Statistical Science
1994, Vol. 9, No. 1, 109–126

Can One See α-Stable Variables and Processes?

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Abstract. In this paper, we demonstrate some properties of α-stable (stable) random variables and processes. It turns out that with the use of suitable statistical estimation techniques, computer simulation procedures and numerical discretization methods it is possible to construct approximations of stochastic integrals with stable measures as integrators. As a consequence we obtain an effective, general method giving approximate solutions for a wide class of stochastic differential equations involving such integrals.

Application of computer graphics provides interesting quantitative and visual information on those features of stable variates which distinguish them from their commonly used Gaussian counterparts. It is possible to demonstrate evolution in time of densities with heavy tails of appropriate processes, to visualize the effect of jumps of trajectories, etc.

We try to demonstrate that stable variates can be very useful in stochastic modeling of problems of different kinds, arising in science and engineering, which often provide better description of real life phenomena than their Gaussian counterparts.

Key words and phrases: Stable distributions, stable processes, stochastic integrals and differential equations with stable integrators, statistical estimation, stochastic modeling, computer simulation.

1. INTRODUCTION

This section contains an overview of the history of research on stable random variates, some remarks on statistical and stochastic modeling and a short summary of the contents of this paper.

1.1 Historical Overview

Let us start with an historical overview of the development of investigations and applications of α-stable random variables and processes. The interested reader looking for documented sources on this topic is referred to some short notes in Breiman (1968 pp. 215, 318) and the recent paper of Fienberg (1992).

The early problem considered by pioneering statisticians of the 18th and 19th centuries was to find the best fit of an equation to a set of observed data points. After some false starts, they hit upon the method of least squares. Legendre's work seemed to be the most influential at the time. Laplace elaborated upon it; and finally in a discussion of the distribution of errors, Gauss emphasized the importance of the normal or Gaussian distribution. Laplace was a great enthusiast of generating functions and solved many complicated probability problems exploiting them. As the theory of Fourier series and integrals emerged in the early 1800s, he and Poisson made the next natural step: applying such representations of probability distributions as a new natural tool for analysis, thus introducing the powerful characteristic function method. Laplace seemed to be especially pleased noticing that the Gaussian density was its own Fourier transform.

In the early 1850s Cauchy, Laplace's former student, became interested in the theory of errors and

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extended the analysis, generalizing the Gaussian formula to a new one,

$$f_N(x) = \frac{1}{\pi} \int_0^\infty \exp(-ct^N)\cos(tx)\,dt,$$

expressed as a Fourier integral with t^N replacing t^2. He succeeded in evaluating the integral (in the non-Gaussian case) only for $N = 1$, thus obtaining the famous Cauchy law defined by the density

$$f_1(x) = \frac{c}{\pi(c^2 + x^2)}.$$

It was realized only much later (in 1919, thanks to Bernstein) that f_N is positive-definite; hence it is a probability density function only when $0 < N \leq 2$. Replacing N by a real parameter α with values in $(0, 2]$, we find that the integral defining functions $f_\alpha = f_\alpha(x)$ is a source of remarkable surprises.

After Cauchy there was a decline in mathematicians' interest in this subject until 1924 when the theory of stable distributions originated with Paul Lévy. When fashion sought the most general conditions for the validity of the central limit theorem, Lévy found simple exceptions to it, namely the class of α-stable distributions with index of stability $\alpha < 2$. The ambiguous name *stable* has been assigned to these distributions because if X_1 and X_2 are random variables having this distribution, then X defined by the linear combination $cX = c_1X_1 + c_2X_2$ has a similar distribution with the same index α for any positive, real values of the constants c, c_1 and c_2 with $c^\alpha = c_1^\alpha + c_2^\alpha$.

Lévy noted that the Gaussian case ($\alpha = 2$) is "singular" because for all $\alpha \in (0, 2)$ all nondegenerate densities $f_\alpha(x)$ have inverse power tails, that is, $\int_{\{|x|>\lambda\}} f_\alpha(x)\,dx \approx C \cdot \lambda^{-\alpha}$ for large λ. Since these distributions have no second moments, the second moment existence condition for the CLT is violated, allowing the possibility of unusual results.

Research concerning stable stochastic processes and models has been directed toward delineating the extent to which they share the features of the Gaussian models and even more significantly toward discovering their own distinguishing and often surprising features [cf. Weron (1984)]. In the last ten years many important results concerning characterizations of different properties of these processes (and of other subclasses of processes with independent increments) have been obtained by several authors. Of particular importance are the results concerning representations involving stochastic integrals. A collection of papers edited by Cambanis, Samorodnitsky and Taqqu (1991) provides a review of the state of the art on stable processes as models for random phenomena. See also Janicki and Weron (1993).

At the same time, there has been an explosive growth in the study of physical and economic systems that can be successfully modeled with the use of stable distributions and processes. Especially infinite moments, elegant scaling properties and the inherent self-similarity property of stable distributions are appreciated by physicists. See Section 4 for more details.

1.2 Statistical and Stochastic Modeling

The terms statistical model and stochastic model may be understood differently and may be ambiguous in some situations. One description of *statistical model* is the following [see Clogg (1992)]

> What statistical methodology refers to in most areas today is virtually synonymous with statistical modeling. A statistical model can be thought of as an equation, or set of equations, that (a) links "inputs" to "outputs"..., (b) have both fixed and stochastic components, (c) include either a linear or a nonlinear decomposition between the two types of components, and (d) purport to explain, summarize or predict levels of or variability in the "outputs".

A very interesting discussion on how to model the progression of cancer, the AIDS epidemic and other real life phenomena is contained in the chapter "Model building: Speculative data analysis" of a book by Thompson and Tapia (1990). The main idea is to derive a stochastic process that describes as closely as possible an investigated problem. Starting from an appropriate system of axioms one has to arrive at a formula (*a stochastic model)* defining this process, construct it explicitly in some way and verify its correctness and usefulness. Appealing to one of the problems they are interested in, Thompson and Tapia (1990), page 214, say, "If we wish to understand the mechanism of cancer progression, we need to conjecture a model and then test it against a data base."

Quite often a stochastic model is a synonym of a stochastic differential equation or of a system of stochastic differential equations. Thanks to Itô's theory of stochastic integration with respect to Brownian motion, it is commonly understood that any continuous diffusion process $\{X(t);\ t \geq 0\}$ with given drift and diffusion coefficients can be obtained as a solution of the stochastic differential equation

$$X(t) = X_0 + \int_0^t a(s,X(s))\,ds + \int_0^t c(s,X(s))\,dB(s), \qquad t \geq 0, \tag{1.1}$$

where $\{B(t);\ t \in [0,\infty)\}$ stands for Brownian motion and X_0 is a given Gaussian random variable. The theory of such stochastic differential equations is well developed [see, e.g., Arnold (1974)] and they are widely applied in stochastic modeling.

However, it is not so commonly understood that a vast class of diffusion processes $\{X(t);\ t \geq 0\}$ with given drift and diffusion coefficients can be described by the stochastic differential equation

$$X(t) = X_0 + \int_0^t a(s,X(s-))\,ds + \int_0^t c(s,X(s-))\,dL_\alpha(s), \qquad t \geq 0, \tag{1.2}$$

where $\{L_\alpha(t);\ t \in [0,\infty)\}$ stands for an α-stable Lévy motion (defined in Section 3) and X_0 is a given α-stable random variable. Note that in general diffusion processes $\{X(t);\ t \geq 0\}$ defined above do not belong to the class of α-stable processes.

As sources of information on modern aspects of stochastic analysis (e.g., on various properties of stochastic integrals and on existence of solutions of stochastic differential equations driven by stochastic measures of different kinds), we recommend, among others, Protter (1990) and Kwapień and Woyczyński (1992).

An application of stochastic differential equations (1.1) or (1.2) to statistical or stochastic model building is not an easy task. So it seems that the use of suitable statistical estimation techniques, computer simulation procedures and numerical discretization methods should prove to be a powerful tool.

> In most of nontrivial cases ... the "closed form" solution is itself so complicated that it is good for little other than as a device for pointwise numerical evaluation. The simulation route should generally be the method of approach for nontrivial time-based modeling. ... Unfortunately, at the present time, the use of the modern digital computer for simulation based modeling and computation is an insignificant fraction of total computer usage. [Thompson and Tapia (1990), pp. 232–233]

We agree with this opinion and add that unfortunately, as far as we know, practical approximate methods for solving stochastic differential equations involving stochastic integrals with stable integrands are only now beginning to be developed.

In our exposition we emphasize the methods exploiting computer graphics. Let us cite Thompson's opinion:

> I feel that the graphics oriented density estimation enthusiasts fall fairly clearly into the exploratory data analysis camp, which tends to replace statistical theory by the empiricism of a human observer. Exploratory data analysis, including nonparametric density estimation, should be a first step down a road to understanding the mechanism of data generating systems. The computer is a mighty tool in assisting us with graphical displays, but it can help us even more fundamentally in revising the way we seek out the basic underlying mechanisms of real world phenomena via stochastic modeling. [Thompson and Tapia (1990), p. xiv]

1.3 Summary of the Contents

This article is structured as follows. After recalling in Sections 2 and 3 some basic properties of α-stable random variables, stochastic integrals with respect to α-stable Lévy motion and stochastic differential equations involving such integrals, we provide a brief survey of examples of stochastic models involving α-stable variables and processes in Section 4: mainly from physics, chemistry and economics. Next, in Sections 5 and 6, we discuss some techniques of computer generation of α-stable random variables and some methods of simulation of approximate solutions of stochastic differential equations of the form (1.2). In Section 7 our method of visualization of such stochastic processes is described. Section 8 contains some theoretical convergence results. Section 9, advertising possible applications of computer methods to empirical model building, provides examples of computer visualizations of solutions of stochastic differential equations driven by α-stable measures.

To the best of our knowledge the figures in this article present some of the first visual representations of such stochastic processes.

2. SOME BASIC PROPERTIES OF α-STABLE RANDOM VARIABLES

Beginning the discussion on properties of some classes of α-stable random variates, we recall briefly the main features of univariate α-stable (or stable) random variables. From the literature on this topic let us mention among others: Feller (1966) and (1971), Lévy (1937), Gnedenko and Kolmogorov (1954), Weron (1984), Zolotarev (1986) or Samorodnitsky and Taqqu (1993). Cf. also Hall (1980) for historical comments.

2.1 Characteristic Function

The most common and convenient way to introduce α-stable random variables is to define their *characteristic function*, which involves four parameters: α—the index of stability, β—the skewness parameter, σ—the scale parameter and μ—the shift. This function is given by

$$\log \phi(\theta) = \begin{cases} -\sigma^\alpha|\theta|^\alpha \{1 - i\beta \operatorname{sgn}(\theta)\tan(\alpha\pi/2)\} + i\mu\theta, & \text{if } \alpha \in (0,1)\cup(1,2], \\ -\sigma|\theta|\{1 + i\beta\frac{2}{\pi}(\operatorname{sgn}\theta)\ln|\theta|\} + i\mu\theta, & \text{if } \alpha = 1, \end{cases} \tag{2.1}$$

where $\beta \in [-1,1]$, $\sigma \in R_+$, $\mu \in R$.

For a random variable X distributed according to the above described rule we will use the notation $X \sim S_\alpha(\sigma,\beta,\mu)$. Notice that $S_2(\sigma,0,\mu)$ and $S_1(\sigma,0,\mu)$ give the Gaussian distribution $\mathcal{N}(\mu,2\sigma^2)$ and the Cauchy distribution, respectively.

2.2 Domain of Attraction of *X*

A random variable X has a stable distribution if and only if it has *a domain of attraction*, that is, if there exists a sequence $Y_1, Y_2, \ldots$ of i.i.d. random variables and sequences $\{d_n\}$ and $\{a_n\}$ of positive real numbers such that

$$\frac{Y_1 + Y_2 + \cdots + Y_n}{d_n} + a_n \Rightarrow X,$$

where $Z_n \Rightarrow X$ means convergence of Z_n to X in distribution.

According to Feller (1971), Theorem VI.1.1, in general we have $d_n = n^{1/\alpha}h(n)$, where the function $h = h(x), x \geq 0$ varies slowly at infinity; the sequence $\{Y_i\}$ is said to belong to the domain of attraction of X, when $d_n = n^{1/\alpha}$. Observe that if Y_i's are i.i.d. random variables with finite variance, then X is Gaussian and we obtain an ordinary version of the CLT.

2.3 Stable Lévy Measure

To justify what was said above one may recall the Lévy–Khintchine representation theorem [see Feller (1966), p. 542]. Let us introduce the *stable Lévy measure*

$$L(dx) = \frac{P}{x^{1+\alpha}} I_{(0,\infty)}(x)\,dx + \frac{Q}{|x|^{1+\alpha}} I_{(-\infty,0)}(x)\,dx,$$

with nonnegative numbers P, Q, and a function

$$\psi(\theta,x) = e^{i\theta x} - 1 - \frac{i\theta x}{1+x^2}.$$

Then for X we have the following representation:

$$E\,\exp(i\theta X) = \begin{cases} \exp\{iM\theta - \sigma^2\theta^2\}, & \text{if } \alpha = 2, \\ \exp\left\{iM\theta + \int_{\mathcal{R}\setminus\{0\}} \psi(\theta,x)\,L(dx)\right\}, & \text{if } 0 < \alpha < 2, \end{cases}$$

where M is real and σ is real and positive.

2.4 Finite and Infinite Moments

The striking feature of α-stable random variables is the behavior of their moments.

1. If we have $X \sim S_\alpha(\sigma,\beta,\mu)$ and $\alpha \in (0,2)$, then
$$E|X|^p < \infty \qquad \text{for } p \in (0,\alpha),$$
and
$$E|X|^p = \infty \qquad \text{for } p \geq \alpha.$$
2. If we have $X \sim S_\alpha(\sigma,0,\mu)$ and $\alpha \in (1,2]$, then $EX = \mu$.

2.5 Density Functions

The main problem when we start to work with α-stable distributions is that, except for a few values of the four parameters describing the characteristic function, their density functions are not known in a simple explicit form. The most interesting exceptions are the Gaussian distribution $S_2(\sigma.0,\mu) \sim \mathcal{N}(\mu,2\sigma^2)$, the Cauchy distribution $S_1(\sigma,0,\mu)$ and the Lévy distribution $S_{1/2}(\sigma,1,\mu)$, whose density

$$f_{1/2}(x) = \left(\frac{\sigma}{2\pi}\right)^{1/2}(x-\mu)^{-3/2}\exp\left\{-\frac{\sigma}{2(x-\mu)}\right\}$$

is concentrated on (μ,∞), that is, $f_{1/2}(x) = 0$ for $x \in (-\infty,\mu]$. In order to obtain *α-stable density functions*, we have to take into account the definition describing characteristic functions of α-stable random variables and to apply the Fourier transform, namely

$$f_\alpha(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)dt.$$

With the use of numerical approximation of this formula via the fast Fourier transform method, it is possible to construct such densities in a general case. Results of computer calculations are presented here in the form of series of computer graphs of such

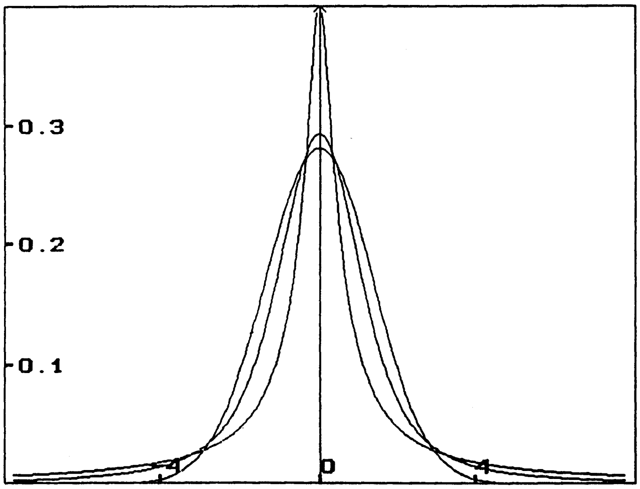

FIG. 2.1. *Three graphs of densities of α-stable random variables* $S_\alpha(1.0, 0.0, 0.0)$ *with* $\alpha \in \{0.7, 1.3, 2.0\}$ *(from top to bottom).*

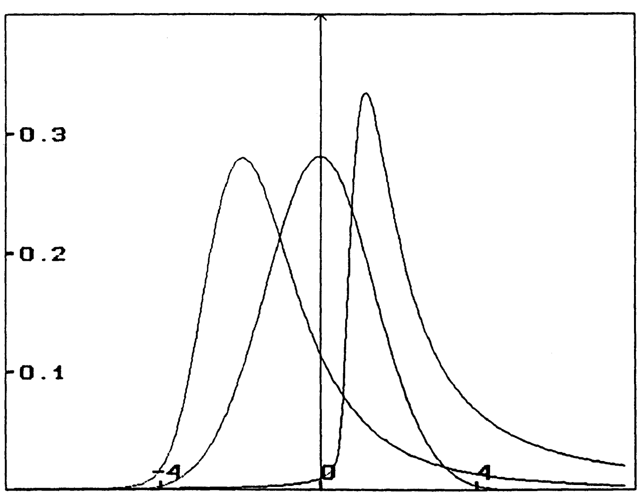

FIG. 2.2. *Three graphs of densities of α-stable random variable* $S_\alpha(1.0, 0.9, 0.0)$ *with* $\alpha \in \{1.3, 2.0, 0.7\}$ *(from left to right).*

densities for different values of parameters α, β (the role of the parameters σ and μ is obvious).

Figures 2.1 and 2.2 present the dependence of stable densities on two parameters α and β, which appear in the definition (2.1). The first shows the dependence on α. In the second one can see an impact of the skewness parameter β on stable densities, resulting in their asymmetry. Observe that when $\alpha = 2$, the Gaussian case, β does not influence the distribution.

REMARK 2.1. All figures presented in this paper demonstrate how different features of α-stable variates depend on α. In order to make the exposition more instructive we fixed three values of α: 2.0, 1.3, 0.7.

2.6 Behavior of Tail Probabilities

Using a CLT-type argument [e.g., Feller (1971), Theorem XVII.5.1] one can prove that if $X \sim S_\alpha(\sigma, \beta, \mu)$ and $\alpha \in (0, 2)$, then

$$\lim_{\lambda \to \infty} \lambda^\alpha P\{X > \lambda\} = C_\alpha \frac{1+\beta}{2} \sigma^\alpha,$$
$$\lim_{\lambda \to \infty} \lambda^\alpha P\{X < -\lambda\} = C_\alpha \frac{1-\beta}{2} \sigma^\alpha,$$

where

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin(x)\, dx \right)^{-1}.$$

In the rest of this paper we restrict ourselves to the symmetric case of $X \sim S_\alpha(1, 0, 0)$ (in what follows such random variables will be denoted by S_α).

The following result gives an exact explicit formula which serves as a basis for two algorithms: an applicable algorithm of computation of tail probabilities and a method of computer simulation of α-stable variables and random measures (see Sections 5 and 7).

THEOREM 2.1. *Let us define in the square of* $(x, y) \in (0, 1) \times (0, 1)$ *the function*

$$f_\alpha(x, y) = \frac{\sin(\alpha\pi(x - \frac{1}{2}))}{\{\cos(\pi(x - \frac{1}{2}))\}^{1/\alpha}} \cdot \left\{ \frac{\cos((1-\alpha)\pi(x - \frac{1}{2}))}{-\log(y)} \right\}^{(1-\alpha)/\alpha}. \tag{2.2}$$

Then we have

$$P\{S_\alpha > \lambda\} = |\{(x, y);\ f_\alpha(x, y) > \lambda\}|, \tag{2.3}$$

where $|A|$ *denotes the volume of a set* A *in* $\mathcal{R}^2$.

A formula more general then (2.2) concerning skewed stable variables $S_\alpha(1, \beta, 0)$ can be found, for example, in Chambers, Mallows and Stuck (1976). It was Kanter (1975) who noticed that a certain integral formula dealing with totally skewed to the right stable random variables ($\beta = 1$ and $\alpha \in (0, 1)$) derived by Ibragimov and Chernin (1959) implies a simulation method similar to that expressed in Theorem 2.1. Then Chambers, Mallows and Stuck (1976) noticed that a formula of Zolotarev's (1966) can be used similarly to give a simulation method for the general case.

3. α-STABLE STOCHASTIC PROCESSES

The starting point for the discussion of α-stable processes is α-stable Lévy motion.

3.1 Stable Lévy Motion

A stochastic process $\{L_\alpha(t);\ t \geq 0\}$ is called *(standard) symmetric α-stable Lévy motion* if :

1. $L_\alpha(0) = 0$ a.s.;
2. $L_\alpha(t)$ has independent increments; and
3. $L_\alpha(t) - L_\alpha(s) \sim S_\alpha((t-s)^{1/\alpha}, 0, 0)$ for any $0 \leq s < t < \infty$.

Observe that α-stable Lévy motion has stationary increments. It is Brownian motion when $\alpha = 2$, precisely $L_2(t) = \sqrt{2}B(t)$. The symmetric α-stable Lévy motion is $1/\alpha$ self-similar. That is for all $c > 0$, the processes $\{L_\alpha(ct);\ t \geq 0\}$ and $\{c^{1/\alpha}L_\alpha(t);\ t \geq 0\}$ have the same finite-dimensional distributions.

3.2 α-stable Measures and Integrals

For any Lebesgue measurable set A in $\mathcal{R}$ its stable independently scattered stochastic measure is defined by the formula

$$L_\alpha(A) \sim S_\alpha\left((m(A))^{1/\alpha}, 0, 0\right),$$

where $m(A)$ denotes Lebesgue measure of A.

Now we want to define an α-stable stochastic integral

$$I(f) = \int_A f(x)L_\alpha(dx) \tag{3.1}$$

on any measurable set $A \subset \mathcal{R}$ and for all $f \in \mathbf{L}^\alpha(m)$, that is, for all measurable functions $f : A \to \mathcal{R}$ satisfying the condition

$$\int_A |\, f(x)\,|^\alpha\, m(dx) < \infty.$$

For a simple function $f(x) = \Sigma_j c_j 1_{A_j}(x)$, where A_j are disjoint measurable sets such that $\bigcup_j A_j = A$, we put

$$I(f) = \int_A f(x)L_\alpha(dx) =_{\mathrm{df}} \sum_j a_j L_\alpha(A_j).$$

The integral $I(f)$ is obviously linear on the space of all simple functions f. The random measure L_α is independently scattered, so the α-stable random variables $L_\alpha(A_1),\ldots, L_\alpha(A_n)$ are independent and consequently $I(f)$ is α-stable, say $I(f) \sim S_\alpha(\sigma_f, \beta_f, \mu_f)$. Consider now any function f in the space $\mathbf{L}^\alpha(m)$. It is easy to construct a sequence of simple functions $\{f^{(n)}\}_{n=1}^\infty$ possessing the following properties:

$$f^{(n)}(x) \to f(x) \text{ for almost all } x \in A;$$

there exists a function $g \in \mathbf{L}^\alpha(m)$ such that

$$|\, f^{(n)}(x)\,| \leq g(x) \quad \text{for any } \ n, x.$$

The sequence of integrals $I(f^{(n)})$, $n = 1, 2, \ldots$ is well defined and converges in probability. Therefore, we define

$$I(f) =_{\mathrm{df}} \lim_{n\to\infty} I(f^{(n)}) \ \text{ in probability.}$$

This definition does not depend on the choice of approximating sequence $f^{(n)}$ and $I(f)$ is linear with respect to f.

For other possible definitions of α-stable integrals consult Kwapień and Woyczyński (1992) or Samorodnitsky and Taqqu (1993).

3.3 Diffusion Processes Driven by Stable Lévy Motion

Applying the above definition of stochastic integral we can define an α-stable diffusion process as a solution of a linear stochastic differential equation with respect to stable Lévy motion of the form

$$\begin{aligned} X(t) = X_0 &+ \int_0^t (a(s) + b(s)X(s-))\, ds \\ &+ \int_0^t c(s)\, dL_\alpha(s) \qquad \text{for} \quad t \in [0, \infty), \end{aligned} \tag{3.2}$$

with $X(0) = X_0$ a given stable random variable.

This linear stochastic equation is of independent interest because, as is easily seen, the general solution belongs to the class of α-stable processes. It may be expressed in the following form:

$$\begin{aligned} X(t) = \Phi(t,0)X_0 &+ \int_0^t \Phi(t,s)\, a(s)\, ds \\ &+ \int_0^t \Phi(t,s)\, c(s)\, dL_\alpha(s), \end{aligned}$$

where $\Phi(t,s) = \exp\left\{ \int_s^t b(u)\, du \right\}$.

The special case of the equation (1.2) involving only integration of deterministic functions with respect to α-stable integrators can be expressed in the following form:

$$\begin{aligned} X(t) = X_0 &+ \int_0^t a(s, X(s-))\, ds \\ &+ \int_0^t c(s)\, dL_\alpha(s) \qquad \text{for } \ t \geq 0. \end{aligned} \tag{3.3}$$

Notice that more general than equations (3.3) and (1.2) are the so-called stochastic differential equations with jumps, involving stochastic integrals with respect to Poisson random measures of suitable point processes with given deterministic intensity measures [see Ikeda and Watanabe (1981)]. In turn, all these stochastic equations are special cases of general stochastic differential equations driven by semimartingales, that is, equations of the form

$$X(t) = X_0 + \int_0^t f(X(s-))\, dY(s), \tag{3.4}$$

where $\{Y(t)\}$ stands for a given semimartingale process.

There is a vast literature concerning this topic [see, e.g., Protter (1990) and the bibliography therein].

To see that the differential equation (1.2) driven by α-stable Lévy motion is a special case of the equation (3.4) with a semimartingale as an integrator, it is enough to notice that α-stable Lévy motion can serve as an example of a semimartingale [see, e.g., Kwapień and Woyczyński (1992)].

This fact allows us to obtain theorems on existence of solutions of stochastic differential equations (3.2) or (3.3) driven by stable measures. It is enough to employ corresponding theorems concerning semimartingales.

4. A SURVEY OF α-STABLE MODELING

We believe that stable distributions and stable processes provide useful models for many phenomena observed in diverse fields. The central limit argument often used to justify the use of a Gaussian model in applications may also be applied to support the choice of a non-Gaussian stable model. That is, if the randomness observed is the result of summing many small effects, and those effects follow a heavy-tailed distribution, then a non-Gaussian stable model may be appropriate. An important distinction between Gaussian and non-Gaussian stable distributions is that the stable distributions are heavy-tailed, always with infinite variance, and in some cases with infinite first moment. Another distinction is that they admit asymmetry, or skewness, while a Gaussian distribution is necessarily symmetric about its mean. In certain applications, then, where an asymmetric or heavy-tailed model is called for, a stable model may be a viable candidate. In any case, the non-Gaussian stable distributions furnish tractable examples of non-Gaussian behavior and provide points of comparison with the Gaussian case, highlighting the special nature of Gaussian distributions and processes.

In order to obtain an appreciation of the basic difference between the Gaussian distribution and a distribution with a long tail, Montroll and Shlesinger (1983b) proposed to compare the distribution of heights with the distribution of annual incomes for American adult males. An average individual who seeks a friend twice his height would fail. On the other hand, one who has an average income will have no trouble discovering a richer person who, with a little diligence, may locate a third person with twice his income, etc. The income distribution in its upper range has a Pareto inverse power tail; however, most of the income distributions follow a log-normal curve. But the last few percent have a stable tail with exponent $\alpha = 1.6$ [see Badgar (1980)], that is, the mean is finite but the variance of the corresponding 1.6-stable distribution diverges.

Failure of the least-squares method of forecasting in economic time series was first explained by Mandelbrot (1963). He introduced a radically new approach based on α-stable processes to the problem of speculative price variation.

Now it is commonly accepted that the distribution of returns on financial assets is non-Gaussian. Mandelbrot (1963) and Fama (1965) proposed the α-stable distribution for modeling stock returns. Mittnik and Rachev (1989) found that the geometric summation scheme provides a better model for describing the stability properties of stock returns computed from the Standard and Poor's (*S and P*) 500 index. The problem of estimating multivariate α-stable distributions has received increasing attention in recent years in modeling portfolios of financial assets; see Mittnik and Rachev (1991) and references therein.

There are many physical phenomena which exhibit both space and time long tails and thus seem to violate the requirements for a Gaussian distribution as a limit in the traditional CLT; see Weron and Weron (1985). However, since these physical systems usually have nice scaling properties (self-similarity) one suspects the use of stable distributions which have long tails, infinite moments and elegant scaling properties to be relevant in the physics of these phenomena. Tunaley (1972) invoked physical arguments to suggest that if the frequency distributions in metallic films are stable, then the observed noise characteristics in them may be understood. Based only on the experimental observation that near second order phase transition, where long tail spatial order develops, Jona-Lasinio (1975) considered stable distributions as a basic ingredient in understanding renormalization group notions in explaining such phenomena. Also see a review article by Cassandro and Jona-Lasinio (1978). Scher and Montroll (1975) connected intermittent currents in certain xerographic films to a stable distribution of waiting times for the jumping of charges out of a distribution of deep traps. This was used to give the first explanation of experiments measuring transient electrical currents in amorphous semiconductors.

Stable distribution of first passage times appears both in the recombination reactions in amorphous materials [Montroll and Shlesinger (1983a)] as well as in the dielectric relaxation phenomena described by the Williams–Watts formula: Montroll and Shlesinger (1984), Montroll and Bendler (1984), Bendler (1984) and Weron (1986). It turns out that the way stable distributions appear here is somewhat more refined and it has been a subject of extensive research in physics; see Scher, Shlesinger and Bendler (1991); Weron (1991); as well as in chemistry; see Plonka (1986, 1991) and Pittel, Woyczyński and Mann (1990).

As examples of the exploration of the stable process models in physical contexts we may cite a few very interesting papers. Doob (1942) and West and Seshadri (1982) examined the response of a linear system driven by stable noise fluctuations and modeled by appropriately constructed stochastic differential equations. Takayasu (1984) demonstrated that the velocity of the fractal turbulence in $\mathcal{R}^3$ is the stable distribution with the index of stability $\alpha = D/2$, where D denotes the fractal dimension of the turbulence and that the diffusion process of particles in the turbulence and that of electrons in a uniformly magnetized plasma both can be approximated by the Lévy process. Mandelebrot and Van Ness (1968) defined fractional Brownian motion. Hughes, Shlesinger and Montroll (1981) examined random walks with self-similar clusters leading to Lévy flights and $1/f$-noise. Some connections between such clustered behavior in space or time of physical processes and fractal dimensionality of Lévy processes were studied by Seshadri and West (1982). Klafter et al. (1990, 1992) described the Lévy walk scheme for diffusions in the framework of continuous time random walks with coupled memories. They concentrated on those Lévy walks which lead to enhanced diffusions. Their approach was based on a modification of the Lévy flights. Schertzer and Lovejoy (1990) made use of the self-similarity property of stable processes in order to make evident the multifractal behavior of some geophysical fields. For computer methods of construction of fractional Brownian motion and other processes mentioned above we refer the reader to Barnsley et al. (1992), pages 42–132.

Stable and infinitely divisible (or Lévy) processes are beginning to attract the interest of mathematicians working in the field of applied probability. Let us mention among others Hardin, Samorodnitsky and Taqqu (1993), Kasahara and Yamada (1991), Kella and Whitt (1991) and McGill (1989).

In this context let us remark that in commonly known probability textbooks only reference to Holtsmark's work from 1915 on the gravitational field of stars (3/2-stable distribution) is made. For example, Feller devotes considerable space to stable distributions in volume 2 of his pair of probability books, but the admission is made that their role in applied sciences seems to be almost nonexistent. The above mentioned and related findings should be viewed as a step forward toward fulfilling Gnedenko and Kolmogorov's (1954) prophecy: "It is probable that the scope of applied problems in which stable distributions will play an essential role will become in due course rather wide".

5. COMPUTER SIMULATION OF α-STABLE RANDOM VARIABLES

Computer methods of generating random distributions of different kinds have existed for a long time; let us mention Devroye (1986) or Bratley, Fox and Schrage (1987) with the references therein. They are based on the application of effective computer generators of uniformly distributed random numbers [see Marsaglia and Zaman (1991)]. From Theorem 2.1 one can easily derive the following algorithm of simulation of symmetric α-stable random variables. It contains as a special case ($\alpha = 2$) the well known Box–Muller method of construction of normally distributed random variables [see, e.g., Bratley, Fox and Schrage (1987), p. 161].

5.1 α-stable Random Variable Generation

To generate a symmetric α-stable random variable $X \sim S_\alpha(1,0,0)$ for $\alpha \in (0,2]$, we may

- generate a random variable V uniformly distributed on $(-\pi/2, \pi/2)$ and an exponential random variable W with mean 1;
- compute $X = \frac{\sin(\alpha V)}{\{\cos(V)\}^{1/\alpha}} \cdot \left\{\frac{\cos(V-\alpha V)}{W}\right\}^{(1-\alpha)/\alpha}$.

We regard this method as a good technique of computer simulation of α-stable stochastic processes and measures, but of course it has its limitations.

5.2 Density Estimators

Problems of smoothing statistical data were studied, for example, by Györfi et al. (1989). Our aim is to recall briefly formulas describing kernel density estimators. So let us suppose that we are interested in a sequence $\{\xi_1, \xi_2, \ldots, \xi_n, \ldots\}$ of i.i.d. random variables distributed according to the law described by an unknown density function and we are given a random sample (a sequence of observed values or realizations) $\{\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(n)}\}$.

The well known Rosenblatt–Parzen method of construction of kernel density estimator $f_n = f_n(x)$ is described by the formula

$$f_n(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{b_n}K\left(\frac{x-\xi^{(i)}}{b_n}\right),$$

for univariate density function $f = f(x)$, and by

$$f_n(x,y) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{b_n^2}K\left(\frac{x-\xi^{(i)}}{b_n}\right)K\left(\frac{y-\zeta^{(i)}}{b_n}\right)$$

in the case of approximation of $f = f(x,y)$ by $f_n = f_n(x,y)$ on the basis of a random sample $\{(\xi^{(1)},\zeta^{(1)}),\ldots,(\xi^{(n)},\zeta^{(n)})\}$.

The problem of optimal selection of the bandwith parameter b_n was discussed by several authors [see, e.g., Hall and Marron (1991) and the references therein].

We believe that this method provides a general, powerful tool for construction of approximate densities of random variates of different kinds. It can also be successfully applied to multidimensional α-stable random samples. There are, however, other methods applicable when other characteristics of such vectors are known. For example, Byczkowski, Nolan and Rajput (1991) developed a method of construction of multidimensional stable densities which is based on numerical integrations of approximate characteristic functions defined by spectral measures having finite number of atoms.

6. APPROXIMATION AND SIMULATION OF α-STABLE INTEGRALS AND STOCHASTIC DIFFERENTIAL EQUATIONS

Numerical methods of approximate solution of the stochastic differential equation (1.1), involving an Itô integral with respect to Brownian motion, have existed for some time [see, e.g., Yamada (1976), Pardoux and Talay (1985) or Kloeden and Platen (1992)]. Up to now these methods focused on such problems as mean-square approximation, pathwise approximation or approximation of expectations of the solution, etc.

Our aim has been to adapt some of these constructive computer techniques, based on discretization of the time parameter t, to the case of equation (3.3) or (1.2). So, looking for an approximation of the process $\{X(t);\ t\in[0,T]\}$ solving such equations, we have to approximate them by a time discretized *explicit scheme* of the form

$$X_{t_i}^{\tau} = \mathcal{F}\left(X_{t_{i-1}}, \Delta L_{\alpha,i}^{\tau}\right), \tag{6.1}$$

where the set $\{t_i = i\tau,\quad i = 0,1,\ldots,I\},\quad \tau = T/I$, describes a fixed mesh on the interval $[0,T]$, $\Delta L_{\alpha,i}^{\tau}$ denotes the stochastic stable measure of the interval $[t_{i-1},t_i)$, that is, an α-stable random variable defined by

$$\Delta L_{\alpha,i}^{\tau} = L_\alpha([t_{i-1},t_i)) \sim S_\alpha(\tau^{1/\alpha},0,0), \tag{6.2}$$

and where $\mathcal{F}$ stands for an appropriate operator defining the method.

Our idea consists of representing the discrete-time process $\{X_{t_i}\}$, solving this discrete system and approximating the solution of equation (3.3) by an appropriate sequence of random samples $\{X_{t_i}(n)\}_{n=1}^N$ calculated with the use of a computer generator of stable random variables. In this way we can obtain kernel estimators of densities of the discrete-time diffusions solving equation (6.1).

Surprisingly, not much has been done in the field of approximation of diffusions driven by stable stochastic measures. The only reference to be found is Janicki, Podgórski and Weron (1993). There are some new, rather sophisticated theoretical results concerning stability theorems for stochastic equations with jumps [see, e.g., Kasahara and Yamada (1991)]. It is possible to derive from them some results on convergence of the approximate method defined in (6.1). In Section 8 we present some rather elementary results on convergence of approximate discrete numerical and statistical methods, related to examples presented in this paper (Sections 7 and 9).

Now we are going to describe briefly a constructive computer method providing approximate solutions to stochastic differential equations involving integrals with respect to stable Lévy motion, introduced in Section 3. An algorithm based on the Euler method consists of the following:

- With a fixed regular mesh on $[0,T]$ approximate the process $\{X(t);\ t\in[0,T]\}$ which solves equation (3.3) by a discrete time process $\{X_{t_i}^{\tau}\}_{i=0}^{I}$ defined by
  1. $X_0^{\tau} = X_0 \sim S_\alpha(\sigma,0,\mu)$;
  2. for $i = 1,2,\ldots,I$,

$$X_{t_i}^{\tau} = X_{t_{i-1}}^{\tau} + Y_{t_i}^{\tau}, \tag{6.3}$$

$$Y_{t_i}^{\tau} = a(t_{i-1},X(t_{i-1}))\,\tau + c(t_{i-1})\Delta L_{\alpha,i}^{\tau}, \tag{6.4}$$

  where $\Delta L_{\alpha,i}^{\tau}$ is defined by (6.2);
  3. the sequence $\{\Delta L_{\alpha,i}^{\tau};\ i = 1,\ldots,I\}$ being a given i.i.d. sequence.
- In order to obtain an appropriate sequence of random samples $\{X_{t_i}(n)\}_{n=1}^N$ it is enough to

replace random variables X_0^τ, $\Delta L_{\alpha,i}^\tau$, $X_{t_i}^\tau$ and $Y_{t_i}^\tau$ above by random samples $\{X_0^\tau(n)\}_{n=1}^N$, $\{\Delta L_{\alpha,i}^\tau(n)\}_{n=1}^N$, $\{X_i^\tau(n)\}_{n=1}^N$ and $\{Y_i^\tau(n)\}_{n=1}^N$, respectively, for $i = 1, 2, \ldots, I$.

An approximate solution to the equation (1.2) can be obtained replacing (6.4) by

$$(6.5)\quad Y_{t_i}^\tau = a(t_{i-1}, X(t_{i-1}))\,\tau + c(t_{i-1}, X(t_{i-1}))\Delta L_{\alpha,i}^\tau.$$

7. COMPUTER VISUALIZATION OF PROCESSES DRIVEN BY α-STABLE MEASURES

In order to obtain a graphical representation of the process $\{X(t);\ t \in [0,T]\}$ defined by (3.3), we propose the following:

1. Fix a rectangle $[0,T]\times[c,d]$, that should include trajectories of $\{X(t)\}$;
2. For each $n = 1, 2, \ldots, n_{\max}$ (with fixed $n_{\max} \ll N$) draw line segments determined by the points $(t_{i-1}, X_{i-1}^\tau(n))$ and $(t_i, X_i^\tau(n))$ for $i = 1, 2, \ldots, I$, constructing $n_{\max}$ approximate trajectories of the process X (thin lines on all figures);
3. Construct kernel density estimators $f_i = f_i^{I,N} = f_i^{I,N}(x)$ of the densities of $X(t_i)$, using for example the optimal version of the Rosenblatt–Parzen method, and their distribution functions $F_i = F_i^{I,N} = F_i^{I,N}(x)$;
4. Fix a few values of a "probability parameter" p_j from $(0, 1/2)$ for $j = 1, 2, \ldots, J$ and for each of them compute two quantiles: $q_{\min}^{i,j} = F_i^{-1}(p_j)$ and $q_{\max}^{i,j} = F_i^{-1}(1-p_j)$ for all $i = 1, 2, \ldots, I$; then draw line segments determined by the points $(t_{i-1}, q_{\min}^{i-1,j})$, $(t_i, q_{\min}^{i,j})$ and $(t_{i-1}, q_{\max}^{i-1,j})$, $(t_i, q_{\max}^{i,j})$ for $i = 1, 2, \ldots, I$ and $j = 1, 2, \ldots, J$, constructing J (varying in time) intervals (thick lines on all figures), that determine subdomains of $\mathcal{R}^2$ to which the trajectories of the approximated process should belong with probabilities $1 - 2p_j$ at any fixed moment of time $t = t_i$.

REMARK 7.1. On the computer screen an interval $[0,T]$ is represented by a few hundred pixels, that is, computer screen "points" (working with VGA graphics card one has exactly 640 pixels). Computer experiments with Brownian motion like processes $\{X(t)\}$ proved that, when simulating them on a finite interval $[0,T]$ with the mesh consisting of 1000, 10,000, 100,000 subintervals (some number of time steps is performed within 1 pixel size), the pictures of approximate trajectories look very much alike.

In our computational experiments, we have found that:

- The most significant errors are induced by approximation of any $X(t_i)$ by a simulated sample $\{X_i^\tau(n)\}_{n=1}^N$ and the only way to improve the situation is to enlarge N making sure to apply computer generators of random numbers with large periods;
- The simplest numerical method of an approximation of stochastic differential equations (the Euler method) performs quite well:
  - it does not bother us that we have to carry out a lot of iterations in time, when we want to obtain a nice graphical representation of curves of any kind ("smooth" quantile lines or "jumping" trajectories),
  - it is the most convenient and effective method to handle stochastic measures more general than those defined by Brownian motion;
- Simulation techniques are incomparable with other techniques (such as, for example, construction of approximate solutions of appropriate Fokker–Planck equations) in situations when we have to manage with long-tail probabilities and when the starting value of an approximated process has a discrete distribution;
- Statistical smoothing techniques providing density estimators perform well, but the problem is to find the best possible value of the bandwidth parameter b_n in practical calculations;
- It is possible to apply statistical methods of estimating parameters α and σ changing in time [see Feuerverger and McDunnough (1981) or DuMouchel (1983)], although our computer experiments did not provide satisfactory results;
- Thanks to some efficient computer algorithms it is possible to make usable quite complicated explicit formulas describing linear and nonlinear regression functions of α-stable random variables [see Hardin, Samorodnitsky and Taqqu (1991)].

7.1 What is to be Learned from Graphical Presentations of Stochastic Processes?

It is clear that in some situations, when we try to construct a stochastic process or to build up an appropriate statistical model, an analytical approach is not usable in practice. In such situations computer simulation techniques may be of particular interest. A combination of suitable statistical estimation techniques, computer simulation procedures and numerical discretization methods seems to be a powerful method at least at a level of preliminary verification of conjectured models. Applying the "scientific visualization" (i.e., computer graphics providing indispensable quantitative information) we try to follow Tufte's advice:

> The computer world has provided us this big set of tools ... I think the way you learn to use the tools is not by contemplating tools but by trying to solve substantive problems. [LePage (1991)]

8. CONVERGENCE OF COMPUTER APPROXIMATIONS

Let us discuss briefly some results on convergence of approximate discrete numerical and statistical methods related to examples in Section 9.

The theory of convergence of discrete time approximations of stochastic integrals and diffusion processes driven by Brownian motion has existed for some time [see e.g. Rootzén (1980), Pardoux and Talay (1985) or Kloeden and Platen (1992)].

The rate of the mean-square convergence of the Euler scheme in the case of equation (1.1) provides the following result.

THEOREM 8.1. *If coefficient functions $a = a(t,x)$ and $c = c(t,x)$ in (1.1) are for all $t \in [0,T]$ uniformly Lipschitz continuous with respect to x, then*

$$E\,|X(T) - X^{\tau}(T)|^2 = O(\tau).$$

This corresponds to the example (9.3).

REMARK 8.1. It is well known that for an α-stable stochastic integral defined by (3.1) the following statement is true.

The sequence $\{I(f^{(n)})\}_{n=1}^{\infty}$ converges to $I(f)$ in probability, if and only if, $f^{(n)} \to f$ in $\mathbf{L}^{\alpha}(m)$, when $n \to \infty$.

This yields the convergence in probability, when $\tau \to 0$, of the process $\{X^{\tau}(t)\}$ defined by

$$X^{\tau}(t) = X^{\tau}_{t_{i-1}} \quad \text{for } t \in [t_{i-1}, t_i) \qquad \text{with } i = 1,2,\ldots,I,$$

and by (6.3) – (6.4) (with $a(t,x) \equiv 0$) to the process $\{X(t)\}$ defined by

$$X(t) = \int_0^t c(s)\, dL_{\alpha}(s) \qquad \text{for all } t \in [0,T].$$

In the case of a linear stochastic equation (3.2) we can provide two results on convergence of the approximate method described in Section 6 related to examples (9.1) and (9.2).

THEOREM 8.2. *The family $\{X^{\tau}(t);\ t \in [0,T]\}$ of approximate solutions of the stochastic equation (3.2) with coefficient functions $a = a(t)$, $b = b(t)$ and $c = c(t)$ continuous on $[0,T]$, converges uniformly in probability to the exact solution $\{X(t);\ t \in [0,T]\}$ of (3.2) on $[0,T]$, when $\tau \to 0$.*

The proof is contained in Janicki, Podgórski and Weron (1993).

Combining some methods yielding convergence of numerical schemes approximating stochastic differential equations presented above with some techniques developed in the theory of nonparametric statistical estimation [see, e.g., Györfi et al. (1989)] one can prove several properties, for example, the following one.

THEOREM 8.3. *Let $f(x,T)$ be the nondegenerate density of the solution $\{X(t)\}$ of the equation (3.2) at $t = T$ and let $f_I^{I,N}(x)$ be its kernel density estimator. Then*

$$\lim_{I,N\to\infty} \int |\, f_I^{I,N}(x) - f(x,T)\,|\, dx = 0 \quad \text{in probability.}$$

At present we do not know any reference to works concerning the rate of convergence of numerical approximations of α-stable integrals or differential equations, the convergence of pathwise approximations, etc. It is possible, however, to derive some convergence theorems from the general results concerning stability theorems for stochastic differential equations with jumps; see, e.g., Kasahara and Yamada (1991) and Janicki, Michna and Weron (1993).

9. EXAMPLES OF STOCHASTIC DIFFERENTIAL EQUATIONS

In this section we provide a few examples of diffusion processes constructed and visualized with the use of our method described in previous sections.

9.1 Visualization of an α-stable Lévy Motion

We find it interesting to illustrate the difference between Brownian and Lévy motions (e.g., the effect of jumps of trajectories) and to demonstrate graphically how α-stable Lévy motion depends on the parameter α.

In order to obtain stable Lévy motion, it is enough to notice that

$$(9.1) \qquad L_{\alpha}(t) = \int_0^t dL_{\alpha}(s), \qquad t \in [0,T].$$

Here on Figures 9.1 through 9.3 we present three examples of this integral for $T = 4.0$. Each of them contains two pairs of quantile lines defined by $p_1 = 0.1$ and $p_2 = 0.25$ and 10 trajectories. In all cases we have chosen I=2000, N=2000. Observe that scaling of the vertical axis is changing with α.

9.2 Test of Stationarity

In order to demonstrate the usefulness of computer graphics, we propose a series of figures al-

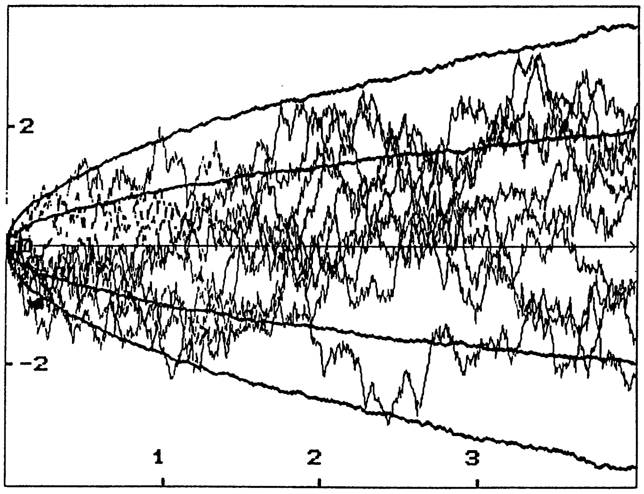

FIG. 9.1. *Stable Lévy motion for* $\alpha = 2.0$.

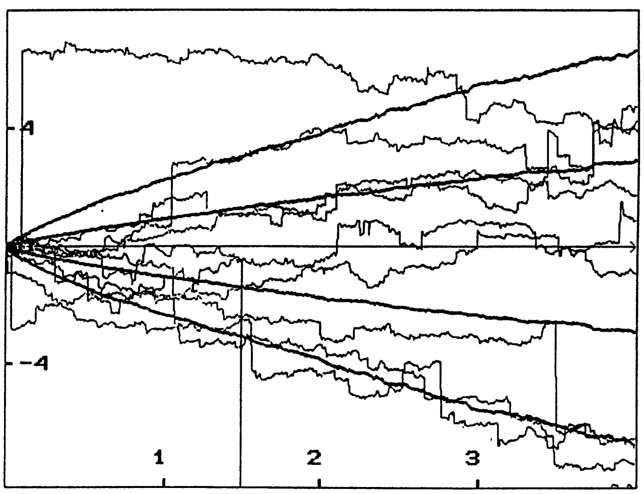

FIG. 9.2. *Stable Lévy motion for* $\alpha = 1.3$.

lowing us to check whether a given Ornstein–Uhlenbeck process is stationary or not.

Let us start with the observation that the Ornstein–Uhlenbeck process $\{X(t);\ t \geq 0\}$, defined by

$$X(t) = e^{-\lambda t} X(0) + \mu \int_0^t e^{-\lambda(t-s)}\, dB(s), \qquad \lambda > 0,\ \mu > 0,$$

with a fixed $X(0) = X_0$ and with a given Brownian motion $\{B(t);\ t \geq 0\}$, is a diffusion process, that is, it can be considered a solution to the equation (1.1) with drift and dispersion coefficients defined as $a(s,X(s)) = -\lambda X(s)$ and $c(s,X(s)) \equiv \mu^2$.

Figures 9.4 through 9.6 show four approximate trajectories of the Ornstein–Uhlenbeck processes $\{X(t);\ t \in [0,1]\}$ with $X(0) \sim \mathcal{N}(0,1)$ for three different values of λ: $\lambda = 1$, $\lambda = 2$ and $\lambda = 4$ with the same value of $\mu = 2$. In all cases the trajectories

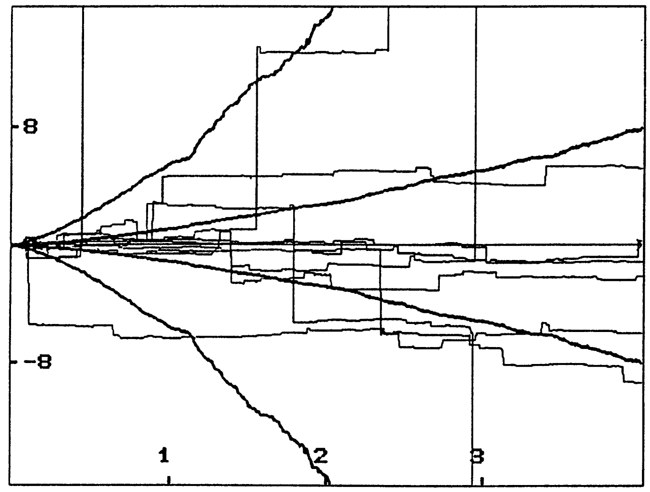

FIG. 9.3. *Stable Lévy motion for* $\alpha = 0.7$.

are included in the same rectangle $(t, X(t)) \in [0,1] \times [-2,2]$. As in all figures in this section the trajectories are represented by thin lines. The two pairs of quantile lines defined by $p_1 = 0.1$ and $p_2 = 0.25$ show that only for the case $\lambda = 2$ and $\mu = 2$ (Figure 9.5) they are "parallel." This means that the quantile lines are time invariant, demonstrating the stationarity of the corresponding Ornstein–Uhlenbeck process (only in this case we have $\mathrm{Var}\, X_0 = 1 = \frac{\mu^2}{2\lambda}$). Moreover, the field of directions of the ordinary differential equation $dx = -\lambda x(t)\, dt$ inserted in these figures helps to estimate the proper value of the parameter λ assuring stationarity of the process (in Figure 9.4 it is too small; in Figure 9.6 it is too large).

This observation helps to estimate this parameter in more complicated situations, for example, when we work with α-stable Ornstein–Uhlenbeck processes, assuming μ and X_0 to be fixed.

9.3 Diffusions with Jumps

For some time there has been a growing interest in various applications of stochastic processes with discontinuous trajectories. Such processes appear in stochastic models defined by the stochastic differential equations with jumps. As an example we recall, after Sørensen (1991), an equation which models the dynamics of a population that grows logistically between disasters. It can be expressed as a nonlinear stochastic differential equation of the form

$$dX(t-) = \varrho X(t-)[K - X(t-)]dt - X(t-)dZ(t) + \vartheta X(t-)dB(t),$$

where $\{Z(t)\}$ is given Poisson compound process and where ϱ, K and ϑ are some fixed parameters.

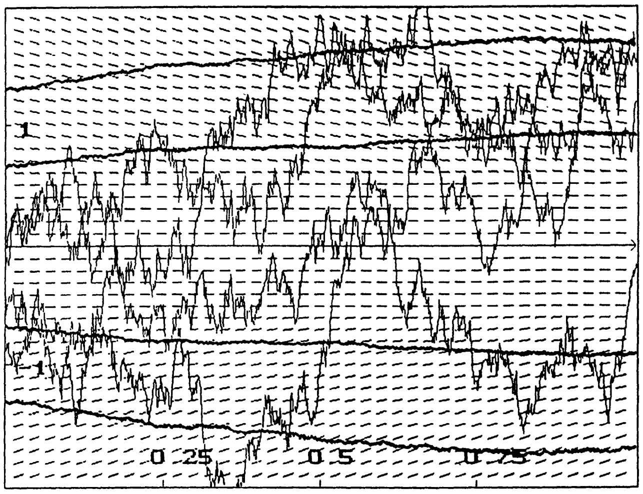

FIG. 9.4. *Case of* $\lambda = 1$, $\mu = 2$.

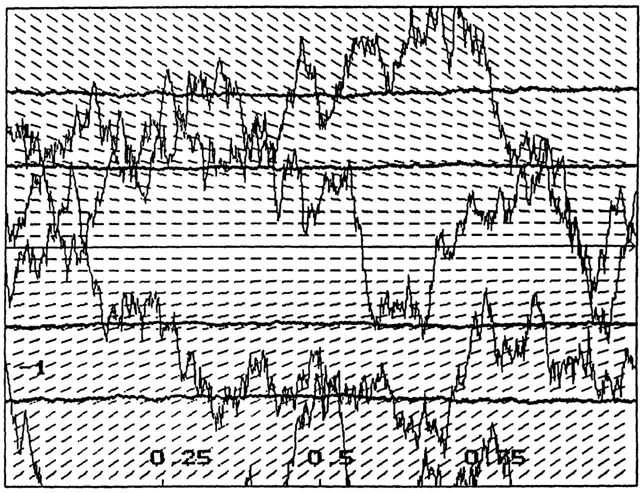

FIG. 9.5. *Case of* $\lambda = 2$, $\mu = 2$.

A vast subclass of diffusion processes with jumps can be constructed with the use of stochastic integrals with α-stable integrators of different kinds (including subordinators, etc.). For example, it could be of interest to consider the logistic model of a population growth of the form

$$dX(t-) = \rho X(t-)[K - X(t-)]dt - \theta X(t-)dM_\alpha(t),$$

where $\{M_\alpha\}$ denotes an appropriately constructed α-stable stochastic measure.

9.4 Resistive-inductive Electrical Circuit

Here we present an example of a linear stable stochastic equation involving this kind of stochastic measure [see West and Seshadri (1982)] that has a nice physical interpretation emphasizing the role of the parameter α.

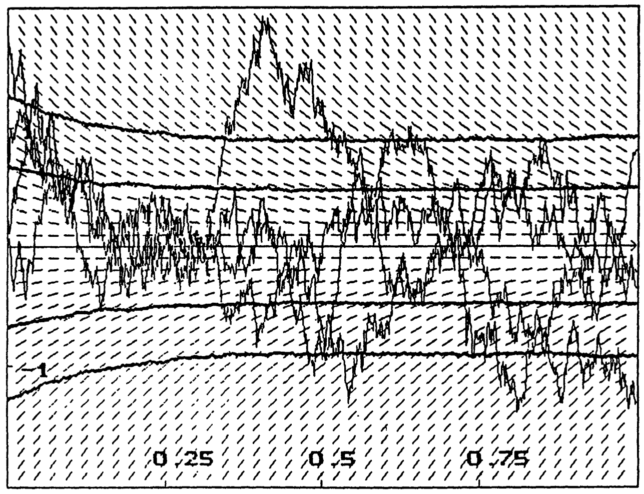

FIG. 9.6. *Case of* $\lambda = 4, \mu = 2$.

The deterministic part of the stochastic differential equation

$$(9.2) \qquad dX(t) = (4\sin(t) - X(t))dt + \frac{1}{2}dL_\alpha(t)$$

can be interpreted as a particular case of an ordinary differential equation

$$\frac{di}{ds} + \frac{R}{L}i = \frac{E}{L}\sin(\gamma s),$$

which describes the resistive-inductive electrical circuit, where i, R, L, E and γ denote, respectively, electric force, resistance, induction, electric power and pulsation. [Similar examples can be found in Gardner (1986).] In order to obtain a realistic model, it is enough to choose for example $R = 2.5[k\Omega]$, $L = 0.005[H]$, $E = 10[V]$, $\gamma = 500[1/s]$ and to rescale real time s using the relation $t = \gamma s$.

Results of computer simulation and visualization of the equation (9.2) with the initial random variable $X(0)$ chosen as an α-stable variable from $S_\alpha(2,0,1)$ for $t \in [0,4]$ and three different values of the parameter $\alpha \in \{2.0, 1.3, 0.7\}$ are included in the following two series of figures. The first series of figures shows the behavior of trajectories in the same way and with the same values of technical parameters as before in the case of α-stable Lévy motion. They contain also a field of directions corresponding to a deterministic part of the equation (9.2), that is, the equation

$$\frac{dx}{dt}(t) = -x(t) + 4\sin(t).$$

This helps us to figure out how the drift acts "against" the diffusion, when t tends to infinity. The second series shows density estimators of $X(4.0)$ for these three values of α.

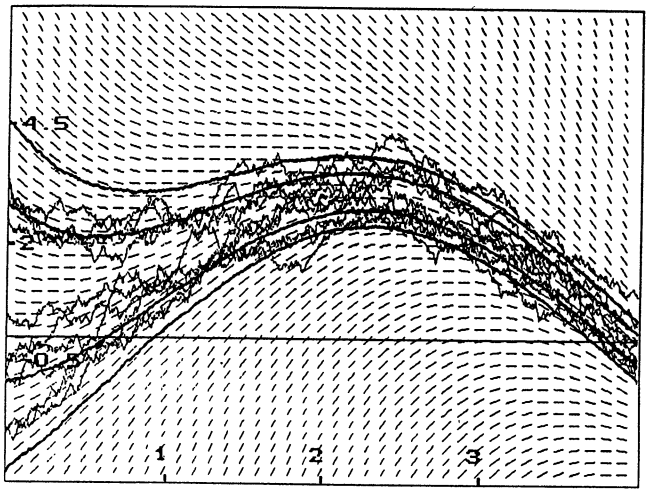

FIG. 9.7. *Resistive-inductive electrical circuit driven by Lévy motion for* $\alpha = 2.0$.

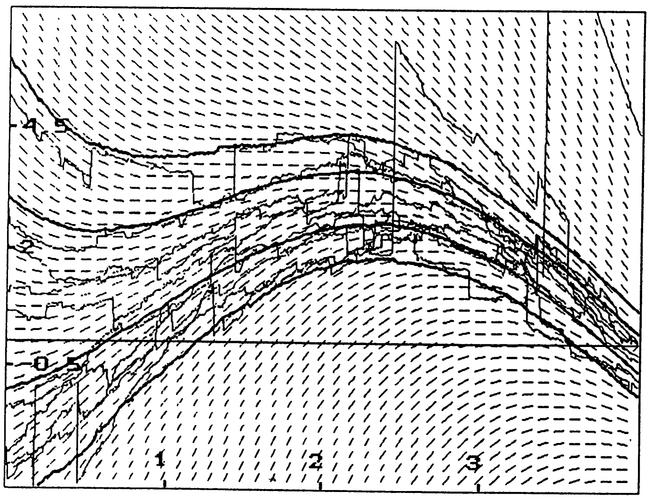

FIG. 9.8. *Resistive-inductive electrical circuit driven by Lévy motion for* $\alpha = 1.3$.

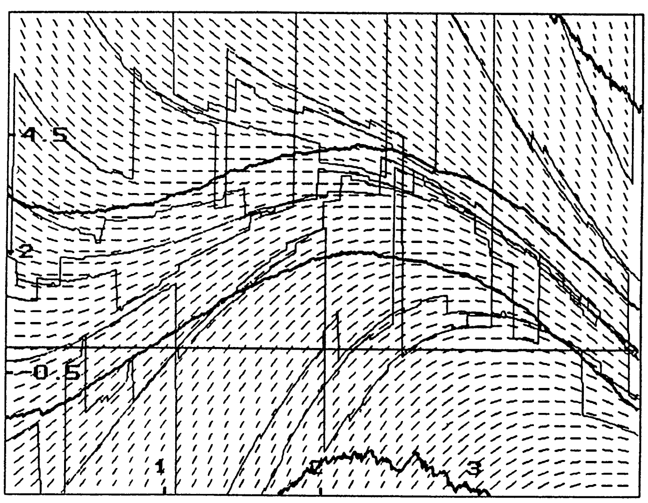

FIG. 9.9. *Resistive-inductive electrical circuit driven by Lévy motion for* $\alpha = 0.7$.

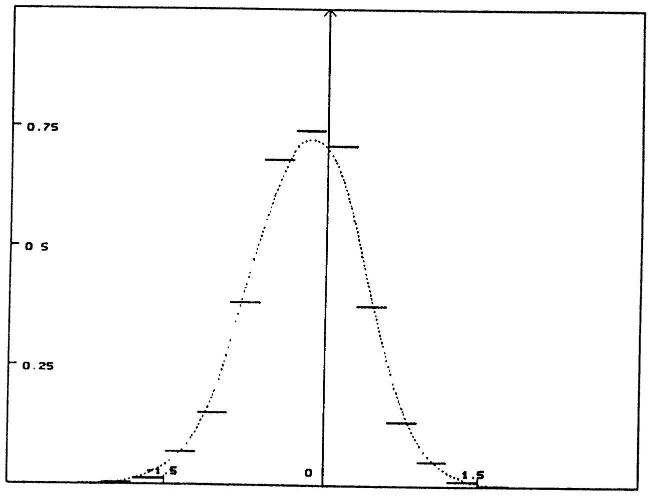

FIG. 9.10. *Density of resistive-inductive electrical circuit driven by Lévy motion for* $\alpha = 2.0$ *at time* $t = 4.0$.

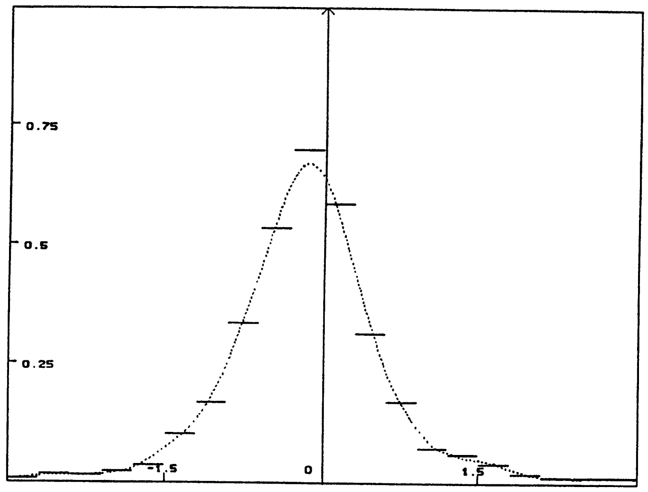

FIG. 9.11. *Density of resistive-inductive electrical circuit driven by Lévy motion for* $\alpha = 1.3$ *at time* $t = 4.0$.

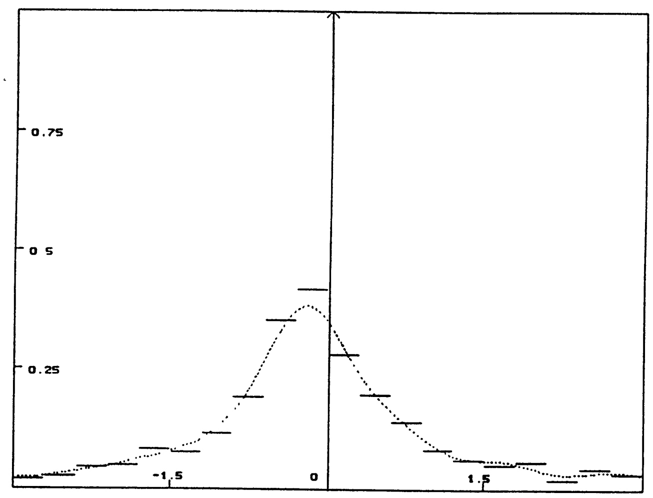

FIG. 9.12. *Density of resistive-inductive electrical circuit driven by Lévy motion for* $\alpha = 0.7$ *at time* $t = 4.0$.

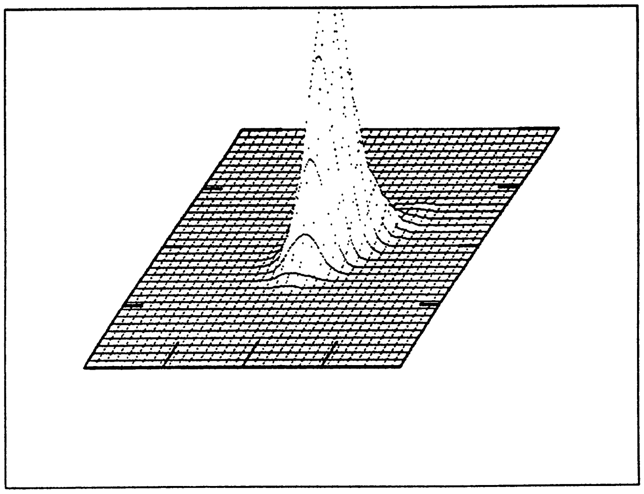

FIG. 9.13. *Density estimator of* $(X(2), Y(2))$.

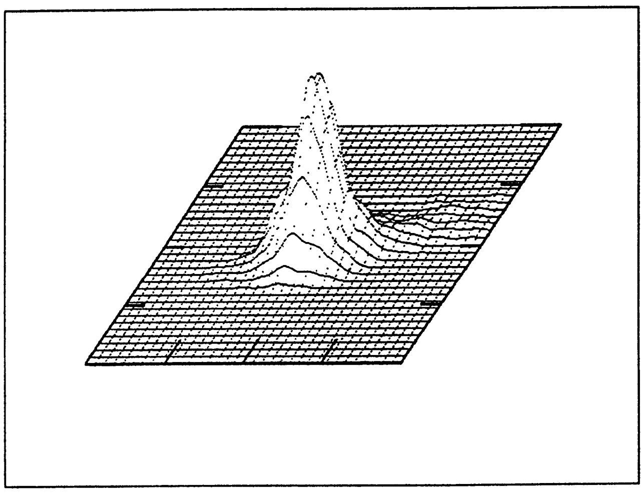

FIG. 9.14. *Density estimator of* $(X(4), Y(4))$.

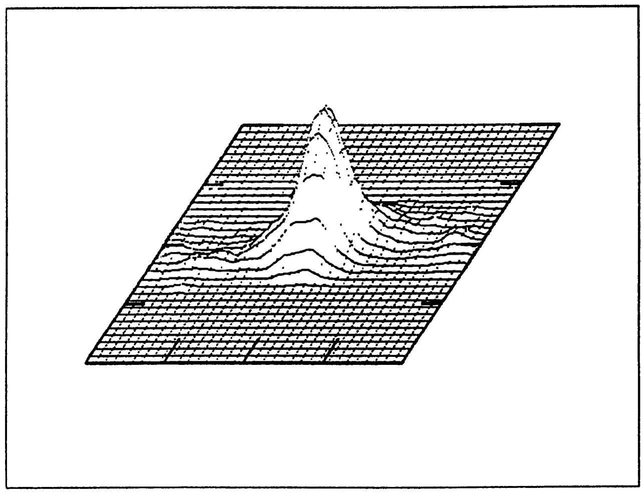

FIG. 9.15. *Density estimator of* $(X(8), Y(8))$.

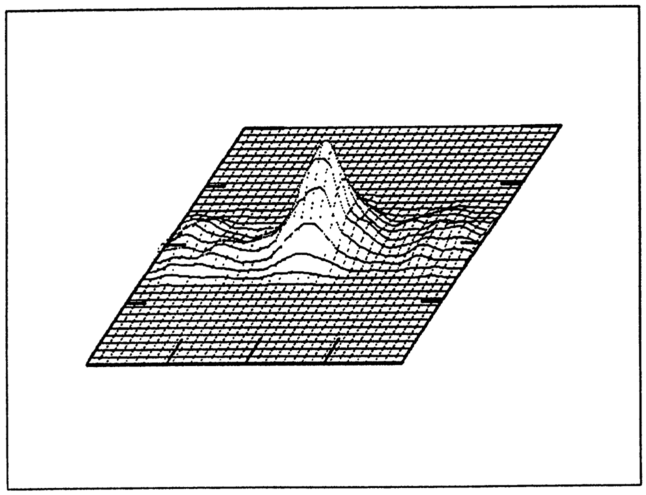

FIG. 9.16. *Density estimator of* $(X(16), Y(16))$.

9.5 Nonlinear Harmonic Oscillator

There are a lot of well known models described by means of stochastic differential equations driven by Brownian motion [see, e.g., Gardiner (1983) or Gardner (1986)]. They still play a very important role in applications.

As an example of a nonlinear stochastic physical model submitted to random external forces described by α-stable "colored noise," we considered a system of two equations describing a harmonic oscillator. We look for a solution $\{(X(t), Y(t)); t \in [0, 16]\}$ of the following system of stochastic differential equations:

$$
(9.3)\quad \begin{aligned} dX(t) &= Y(t)dt, \\ dY(t) &= \{-\sin(X(t)) - \tfrac{1}{2}Y(t)\}dt + dL_\alpha(t), \end{aligned}
$$

where $X(0) = 0$ a.s. and $Y(0) = 1$ a.s.

Notice that, of course, $X(t)$ and $Y(t)$ are not independent and the joint distributions of $(X(t), Y(t))$ are not α-stable. The dependence of solutions of (9.3) on the parameter α is similar to that discussed in connection with the previous example, see Figures 9.7 through 9.12. Therefore, we restrict ourselves to presentation of the Gaussian case only.

This example demonstrates that a wide class of nonlinear multidimensional problems can be successfully solved with the use of computer simulation and visualization techniques when analytical calculations are inaccessible.

In Figures 9.13 through 9.16, we present the evolution in time of joint densities of $(X(t), Y(t))$, solving the system of equations (9.3). They contain estimators of these densities for $t \in \{2.0, 4.0, 8.0, 16.0\}$. All of them were obtained with the use of the same technical parameters (e.g., defining kernel function) and their graphs are included in the same part of $\mathcal{R}^3$; their domains are cut to the rectangle $[-8,8]\times[-8,8]$. It is impossible to construct the density of $(X(0), Y(0))$, which is of the form of a product of two Dirac delta functions: $\delta(0) \times \delta(1)$, or even for values of t close to 0, so the first value for which we decided to present a density estimator in the chosen frame in $\mathcal{R}^3$ was $t = 2$. To give an idea of scaling of the vertical axis let us mention that the maximum of the density for $t = 16$ is about 0.090.

With the use of the computer methods presented in this paper, it is possible to construct successfully approximate solutions of much more complicated and more general problems than (9.1), (9.2) and (9.3), chosen here as expository examples. For more sophisticated problems consult Janicki and Weron (1993).

10. CONCLUSIONS

The powerful tools of stochastic calculus are finding their way into many branches of applied probability and statistics, enabling analysis of more complicated models than could be handled earlier. In particular, understanding stable distributions and processes has become increasingly important. The Gaussian model, long used in the past to describe many random phenomena because of its versatibility and mathematical simplicity, is not universally applicable. One reason for this is that the Gaussian model does not allow for the large random fluctuations found in many important phenomena in science and engineering. A number of such phenomena (gravitational fields of stars, transport of electric charges in amorphic materials, dielectric relaxation, etc.) have been discussed in Section 4.

Looking ahead, while a great deal still remains to be done to improve our understanding of stable models, we tried to convince the reader that in spite of the lack of practically useful analytical methods it is possible to demonstrate different properties of α-stable random variables and processes with the use of computer methods (see Section 9). We hope that the illustrative examples given in this paper show how the application of proper statistical methods, computer simulation techniques and computer graphics provides interesting quantitative and visual information on those features of α-stable models which distinguish them from their commonly used Gaussian counterparts. We also believe that the included figures will permit the reader to find his (or her) own answer to the title question.

This article was written as an open invitation to the wonderful world of α-stable variables and processes and was influenced by the opinion that computers provide powerful tools for construction of solutions to modern statistical and stochastic models of real life phenomena [see, e.g., Lehmann (1990) or Bickel and Le Cam (1990)].

ACKNOWLEDGMENTS

We would like to thank Professor Robert E. Kass and the referees for comments that helped us to improve the original version of this paper.

This research was supported in part by KBN Grant 2 1153 91 01 and by NSF Grant INT 92-20285.

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