Comment

Ann F. S. Mitchell

Without doubt many interesting developments and applications can be expected to arise from this elegant and clear exposition of saddlepoint methods in statistics.

I have little to add except to comment on the similarities and dissimilarities in the geometry of exponential families and transformation models. To achieve the level of clarity of Nancy Reid's presentation would require a lengthy and relatively technical introduction to the basic ideas and concepts of the differential geometry approach to statistics. As a compromise, albeit a relatively unsatisfactory one, the reader is referred to Amari (1985) and Barndorff-Nielsen, Cox and Reid (1986).

Although the exponential families are flat with respect to the ± 1 connections of the family of α-connections described by Chentsov (1972), Dawid (1975, 1977) and Amari (1982), transformation models have constant scalar curvature but are not necessarily flat for any α-value. The following well known two-parameter transformation models, which are not exponential families, illustrate what can happen. The parameters are the location and scale parameters. The Cauchy family is particularly unusual. It has constant negative scalar curvature, which does not depend on α and is thus never flat. However, the corresponding Student's t family on k ($k > 1$) degrees of freedom is flat when

$$\alpha = \pm \frac{k+5}{k-1}.$$

In the limit, as $k \to \infty$, we get $\alpha = \pm 1$, the appropriate values for a normal family that is both an exponential family and a transformation model. The logistic family is not flat for any value of α. These examples are special cases of a subclass of transformation models, studied by Mitchell (1988), and are the univariate analogue of the well known class of multivariate elliptic distributions.

Flatness plays an important role in defining unique estimators based on projections and is discussed extensively by Amari (1985), Lauritzen and Picard (1987) and Lauritzen (1987).

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ADDITIONAL REFERENCES

Barndorff-Nielsen, O. E., Cox, D. R. and Reid, N. (1986). The role of differential geometry in statistical theory. *Internat. Statist. Rev.* **54** 83–96.

Chentsov, N. N. (1972). *Statistical Decision Rules and Optimal Inference* (in Russian). Nauka, Moscow. English translation (1982). *Translation of Mathematical Monographs* **53**. Amer. Math. Soc., Providence, R. I.

Dawid, A. P. (1975). Discussion of "Defining the curvature of a statistical problem (with applications to second order efficiency)" by B. Efron. *Ann. Statist.* **3** 1231–1234.

Dawid, A. P. (1977). Further comments on some comments on a paper by Bradley Efron. *Ann. Statist.* **5** 1249.

Lauritzen, S. L. (1987). Conjugate connections in statistical theory. *Proc. of the Geometrization of Statistical Theory*. Dept. Mathematics, Univ. Lancaster, 28–31 October.

Lauritzen, S. L. and Picard, D. (1987). On the uniqueness of projections in statistical manifolds. Unpublished.

Mitchell, A. F. S. (1988). Statistical manifolds of univariate elliptic distributions. *Internat. Statist. Rev.* **56** 1–16.

Rejoinder

N. Reid

I would like to thank all the discussants for their contributions. Many of the discussants, and several colleagues, also sent me helpful comments on earlier versions of the paper.

It is a measure of the interest in and potential of saddlepoint methods that I expect this review will soon be out of date!

I am grateful to Hougaard and to Hinkley and Wang for providing additional numerical work to supplement Figure 1. Srivastava and Yau (1987) have also derived the saddlepoint expansion for the noncentral χ^2 distribution. In work as yet unpublished, Yau has derived the saddlepoint approximation for the density and Lugannani and Rice's tail area approximation for several statistics arising in multivariate analysis, such as Wilk's lambda, the Durbin-Watson statistic and ratios of quadratic forms.

These and other examples, such as in Davison and Hinkley (1988), are so impressive than one is naturally tempted to find examples where the saddlepoint approximation doesn't work. As Kass points out, distributions that are far from normal might be expected to be poorly approximated by saddlepoint techniques. Aside from taking very small ($n = 1$) samples from anomalous distributions such as the uniform or a multimodal distribution (Feuerverger, 1988), it seems difficult to find such examples. The extreme value distributions, or rather statistics with limiting extreme value distributions, are natural candidates, but even here the situation is not clear. For example, the limiting distribution of the minimum of a sample of uniforms will be well approximated by the renormalized saddlepoint approximation.

Feuerverger (1988) has also investigated the possibility of using the empirical cumulant generating function, defined in equation (A1) in Hinkley and Wang's discussion, to estimate the saddlepoint approximation to the true density or distribution function. Here the situation is less promising than in estimating the bootstrap distribution; the relative error of the estimated saddlepoint approximation is at best $O_p(n^{-1/2})$, and this is not uniform. I don't think this is inconsistent with Wang's result; the bootstrap approximation to the distribution of $\bar{X} - \mu$ is accurate to $O_p(n^{-1})$ only if $\bar{X} - \mu$ is studentized (Hartigan, 1986).

I would like to thank the discussants, particularly Barndorff-Nielsen, Daniels and Tierney, for additional historical notes and references. I hope the list of references is now a fairly complete guide to the statistical literature on second order asymptotics, and apologize to authors whose contributions I have overlooked. Davison has drawn my attention to the papers by Cordeiro (1983, 1985, 1987) on Bartlett factors for generalized linear models; these references should have been included in Section 4. Further discussion of the geometric interpretation of Bartlett factors is given in Ross (1987). Hayakawa's (1977) result is corrected in Harris (1986).

My interest in the saddlepoint approximation originated with an interest in the many recent developments in what might be termed "improved likelihood-based inference." In much of this work Barndorff-Nielsen's formula plays a central role, although as he points out, not always as a saddlepoint approximation. It is, I think, quite striking that the likelihood functions at the points on the sample space give an approximate density that can be used for inference directly. The connection between the renormalized likelihood function and Laplace's method is further discussed in Fraser (1988).

As noted by Mitchell, in the discussion of likelihood-based inference and higher order asymptotics, the geometry of the model seems increasingly to play a crucial role. It was unfortunately not possible to include discussion of this very interesting area in the present paper. The geometrical interpretation of terms arising in Taylor series expansions, initiated by McCullagh and Cox (1986) in their discussion of the Bartlett factor, and extended in Barndorff-Nielsen (1986c) and Murray (1988), seems likely to lead to further understanding of the role of higher order asymptotics in inference.

The discussants seem divided over whether or not it will soon be easy to implement saddlepoint approximations and other second order corrections in practical problems, with Daniels and Hougaard possibly the most optimistic, and Kass rather more doubtful. I am myself quite optimistic; I think that it will soon be possible, and eventually even easy, to compute corrections like the Bartlett factor in a relatively straightforward manner.

Preparing this review has been very rewarding, not least because of the interest and enthusiasm with which the topic has been received by a great many colleagues. I would particularly like to thank Professor DeGroot for his encouragement and his editorial efforts.

ADDITIONAL REFERENCES

BARNDORFF-NIELSEN, O. E. (1986c). Strings, tensorial combinants, and Bartlett adjustments. *Proc. Roy. Soc. London Ser. A* **406** 127–137.

CORDEIRO, G. M. (1983). Improved likelihood ratio statistics for generalized linear models. *J. Roy. Statist. Soc. Ser. B* **45** 404–413.

CORDEIRO, G. M. (1985). The null expected deviance for an extended class of generalized linear models. *Generalized Linear Models. Lecture Notes in Statist.* **32** 27–34. Springer, New York.

CORDEIRO, G. M. (1987). On the corrections to likelihood ratio statistics. *Biometrika* **74** 265–274.

FEUERVERGER, A. (1988). On the empirical saddlepoint. Technical Report 8, Dept. Statistics, Univ. Toronto.

FRASER, D. A. S. (1988). Normed likelihood as saddlepoint approximation. *J. Multivariate Anal.* To appear.

HARRIS, P. (1986). A note on Bartlett adjustments to likelihood ratio tests. *Biometrika* **73** 735–737.

HARTIGAN, J. A. (1986). Discussion of "Bootstrap methods for standard errors, confidence intervals and other measures of statistical accuracy" by B. Efron and R. Tibshirani. *Statist. Sci.* **1** 75–77.

HAYAKAWA, T. (1977). The likelihood ratio criterion and the asymptotic expansion of its distribution. *Ann. Inst. Statist. Math.* **29** 359–378.

MURRAY, M. K. (1988). Coordinate systems and Taylor series in statistics. *Proc. Roy. Soc. London Ser. A* **415** 445–452.

ROSS, W. H. (1987). The expectation of the likelihood ratio criterion. *Internat. Statist. Rev.* **55** 315–330.

SRIVASTAVA, M. S. and YAU, W. K. (1987). Comparison of approximate saddlepoint and saddlepoint with Edgeworth expansions. Technical Report 4, Dept. Statistics, Univ. Toronto.