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CHAPTER 17

Conformal transformation method for irrigation Dirichlet problem, *by F. Ndiaye, B.M. Ndiaye, M. Ndiaye, D. Seck and I. Ly*

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Abstract. In this paper, an identification method of a real connected set $\Sigma \subset \mathbb{R}^2$ is presented in the case of irrigation Dirichlet problem. We use conformal mapping tools in Complex Analysis to approximate Σ, the unknown. Some numerical tests are given as illustrations.

Keywords. Inverse problems; conformal mapping; shape optimization; optimal mass transport; conformal mappings.
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1. Introduction

This paper deals with an optimization problem that consists in recognizing an optimal connected region Σ included in a domain $\Omega \subset \mathbb{R}^2$. Namely, we consider the irrigation Dirichlet problem that is formulated as follows:

Let Ω be a given subset of $\mathbb{R}^2$, is it possible to find a subset $\Sigma \subset \Omega$ such that

$$\int_\Omega dist(x, \Sigma) \text{ be minimal ? } dist(x, \Sigma)$$

means the distance from the position $x \in \mathbb{R}^2$ to the set Σ.

This problem and other ones in the same type have been extensively studied in the last fifteen years. Among them, one can quote works due to Buttazzo (2002), Buttazzo (2003), Buttazzo (2009), Bernot (2005), Bernot (2009), Xia (2013).

The approach used to get interesting and relevant results by these latter is optimal mass transport theory. The irrigation Dirichlet problem can be seen as a shape optimization one where Σ is to be looked for in a topological space under the constraint on the Hausdorff measure of Σ.

It is worth mentioning that the irrigation Dirichlet problem finds interpretation in a variety of areas. In as much as we know in Economy, Urban Network and Biology (see Buttazzo (2002), Buttazzo (2003), Buttazzo (2009), Bernot (2005), Bernot (2009), Xia (2013)) and references therein for more details.

In this work, we endeavor to use finite geometrical measure to approximate the set Σ whose existence has been proven under weak hypotheses. Nevertheless we will remind one of the main existence theorem in this paper.

In Ndiaye (2014), the authors proposed in the case of overdetermined eigenvalue problem, a recognition method of an unknown boundary of a domain using conformal mappings.

The idea of using conformal mappings tools is not new, see for instance Kress (2004), Kress (2002), Nehari (1952), Laura (2003), Dambrine (2007). But its use in inverse and geometrical shape problems remains of interest.

What comes next consists in section 2, section 3 and end with a list of references. Section 2 is devoted to the setting and to remind the existence of minimum result. Section 3 aims are to show our approximation approach of Σ and numerical examples.

2. Preliminary

Let us begin by the position of the problem followed by a few hints on the existence theorem of the optimal set.

2.1. Position of problem. . Given $\Omega \subset \mathbb{R}^2$, our aim is to determine an optimal irrigation region $\Sigma \subset \Omega$ to be recognized, subject to some constraints.

The problem to address will then be to find an univalent conformal mapping $\phi : D_1 \to \Omega$ where D_1 stands for the two dimensional unit disc, minimizing the following integral

$$J(\Sigma) = \int_{\overline{\Omega}} dist(x, \Sigma) dx \tag{2.1}$$

be minimal? Otherwise:

Does it exist $\Sigma \in \Theta$, $\Theta = \left\{\Sigma \subset \Omega \subset \mathbb{R}^2 \ \dim(\Sigma) = 1 \, \mathcal{H}^1(\Sigma) \leq l, \, l > 0 \text{ fixed}\right\}$ such that

$$\min_{\Sigma \in \Theta} \int_{\overline{\Omega}} dist(x, \Sigma) dx \tag{2.2}$$

makes sense, where $\mathcal{H}^1$ stands for the one dimensional Hausdorff measure and

$$dist(x, \Sigma) := \inf_{y \in \Sigma} d(x, y).$$

It is important to underline that the natural constraints on Σ are:

(1) The Hausdorff measure of Σ, $\mathcal{H}^1(\Sigma)$ is given and finite;
(2) Σ varies in class of closed sets of $\overline{\Omega}$, with finite number of connected components.

It is clear that if one of the above constraints is removed then minimum is trivially zero. In the sequel we shall consider the problem (2.2) with length constraint non-zero.

2.2. Existence of optimal sets. Let be l fixed positive real number and Ω a bounded subset of $\mathbb{R}^2$ with Lipschitzian boundary. Let us consider the following optimization problem

$$\min\{J(\Sigma) : \Sigma \subset \overline{\Omega} \text{ connected, closed, } \mathcal{H}^1(\Sigma) \leq l\} \tag{2.3}$$

where J the functional defined in (2.1), then we have the following existence theorem

THEOREM 46. *The problem (2.3) admits a solution*

Proof of Theorem 2.3. See Buttazzo (2003), Buttazzo (2002). □

3. Identification of the domain by conformal mapping method

In this section, we shall consider several situations where the optimal solution Σ_{opt} of problem (2.3) satisfies some topological and geometrical configurations. In each case we shall aim to propose an approximation method relying on conformal mapping tools. In fact we shall consider the following four cases for Σ_{opt} :

- the closed curve,
- the not closed curve,
- the two branches case .
- it is a fork with several branches.

Our method is to determine transformations of multi-connected domains. So, we will be in front of two problems :

(1) first, how to proceed in order to get a continuous conformal mapping and that keeps connectivity order of a domain? One idea is to look for introducing a canonical domain for each order of connectivity;
(2) second, classical Riemann theorem can not be applied in a multi-connected domain.

To overcome these difficulties, we define for each case a canonical domain that determines the geometrical shape of the multi-connected domain.

3.1. Case where Σ is closed. Let us denote by the Γ_1 unit circle and Γ_R the circle centered at the same origin with radius R.

By Riemann's theorem the unit disk D_1 can be transformed into Ω. So, $\partial\Omega$ is the conformal representation of Γ_1 and by continuity Σ is the conformal

representation of Γ_R, with $(0 < R < \dfrac{l}{2\pi} < 1)$.
This type of conformal mapping is called a canonical one and it solves some extremum problems in the class of univalent functions.

According to Nehari (1952), these properties of extremum make it possible to prove existence of the conformal mapping and this from to compactness of the family of univalent functions.

Denoting Φ this conformal mapping and suppose that Φ transforms the unit disk D_1 into Ω; Γ_1 into $\partial\Omega$ and Γ_R into Σ. Now the main goal is to see how to proceed in order to have a good approximation of Φ univalent such that:

$$\left\{ \begin{array}{l} \Phi(\Gamma_1) = \partial\Omega \\ \\ \Phi(\Gamma_R) = \Sigma \end{array} \right. \tag{3.1}$$

After transformation it is easy to see that problem (2.2) is equivalent to:

$$\min \int_{\Omega} \inf_{y\in\Sigma} ||x-y||^2 dx = \min \int_0^{\frac{l}{2\pi}} (\int_0^{2\pi} ||\Phi(\rho e^{i\theta}) - \Phi(Re^{i\theta})||^2 d\theta)\rho d\rho, \tag{3.2}$$

with

$$(0 < R < 1 \,\text{and}\, R < \rho < \frac{l}{2\pi}).$$

Let us introduce the following notation

$$J(\Phi) = \int_0^{\frac{l}{2\pi}} (\int_0^{2\pi} ||\Phi(\rho e^{i\theta}) - \Phi(Re^{i\theta})||^2 d\theta)\rho d\rho. \tag{3.3}$$

satisfying $\Phi'(z) \neq 0$ for $z \in D_1$. Let

$$\Phi(z) = P(x,y) + iQ(x,y).$$

By putting

$$x = r\cos\theta \text{ and } y = r\sin\theta,$$

we have

$$\Phi(z) = P(r,\theta) + iQ(r,\theta),$$

with P and Q being the real and imaginary parts respectively of Φ. We have

$$\begin{aligned} &z \in \Gamma_1 \Leftrightarrow \Phi(e^{i\theta}) \in \partial\Omega \text{ et} \\ &z \in \Gamma_R \Leftrightarrow \Phi(Re^{i\theta}) \in \Sigma. \end{aligned} \tag{3.4}$$

The problem (2.1) is then equivalent to:

find Φ^* an univalent conformal mapping on $\overline{D}_1$ and $0 < R < 1$, such that

$$\begin{cases} \Phi^*(\Gamma_1) &= \partial\Omega \\ \Phi^*(\Gamma_R) &= \Sigma \\ J(\Phi^*) &= \min \int_R^{\frac{l}{2\pi}} (\int_0^{2\pi} ||\Phi(\rho e^{i\theta}) - \Phi(Re^{i\theta})||^2 d\theta)\rho d\rho \end{cases} \tag{3.5}$$

Actually, from the origin of the problem, one can deduce that $J(\Phi^*) = 0$.

Since it is not easy to solve the problem, numerically, we introduce an approached optimization problem. Let $\mathbf{T}$, be the set of all conformal mappings defined on $\overline{D}_1$ such that (3.1) is satisfied.

Any $\Phi \in \mathbf{T}$ is expressed as follows:

$$\Phi(z) = \sum_{n=0}^{+\infty} a_n z^n$$

where $a_n = \alpha_n + i\beta_n$ with $\alpha_n, \beta_n \in \mathbb{R}$ and $z = re^{i\theta}$ with $0 < r \leq 1$. In polar coordinates, we have

$$P(r,\theta) := \Re(\Phi(z)) = \alpha_0 + \sum_{n=1}^{+\infty} r^n(\alpha_n \cos n\theta - \beta_n \sin n\theta)$$

and Cauchy conditions imply

$$Q(r,\theta) := \Im(\Phi(z)) = \beta_0 + \sum_{n=0}^{+\infty} r^n(\alpha_n \sin n\theta + \beta_n \cos n\theta).$$

One can suppose that $\beta_0 = 0$ without loosing generality in the reasoning.

3.1.1. *Minimization of the approximate functional.* Since we want to set up a numerical algorithm; we go back to the finite dimension with the introduction of the space of trigonometric polynomials (see Ndiaye (2014)). Let

$$\mathbf{T}_N := \left\{\Phi_N \text{ defined on } \overline{D}_1 \ / \ \Phi_N(z) = \sum_{n=0}^{+N} a_n^N z^n \ , \ a_n^N \in \mathbb{C}\right\},\ N \in \mathbb{N}.$$

$\mathbf{T}_N$ is a subset of $\mathbf{T}$; it can be represented also as a subspace of $\mathbb{R}^{2N+1}$. Let $\Phi_N(z) = P_N(r,\theta) + iQ_N(r,\theta)$, where

$$J(\Phi_N) = \int_0^{\frac{l}{2\pi}} (\int_0^{2\pi} ||\Phi_N(\rho e^{i\theta}) - \Phi_N(Re^{i\theta})||^2 d\theta)\rho d\rho. \tag{3.6}$$

The minimization of J on $\mathbf{T}_N$ is equivalent to minimize the functional J on $\mathbb{R}^{2N+1}\times]0,1[$. Then the functional J is defined by

$$\begin{aligned} J : \mathbb{R}^{2N+1}\times]0,1[&\to \mathbb{R} \\ \overrightarrow{X} &\mapsto J(\Phi_N) := J(\overrightarrow{X}) \end{aligned} \tag{3.7}$$

where $\overrightarrow{X} = (\alpha_0, ..., \alpha_N, \beta_1, ..., \beta_N, R)$. The function Φ_N is obtained as soon as the coordinates of the vector $\overrightarrow{X}$ are known. We have the following result.

Proposition 5. *The functional $J(\overrightarrow{X})$ admits at least a local minimum in $\mathbb{R}^{2N+1}\times]0,1[$, moreover $J'(\overrightarrow{X}) = 0$.*

proof of Proposition 5: It derives from the classical optimization theory. Let us assume that $\Phi_N(z)$ is a solution of the problem

$$\min_{\overrightarrow{X}\in\mathbb{R}^{2N+1}\times]0;1[} J(\overrightarrow{X}).$$

Now, the objective is to show that the sequence $(\Phi_N)_N$ converges to Φ^*, where Φ^* is solution to the problem $\min\limits_{\Phi\in\mathbf{T}} J(\Phi)$, or converges at least to Φ_0 such that $J'(\Phi_0) = 0$. That is a solution to the minimization problem on $\mathbf{T}$.

3.1.2. *Convergence results.* Before presenting some propositions let us recall we have the existence of a conformal mapping Φ^* such that $\Phi^*(\Gamma_1) = \partial\Omega, \Phi^*(\Gamma_R) = \Sigma$ and then $J(\Phi^*) = 0$.

Proposition 6. *Let $\Phi \in \boldsymbol{T}$. We assume that:*

- *Φ^* is of class $\mathcal{C}^2$ and and a diffeomorphism, defined from $\overline{D}_1$ to $\overline{\Omega}$.*
- *for $n > 0$ $a_n = o(\frac{1}{n^3})$ where a_n is the nth coefficient of power expansion of Φ^*.*

Then

Φ^N converges uniformly to Φ^* on $\overline{D}_1$ where Φ^N is the truncated function of Φ^* : that is, $\Phi^N(z) = \sum\limits_0^N a_n^* z^n$ and $\Phi^*(z) = \sum\limits_0^{+\infty} a_n^* z^n$.

Proof of Proposition 6 (see Seck (1996)).

Proposition 7. *$J(\Phi^N)$ converges to $J(\Phi^*)$ when N goes to $+\infty$ and*

$$\lim_{N\to+\infty} J(\Phi_N) = 0$$

.

Proof of Proposition 7. We have

$$J(\Phi^N) = \int_0^{\frac{l}{2\pi}} (\int_0^{2\pi} ||\Phi^N(\rho e^{i\theta}) - \Phi^N(Re^{i\theta})||^2 d\theta)\rho d\rho.$$

On Γ_1, (Φ^N) converges uniformly to Φ^* when N goes to $+\infty$. Since $\Re\Phi$ and $\Im\Phi$ are continuous functions, $\Re\Phi^N$ and $\Im\Phi^N$ which respectively converge uniformly to $\Re\Phi^*$ and $\Im\Phi^*$ when N go to $+\infty$ and $[0, 2\pi], [0, \frac{l}{2\pi}]$ are compact sets, we have:

$$\lim_{N\to+\infty} J(\Phi^N) = \int_0^{\frac{l}{2\pi}} (\int_0^{2\pi} ||\Phi^N(\rho e^{i\theta}) - \Phi^N(Re^{i\theta})||^2 d\theta)\rho d\rho \tag{3.8}$$

$$= \int_0^{\frac{l}{2\pi}} (\int_0^{2\pi} ||\Phi^*(\rho e^{i\theta}) - \Phi^*(Re^{i\theta})||^2 d\theta)\rho d\rho = J(\Phi^*) \tag{3.9}$$

So $J(\Phi^N)$ converges to $J(\Phi^*)$. Since $\Phi^N \in \mathbf{T}_N$, then we have $J(\Phi^N) \geq J(\overrightarrow{X}) = J(\Phi_N)$. But

$$\lim_{N\to+\infty} J(\Phi^N) = J(\Phi^*) = 0.$$

Thus we get

$$\lim_{N\to+\infty} J(\Phi_N) = 0.$$

THEOREM 47. (1) *There exists a subsequence* $(\Phi_{N_k})_{N_k}$ *of* $(\Phi_N)_N$ *and* Φ_0 *such that* Φ_{N_k} *converges uniformly to* Φ_0 *on all compact of* C_R.

(2) $\exists\ \Phi_1 \in L^2(\Gamma_1)$ *such that* Φ_{N_k} *converges weakly to* Φ_1 *in* $H^1(\Gamma_1)$*;*

(3)

$$\Phi_0(z) = \frac{1}{2i\pi}\int_{\Gamma_1} \frac{\Phi_1(\xi)}{(\xi - z)}d\xi \ \ \forall z \in D_1. \tag{3.10}$$

Then Φ_0 *can be developed as a integer power series on* D_1

$$\Phi_0(z) = \sum_0^{+\infty} c_n z^n, \tag{3.11}$$

with

$$c_n = \frac{1}{2i\pi}(\int_{\Gamma_1} \frac{\Phi_1(\xi)}{\xi^{n+1}}d\xi,\ n \in \mathbb{N}. \tag{3.12}$$

Proof of theorem See Theorem 47(also Seck (1996)).

COROLLARY 17.

$$c_n = (\Phi_1, e^{in\theta})_{L^2(\Gamma_1)\times L^2(\Gamma_1)} \ \text{ if } \theta \in [0, 2\pi] \tag{3.13}$$

where $(,)$ *is the scalar product in* L^2*,*

$$(\Phi_1, e^{in\theta})_{L^2(\Gamma_1)\times L^2(\Gamma_1)} = (\Phi_0, e^{in\theta})_{L^2(\Gamma_1)\times L^2(\Gamma_1)} \ \text{ si } \theta \in [0, 2\pi],$$

$\Phi_0(e^{i\theta})$ *is defined almost everywhere in* $[0, 2\pi]$.

And for almost all $\theta \in [0, 2\pi]$*, we have*

$$\Phi_0(e^{i\theta}) = \sum_0^{+\infty} a_n e^{in\theta}. \tag{3.14}$$

Proof of Corollary 17 (see Seck (1996); Ndiaye (2014))

Finally, we are in a position to propose a numerical algorithm in two dimension to approximate Σ. The algorithm is derived from the minimization of the functional J defined in (3.7) on $\mathbb{R}^{2N+1}\times]0,1[$.

As soon $\overrightarrow{X} = (\alpha_0, ..., \alpha_N, \beta_1, ..., \beta_N, R) \in \mathbb{R}^{2N+1}\times]0,1[$, is known the trigonometric polynomial Φ_N will be. So we are going to solve numerically the following optimization problem:

$$\min_{0<R<1, R<\rho<\frac{l}{2\pi}} \int_0^{\frac{l}{2\pi}} \int_0^{2\pi} \Big\{ (\sum_{n=0}^{N} (\alpha_n \cos n\theta - \beta_n \sin n\theta)(\rho^n - R^n))^2 + (\sum_{n=0}^{N} (\alpha_n \sin n\theta + \beta_n \cos n\theta)(\rho^n - R^n))^2 \Big\} d\theta \rho d\rho \tag{3.15}$$

that is equivalent to:

(3.16)
$$\min \int_0^{\frac{l}{2\pi}} \int_0^{2\pi} \Big\{ \sum_{n=0}^{N} (\alpha_n^2 + \beta_n^2)(\rho^n - R^n)^2 + 2 \sum_{i,j=0; i\neq j}^{N} \Big\{ (\alpha_i\alpha_j + \beta_i\beta_j)\cos(i-j)\theta + (-\alpha_i\beta_j + \alpha_j\beta_i)\sin(j-i)\theta \Big\} (\rho^i - R^i)(\rho^j - R^j) \Big\} d\theta \rho d\rho.$$

with $0 < R < 1, R < \rho < \dfrac{l}{2\pi}$. The second term in the above integral vanishes. So the problem becomes:

$$\min_{0<R<1, 0<\rho<\frac{l}{2\pi}} \int_0^{\frac{l}{2\pi}} 2\pi \sum_{n=1}^{N} (\alpha_n^2 + \beta_n^2)(\rho^n - R^n)^2 \rho d\rho. \tag{3.17}$$

But it is difficult to solve explicitly this above problem with $2N+2$ unknown variables. To overcome this issue, one looks for properties of polynomial $P(\rho) = 2\pi \sum_{n=1}^{N} (\alpha_n^2 + \beta_n^2)(\rho^n - R^n)^2 \rho$ defined on $\left[0, \dfrac{l}{2\pi}\right]$. There are a finite number of $\rho_i, i \in I, |I| < \infty$ where P reaches minimum because P is not constant.

And among these we are interested by the biggest one that we note ρ_0.

In this study, we solve all the problems with Matlab version R2017a , on a 64-bit Xeon processor 3.2GHz processor running Linux on Ubuntu 16.04. The simulations results for the closed curve, for $N = 5$ points, are summarized in table 1. The approximate shapes of $\partial\Omega$ and Σ are shown in figure 1.

α_n	β_n	R
-8.5020735e-05	-1.3405927e-05	5.3277319e-01
9.2134725e-06	1.6641021e-05	
-6.7391023e-06	-9.5126354e-06	
-1.5003233e-06	-7.8307508e-07	
2.2893080e-06	8.2571621e-06	

TABLE 1. RESULTS FOR THE CLOSED CURVE. results for the closed curve

3.2. Case where Σ is not a closed curve. Let us assume that Φ transforms Γ_1 into $\partial\Omega$ and the set $[-1; 0] \times \{0\}$, denoted Γ_2 is transformed into Σ. So the main goal now is to find the injective conformal mapping Φ such that

$$\begin{cases} \Phi(\Gamma_1) = \partial\Omega \\ \Phi(\Gamma_2) = \Sigma. \end{cases} \tag{3.18}$$

So problem (2.2) is equivalent to:

$$\min \int_\Omega \inf_{y\in\Sigma} ||x-y||^2 d\sigma = \min \int_{-1}^{0} \Big(\int_0^{2\pi} ||\Phi(e^{i\theta}) - \Phi(R)||^2 d\theta \Big) R dR, \tag{3.19}$$

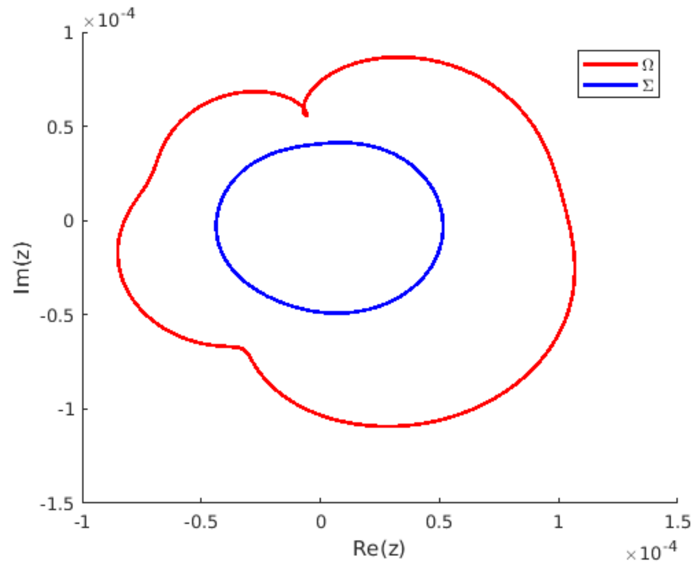


FIGURE 1.CASE CLOSED . Case closed

Let $C_* := \{x \in \mathbb{R}^2,\ \ x \in D_1 \setminus \Gamma_2\}$ and

$$J(\Phi) = \int_{-1}^{0} (\int_{0}^{2\pi} ||\Phi(e^{i\theta}) - \Phi(R)||^2 d\theta) R dR \tag{3.20}$$

Since C_* is multi-connected then, according to Nehari (1952), there exists a unique univalent, analytical function Φ with $\Phi' \neq 0$, such that problem (3.19) has a solution. We use the same method as in the case where Σ is closed curve to get Φ.

Let us set **W** the set of univalent functions Φ on $\overline{C}_*$, with $\Phi'(z) \neq 0$, such that the problem (3.19) has a solution.

Let $\Phi(z) = u(x,y) + iv(x,y)$, $x = r\cos\theta$ and $y = r\sin\theta$,

$$\Phi(z) = u(r,\theta) + iv(r,\theta),$$

where u and v being real and imaginary parts of Φ. We have

$$\begin{aligned} &z \in \Gamma_1 \Leftrightarrow \Phi(e^{i\theta}) \in \partial\Omega \\ &z \in \Gamma_2 \Leftrightarrow \Phi(R) \in \Sigma, \quad R \in]-1;0]. \end{aligned} \tag{3.21}$$

The problem (3.19) is equivalent to the finding of an univalent conformal mapping Φ^* conformal mapping on $\overline{C}_*$ such that

$$\text{(3.22)} \qquad \begin{cases} \Phi^*(\Gamma_1) & = & \partial\Omega \\ \Phi^*(\Gamma_2) & = & \Sigma \\ J(\Phi^*) & = & \min \int_{-1}^{0}(\int_0^{2\pi} ||\Phi(e^{i\theta}) - \Phi(R)||^2 d\theta)RdR. \end{cases}$$

According to Nehari (1952), there exists $\Phi \in \mathbf{W}$ unique solution of problem (3.22). Let us suppose that $\Phi(z)$ is written as follows

$$\text{(3.23)} \qquad \Phi(z) = \Phi(z, R) = \begin{cases} \sum_{-\infty}^{+\infty} a_n(z-R)^n \ \text{ with } \ z \in C_* \ \text{ and } \ R \in [0;1[\\ \Phi(R) = R \end{cases}$$

where $a_n = \alpha_n + i\beta_n$ with $\alpha_n, \beta_n \in \mathbb{R}$ and $z = re^{i\theta}$.
Factors that go to 0 and ∞ more slowly than any power of $z - R$ may appear in the main term. Hence the conformal mapping does not have this form. According to (Lavrentiev (1972), p. 173), $\Phi(z)$ is written as:

$$\text{(3.24)} \qquad \begin{cases} \Phi(z) = z\ln(\dfrac{1}{-z}) \ \text{ if } \ z \neq 0 \\ \Phi(0) = 0 \end{cases}$$

where the branch of the logarithm is characterized by the condition $0 \leq \arg z \leq \pi$. It is easy to see that for R small enough, it transforms:

(1) the segment $[R, 0] \times \{0\}$ of the radial line $[-1; 0] \times \{0\}$ on the curve arc Σ defined by:

$$u(0,0) = v(0,0) = 0 \ \ on \ \ O$$

and

$$\text{(3.25)} \qquad \begin{cases} u(R,\theta) = R\ln(\dfrac{1}{-R}) \\ v(R,\theta) = 0 \ \text{ for } \ -1 < R < 0 \ \text{ and } \ \theta = 0. \end{cases}$$

(2) unit circle $|z| = 1$ on $\partial\Omega$ that is a curve arc.

According to the correspondence principle of boundaries, for small R, the function defined in (3.24) transforms in a univalent and conformal way the unit circle into $\partial\Omega$ and $]-1, 0]$ on Σ. The curve arc is connected to segment

$$[R\ln(\frac{1}{-R}), 0] \times \{0\}, \quad -1 < R < 0,$$

at point $(0, 0)$ smoothly. Finally, we can deduce from this process an approximation of the shape and the topology of Σ.

The approximation shape of Σ is obtained as follows : The segment $]-1, 0]\times \{0\}$ is transformed into

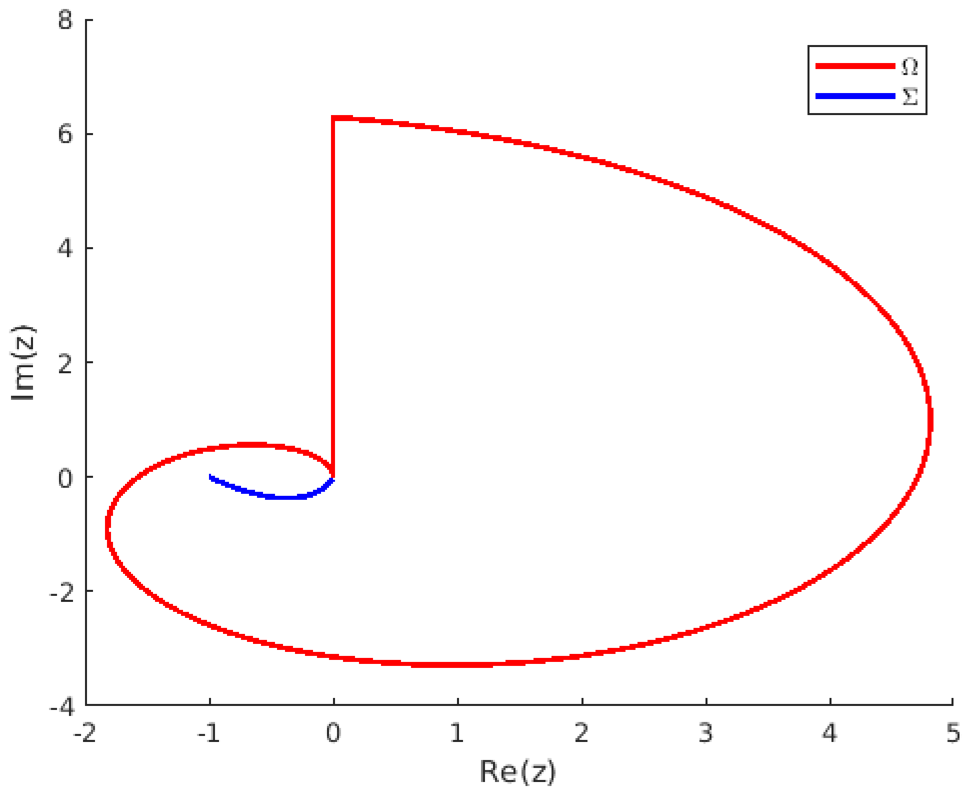
$$\Sigma = \Big\{(u, v); \ (u, v) \in \mathbb{R}^2/u = R\ln(\frac{1}{-R}) \text{ and } v = 0 \text{ with } -1 < R < 0\Big\}. \tag{3.26}$$

And the unit circle is transformed into $\partial\Omega$, defined for $\theta \in]-\pi, \pi[$, by all points that have for affix $Z = X(\theta) + iY(\theta)$, with:

$$\begin{cases} X(\theta) = (\pi + \theta)\sin\theta \\ \\ Y(\theta) = -(\pi + \theta)\cos\theta. \end{cases} \tag{3.27}$$

So the approximated shapes of $\partial\Omega$ and Σ are shown in figure 2. The Σ in blue and and $\partial\Omega$ in red, are represented for values $-1 < R < 0$ and $\theta \in]-\pi, \pi[$.

FIGURE 2.CASE NO CLOSED . Case no closed



3.3. Case where Σ is a set with two branches. In this part we consider that Σ is the union of Σ_1 and Σ_2 $(\Sigma = \Sigma_1 \cup \Sigma_2)$.

Let $\Gamma_3 =]-1,0] \times \{0\}$ and $\Gamma_4 = \{-\frac{1}{2}\}\times]0, \frac{\sqrt{3}}{2}[$ such that $\Gamma_2 = \Gamma_3 \cup \Gamma_4$.

Let us assume that Φ transforms Γ_1 $\partial\Omega$, Γ_3 into Σ_1 and Γ_4 into Σ_2 so that Γ_2 is transformed into Σ. We have:

$$\left\{ \begin{array}{l} \Phi(\Gamma_1) = \partial\Omega \\ \\ \Phi(\Gamma_2) = \Sigma \end{array} \right. \tag{3.28}$$

The problem (2.2) is equivalent to:

$$\min \int_{\Omega} \inf_{y\in\Sigma} ||x-y||^2 dx = \min \int_{\Omega} \inf \left(\inf_{y\in\Sigma_1} ||x-y||^2 dx, \inf_{y\in\Sigma_2} ||x-y||^2 dx \right) \tag{3.29}$$

which is also equivalent to:

$$\inf \Big\{ \min_{\substack{-1<\rho<0,\\ t\in]\epsilon,0[,\\ \alpha\in]-\pi,\pi[}} \int_{\rho}^{t} \int_{-\alpha}^{\alpha} (R\ln(\frac{1}{-R}) - (\pi+\theta)\sin\theta)^2 + ((\pi+\theta)\cos\theta)^2 d\theta R dR,$$
$$\min_{\substack{0<\lambda<1-\delta\\ \beta\in]-\pi,\pi[}} \int_{0}^{\lambda} \int_{-\beta}^{\beta} ||\Phi(e^{i\theta}) - \Phi(-\frac{1}{2}+i\frac{\sqrt{3}}{2}t)||^2 d\theta t dt \Big\} \tag{3.30}$$

where δ is a positive real and small enough.

We have two intermediate problems before solving the problem (3.29):

$$\min_{\substack{-1<\rho<0,\\ t\in]\epsilon,0[,\\ \alpha\in]-\pi,\pi[}} \int_{\rho}^{t} \int_{-\alpha}^{\alpha} (R\ln(\frac{1}{-R}) - (\pi+\theta)\sin\theta)^2 + ((\pi+\theta)\cos\theta)^2 d\theta R dR, \tag{3.31}$$

where $\epsilon < 0$, and

$$\min_{\substack{0<\lambda<1-\delta\\ \beta\in]-\pi,\pi[}} \int_{0}^{\lambda} \left(\int_{-\beta}^{\beta} ||\Phi(e^{i\theta}) - \Phi(-\frac{1}{2}+i\frac{\sqrt{3}}{2}t)||^2 d\theta \right) t dt \tag{3.32}$$

The cases where x belong to Σ_1 or Σ_2 are trivial. So we suppose that x does not belong to Σ_1 or Σ_2. Then the problem (3.31) is already studied in the previous section. And it remains to investigate problem (3.32).

It is already known from the previous section that $\Phi(z) = z\ln(\frac{1}{-z})$ transforms the unit circle $|z| = 1$ into $\partial\Omega$ that is a curve arc whose points have affix $z' = -i(\theta+\pi)e^{i\theta}$ and the segment $]-R, 0] \times \{0\}$ into a curve arc whose the equation is $z' = R\ln(\frac{1}{-R}),\ \ -1 < R < 0.$

It is well known that for all $z \in \mathbb{C} \setminus \mathbb{R}_-$:

$$\ln z = \ln(x+iy) = \ln(\sqrt{x^2+y^2}) + 2i\arctan\frac{y}{x+\sqrt{x^2+y^2}}.$$

And then it is easy to see that it transforms the segment $\{-\dfrac{1}{2}\}\times]0,\frac{\sqrt{3}}{2}[$ into the curve arc

$$\Sigma_2=\{(-\frac{1}{2}+i\frac{\sqrt{3}}{2}t)\left(\ln\sqrt{\frac{1}{4}+\frac{3}{4}t^2}+2i\arctan\frac{\frac{\sqrt{3}}{2}t}{-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{3}{4}t^2}}\right)/t\in]\epsilon,0[\} \tag{3.33}$$

$$\Sigma_2=\left\{w,w\in\mathbb{C}/\ \ w=\Re(w)+i\Im(w)\right\}$$

where

$$\Re(w)=-\frac{1}{2}\ln\sqrt{\frac{1}{4}+\frac{3}{4}t^2}-\sqrt{3}t\arctan\frac{\frac{\sqrt{3}}{2}t}{-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{3}{4}t^2}},$$

and

$$\Im(w)=-\arctan\frac{\frac{\sqrt{3}}{2}t}{-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{3}{4}t^2}}+\frac{\sqrt{3}}{2}t\ln\sqrt{\frac{1}{4}+\frac{3}{4}t^2},\ \ t\in]0,1[.$$

So Problem (3.32) is equivalent to

$$\begin{aligned}\min_{\substack{0<\lambda<1-\delta,\\ \beta\in]-\pi,\pi[}}\int_0^{\lambda}\int_{-\beta}^{\beta}|(\theta+\pi)\sin\theta+\frac{1}{2}\ln\sqrt{\frac{1}{4}+\frac{3}{4}t^2}+\\ \sqrt{3}t\arctan\frac{\frac{\sqrt{3}}{2}t}{-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{3}{4}t^2}}+\\ i\left(-(\theta+\pi)\cos\theta+\arctan\frac{\frac{\sqrt{3}}{2}t}{-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{3}{4}t^2}}-\frac{\sqrt{3}}{2}t\ln\sqrt{\frac{1}{4}+\frac{3}{4}t^2}\right)|^2t\,d\theta\,dt\end{aligned} \tag{3.34}$$

which is the same to

$$\min_{\substack{0<\lambda<1-\delta,\\ \beta\in]-\pi,\pi[}}J(\lambda,\beta) \tag{3.35}$$

where

$$J(\lambda,\beta) = \int_0^{\lambda}\int_{-\beta}^{\beta}\left((\theta+\pi)\sin\theta + \frac{1}{2}\ln\sqrt{\frac{1}{4}+\frac{3}{4}t^2} + \sqrt{3}t\arctan\frac{\frac{\sqrt{3}}{2}t}{-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{3}{4}t^2}}\right)^2 + \left(-(\theta+\pi)\cos\theta + \arctan\frac{\frac{\sqrt{3}}{2}t}{-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{3}{4}t^2}} - \frac{\sqrt{3}}{2}t\ln\sqrt{\frac{1}{4}+\frac{3}{4}t^2}\right)^2 d\theta\, t\, dt \tag{3.36}$$

J is a continuous function on $[0,1-\delta]\times[-\pi+\delta,\pi-\delta], \delta>0$ small enough. Then there are

$$\lambda_0 \in]0,1[\text{ and } \beta_0 \in]-\pi,\pi[.$$

realizing the minimum for J.

We can thus conclude that problem (3.30) admits a solution. And a numerical study comparing the two respective minima of the problems (3.31) and (3.34) give us a approximated shape of Σ.

The simulations results for the two branches, are given in table 2, and the approximate shapes of $\partial\Omega$ and Σ are shown in figure 3.

λ	β
9.9999696e-01	-3.1415863e+00

TABLE 2. RESULTS FOR TWO BRANCHES. results for two branches

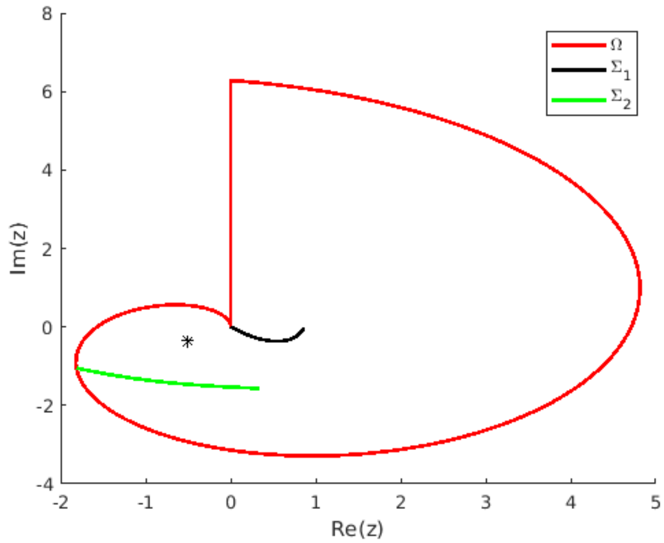


FIGURE 3. CASE OF FORK GRAPH WITH TWO BRANCHES . Case of fork graph with two branches

3.4. Case where Σ is a fork graph with several branches. In this part we consider that Σ is the union of $\Sigma^{'}$ and Σ_k with $k \in [-6;6] \cap \mathbb{Z}$.

Let' us set $\Gamma^{'} =]-1,0] \times \{0\}$ and $\Gamma_k = \{Re^{ik\frac{\pi}{12}},\ R \in]0;1[$ and $k \in [-6;6] \cap \mathbb{Z}\}$.

One supposes that Φ transforms Γ_{D_1} the boundary of D_1 into $\partial\Omega$, $\Gamma^{'}$ into $\Sigma^{'}$ and the radial lines Γ_k into branches Σ_k such that the fork graph are transformed into Σ.

We have:

$$\begin{cases} \Phi(\Gamma_{D_1}) = \partial\Omega \\ \Phi(\Gamma^{'}) = \Sigma^{'} \\ \Phi(\Gamma_k) = \Sigma_k,\ \ k \in [-6;6] \cap \mathbb{Z}. \end{cases} \tag{3.37}$$

The problem (2.2) is equivalent to:

$$\min \int_\Omega \inf_{y\in\Sigma} ||x-y||^2 dx = \min \int_\Omega \inf(\inf_{y\in\Sigma'} ||x-y||^2 dx, \inf_{y\in\Sigma_k} ||x-y||^2 dx) \tag{3.38}$$

which is equivalent also to:

$$\inf(\min \int_\rho^t \left(\int_{-\alpha}^{\alpha} \left[(R\ln(\frac{1}{-R}) - (\pi+\theta)\sin\theta)^2 + ((\pi+\theta)\cos\theta)^2 \right] d\theta \right) RdR,$$
$$\min \int_0^\lambda \left(\int_{-\beta}^{\beta} |\Phi(e^{i\theta}) - \Phi(Re^{ik\frac{\pi}{12}})|^2 d\theta \right) RdR), \ k \in [-6;6]\cap\mathbb{Z} \tag{3.39}$$

So we have two intermediate problems below, to solve:

$$\min \int_\Omega \inf_{y\in\Sigma'} ||x-y||^2 dx = \tag{3.40}$$
$$\min \int_\rho^t \left(\int_{-\alpha}^{\alpha} \left[(R\ln(\frac{1}{-R}) - (\pi+\theta)\sin\theta)^2 + ((\pi+\theta)\cos\theta)^2 \right] d\theta)RdR \right),$$

and

$$\min \int_\Omega \inf_{y\in\Sigma_k} ||x-y||^2 dx = \min \int_0^\lambda \left(\int_{-\beta}^{\beta} |\Phi(e^{i\theta}) - \Phi(Re^{ik\frac{\pi}{12}})|^2 d\theta \right) RdR, \tag{3.41}$$
$$k \in [-6;6]\cap\mathbb{Z}.$$

We assume in the sequel that x does not belong to Γ' or $\Gamma_k,\ k\in[-6;6]\cap\mathbb{Z}$.

To get more information about this minimization problem it suffices to solve separately (3.40) and (3.41). We have already seen from the previous section that $\Phi(z) = z\ln(\frac{1}{-z})$ transforms:

- the unit circle $|z| = 1$ into $\partial\Omega$ which is a curve arc,
- the segment $]R,0]\times\{0\}, -1 < R < 0$ into a curve arc defined by:

$$\Sigma' = \left\{ w, w\in\mathbb{C}/\ w = R\ln(\frac{1}{-R}),\ \ R\in]-1,0[\right\} \tag{3.42}$$

- the branches $\Gamma_k := \{Re^{i\theta_k},\ R\in]0;\lambda[\ 0<\lambda<1-\epsilon,\ \theta_k = k\frac{\pi}{12}$ and $k\in[-6;6]\cap\mathbb{Z}\}$ into Σ_k defined by:

$$\Sigma_k = \left\{ w, w\in\mathbb{C}/\ \Re w = R\cos(\theta_k)\ln(\frac{1}{R}) + R(\theta_k+\pi)\sin(\theta_k) \text{ and} \right.$$

$$\Im w = R\sin(\theta_k)\ln(\frac{1}{R}) - R(\theta_k+\pi)\cos(\theta_k)\Big\}. \tag{3.43}$$

Then, problem (3.40) is equivalent to:

$$\min \int_\rho^t \left(\int_{-\alpha}^{\alpha} \left[R^2 \ln^2(\frac{1}{-R}) - 2R(\pi+\theta)\sin\theta \ln(\frac{1}{-R}) + (\pi+\theta)^2 \right] d\theta \right) RdR, \tag{3.44}$$

and (3.41) to:

$$\min \int_0^\lambda (\int_{-\beta}^{\beta} \left[(\pi+\theta)\sin\theta + R\ln R\cos(\frac{k\pi}{12}) - R(\frac{k\pi}{12}+\pi)\sin(\frac{k\pi}{12}) \right]^2 +$$

$$\left[-(\theta+\pi)\cos\theta + R\sin(\frac{k\pi}{12})\ln R + R(\frac{k\pi}{12}+\pi)\cos(\frac{k\pi}{12}) \right]^2 d\theta)RdR. \tag{3.45}$$

with

$$0 < \lambda < 1-\epsilon, \; k \in [-6;6] \cap \mathbb{Z} \text{ and } \beta \in]-\pi,\pi[.$$

By similar arguments as in the previous case we conclude that the approximation of Σ is possible.

The simulations results for the several branches, are given in table 3, and the approximate shapes of $\partial\Omega$ and Σ are shown in figure 4.

λ	β	k
9.9999871e-01	-3.1424208e+00	6

TABLE 3. RESULTS FOR TWO BRANCHES. results for two branches

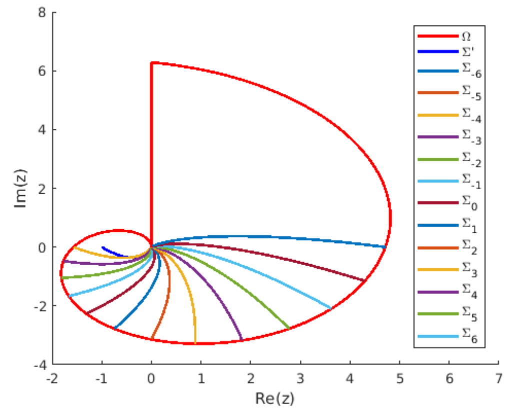


FIGURE 4.CASE OF FORK GRAPH WITH SEVERAL BRANCHES . Case of fork graph with several branches

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