\theta}, \nabla' w)]\, ds\,, \tag{4.11}$$

for $\forall w \in H_0^1(\omega)$.

Similarly, we integrate (3.1) between 0 and t, we have

$$\begin{aligned}
&(\tilde{M}_\varepsilon u, v)_{L^2(\omega)} \\
&\quad = (\tilde{M}_\varepsilon u_0, v)_{L^2(\omega)} - \int_0^t \Big[\frac{1}{\varepsilon}((\tilde{N}_\varepsilon u\cdot\nabla)\tilde{N}_\varepsilon u, v) \\
&\qquad + ((\tilde{M}_\varepsilon u\cdot\nabla')\tilde{M}_\varepsilon u, v)_{L^2(\omega)} + \nu(\nabla'\tilde{M}_\varepsilon u, \nabla' v)_{L^2(\omega)} \\
&\qquad + 2f(k\times\tilde{M}_\varepsilon u, v)_{L^2(\omega)} + \frac{2f}{\varepsilon}(k\times\tilde{N}_\varepsilon u, v)\Big]\, ds\,,
\end{aligned} \tag{4.12}$$

for all $v \in \{\varphi \in C_0^\infty(\omega)^2;\ \mathrm{div}'\,\varphi = 0 \text{ in } \omega\}$ and $t \in [0, T]$. We calculate such as the case of $M_\varepsilon\theta$, and obtain

$$\int_0^t ((\tilde{M}_\varepsilon u\cdot\nabla')\tilde{M}_\varepsilon u, v)\, ds \to \int_0^t ((\hat{u}\cdot\nabla')\hat{u}, v)\, ds\,,$$

$$\int_0^t (\nabla'\tilde{M}_\varepsilon u, \nabla' v)\, ds \to \int_0^t (\nabla'\hat{u}, \nabla' v)\, ds$$

and

$$\left|\int_0^t \frac{1}{\varepsilon}((\tilde{N}_\varepsilon u\cdot\nabla)\tilde{N}_\varepsilon u, v)\, ds\right| \to 0$$

as $\varepsilon \to 0$. For the Coriolis term, by (4.7) and Proposition 2.1, we have

$$\int_0^t 2f(k \times \tilde{M}_\varepsilon u, v)\, ds = \int_0^t (b(\tilde{M}_\varepsilon u), v)\, ds \to \int_0^t (b(\hat{u}), v)\, ds\,,$$

and

$$\left| \int_0^t \frac{2f}{\varepsilon}(k \times \tilde{N}_\varepsilon u, v)\, ds \right| \le c\, \varepsilon^{\frac{1}{2}} \|\tilde{N}_\varepsilon u\|_{L^2(0,T;\, H^1(\Omega_\varepsilon))} |v|_{\mathcal{C}(\bar{\omega})} \to 0\,,$$

as $\varepsilon \to 0$.

Then the right-hand side in (4.12) converges to

$$(\tilde{u}_0, v)_{L^2(\omega)} - \int_0^t [((\hat{u} \cdot \nabla')\hat{u}, v)_{L^2(\omega)} + \nu(\nabla'\hat{u}, \nabla' v)_{L^2(\omega)} + (b(\hat{u}), v)_{L^2(\omega)}]\, ds\,,$$

which is equal to $(\hat{u}, v)_{L^2(\omega)}$ by (4.7), as $\varepsilon \to 0$ (t is fixed). Hence we obtain

$$(\hat{u}, v) = (\tilde{u}_0, v) - \int_0^t [((\hat{u} \cdot \nabla')\hat{u}, v) + \nu(\nabla'\hat{u}, \nabla' v) + (b(\hat{u}), v)]\, ds\,, \tag{4.13}$$

for $\forall v \in \tilde{\mathbf{V}}$, because $\tilde{\mathbf{V}}$ is the completion of the space $\{\varphi \in C_0^\infty(\omega)^2;\ \mathrm{div}'\,\varphi = 0 \text{ in } \omega\}$ under the $H^1(\omega)^2$ norm.

Hence, because of (4.11) and (4.13), we can see that $\{\hat{u}, \hat{\theta}\}$ is the weak solution of (1.9)–(1.13).

Finally, from (4.3) and (4.4), we obtain the weak time-continuity $\hat{\theta} \in \mathcal{C}(0, T;\ H^{-1}(\omega))$ and $\hat{u} \in \mathcal{C}(0, T;\ \tilde{\mathbf{V}}^*)$, so that the initial conditions make sense. □

References

[1] AZÉRAD, P. and F. GUILLEN, Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics, Siam J. Math. Anal. **33** (2001), 847–859.

[2] IFTIMIE, D. and RAUGEL, G., Some results on the Navier-Stokes equations in thin 3D domains, J. Differential Equations **169** (2001), 281–331.

[3] FIFE, P. C. and D. D. JOSEPH, Existence convective solutions of the genelarized Bénard problem which are analytic in there norm, Arch. Rational Mech. Anal. **33** (1969), 116–138.

[4] KAGEI, Y., On weak solutions of nonstationary Boussinesq equations, Differential and Integral Equations, **6** (1993), 587–611.

[5] LIONS, J. L., R. TEMAM and S. WANG, On the equation of the large-scale ocean, Nonlinearity **5** (1992), 1007–1053.

[6] LIONS, J. L., R. TEMAM and S. WANG, New formulations of the primitive equations of atmosphere and applications, Nonlinearity **5** (1992) 237–288.

[7] MORIMOTO, H., Non-stationary Boussinesq equations, J. Fac. Sci. Univ. Tokyo Sect. IA Math **39** (1992), 61–75.

[8] REBOLLO, T. C., R. LEWANDOWSKI and E. C. VERA, Analysis of the hydrostatic approximation in oceanography with compression term, Model. Math. Anal. Numer. **34** (2000), 525–537.

[9] TEMAM, R., *Navier-Stokes Equations*, North-Holland, Elsevier, Amsterdam, 1987.

[10] TEMAM, R. and M. ZIANE, Navier-Stokes equations in three-dimensional thin domains with various boundary conditions, Adv. Differential Equations **1** (1996), 499–546.

[11] TEMAM, R. and M. ZIANE, Navier-stokes equations in thin spherical domains, Contemp. Math. **209** (1997), 281–314.

[12] SAMELSON, R., TEMAM, R. and WANG, S., Some mathematical properties of the planetary geostrophic equations for large-scale ocean circulation, Applicable Analysis **70** (1998), 147–173.

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