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CHAPTER 5

Sine-Skewed Cardioid Distribution, *by M. Ahsanullah*

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Abstract. Azzalini (1985) introduced a family of skew symmetric distributions. His technique has been applied to skew many continuous distribution defined on the entire real axis. Very few people worked on circular or on distributions defined on finite intervals. In this paper it is considered a sine-skewed cardioid distribution generated by perturbation of symmetric Cardioid distribution. Several basic properties of the sine-skewed cardioid distribution will be presented. Based on the truncated moment some characterizations of this distribution are given.

Keywords. Skew distribution; circular distribution; cardioid distribution, symmetric distribution; characterization
AMS 2010 Mathematics Subject Classification. 62EXX; 62E10; 62E15.

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1. Introduction

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Azzalini (1985) showed that if f is a probability density function (pdf) on the real line which is symmetric about a given point, say zero (without loss of generality) and G is a cumulative distribution function (cdf) such that G' exists for all x and is a probability density which is symmetric about 0, then

$$2f(x)G(w(x)),\ x \in \mathcal{R}$$

is a skew pdf over the real line for any odd function w. Let $f(\theta)$ and $g(\theta)$ be two circular *pdf*'s both symmetric about 0 and

$$G(\theta) = \int_{-\pi}^{\theta} g(u)du,$$

then

$$f_{w(\theta)} = 2f(\theta)G(w(\theta)), \tag{1.1}$$

is a skew circular distribution if $w(\theta) = w(-\theta) = w(\theta + 2k\pi)$, $-\pi < \theta < \pi$ and k is any integer. We can consider, for $|\lambda| \leq 1$,

$$f_\lambda(\theta) = f(\theta)(1 + \lambda \sin\theta), \tag{1.2}$$

as a skew circular distribution. The circular distribution defined in Formula (1.2) is discussed by Umbach and Jammalamadaka (2009) and Abe and Pewsey (2011). Ahsanullah (2016) presented some distributional properties and characterizations of a semi circular distribution.

The pdf $f_{sc}(\theta)$ of the sine-skewed cardioid distribution is defined, for $(\lambda, \rho) \in [-1, 1]^2$, by

$$f_{sc}(\theta, \rho, \lambda) = \frac{1}{2\pi}(1 + \rho\cos\theta)(1 + \lambda\sin\theta),\ -\pi < \theta < \pi. \tag{1.3}$$

Though the cardioid distribution is unimodal but sine-skewed cardioid distribution is bimodal for some combinations of the parameters. For various properties of this distribution, see Mardia and Jupp (1999) and Fisher

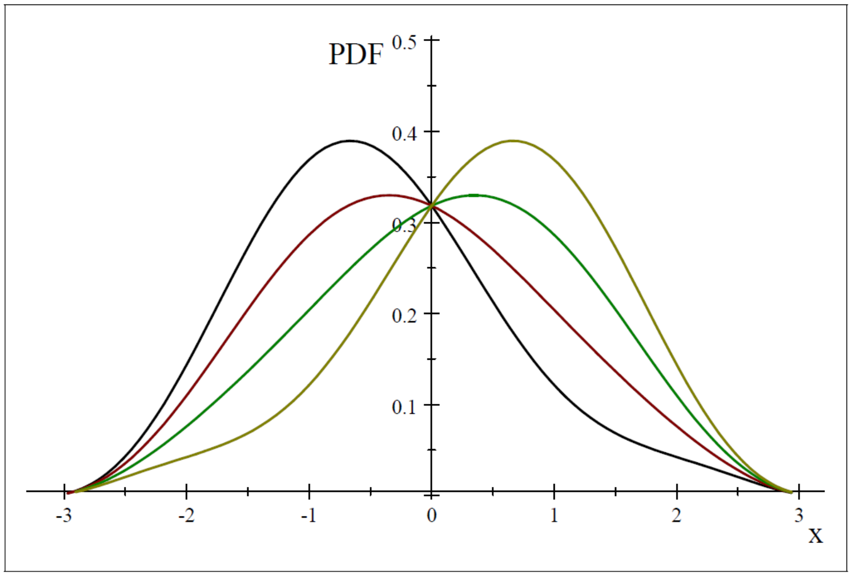

FIGURE 1. PDF of $f_{sc}(x,1,\lambda)$,Black-$\lambda=-0.6$, Red- $\lambda=-0.2$, Green-$\lambda=0.2$ and Brown-$\lambda=0.6$

(1993). In this paper some basic properties and characterizations of the sine-skewed distribution will be presented.

2. Main Results

The cdf $F_{sc}(x,\rho,\theta)$ of the sine-skewed cardioid distribution is

$$F_{sc}(\theta,\rho,\lambda)=\frac{1}{2}+\frac{1}{2\pi}(\theta-\lambda\cos\theta-\lambda+\rho\sin\theta-\frac{\lambda\rho}{4}\cos 2\theta+\frac{\lambda\rho}{4}).$$

Figure 2 gives the pdf of $f_{sc}(x,\rho,\lambda)$ for $\rho=1$ and $\lambda=-0.6,-0.2,0.2$ and 0.6. The percentage points of $F_{sc}(x,1.\lambda)$ are given for $\lambda=-0.6,-0.2,0.2$ and 0.6 in Table 1.

From the Table 1, it is evident that if x_p is the pth percentage of $f_{sc}(x,1,\lambda)$, $0<p<1$, then $-x_p$ is the percentage of $f_{sc}(x,1,-\lambda)$ for all $\lambda\geq 0$. Let $\alpha_m=E(\cos^m\theta)$ and $\beta_m=E(\sin^m\theta)$, then

p/λ	-0.6	-0.2	0.2	0.6
0.1	-1.7362	-1.6068	-1.3904	-0.9492
0.2	-1.3526	-1.1624	0.85577	-0.3922
0.3	-1.0553	-0.8140	-0.4561	-0.0286
0.4	-0.7895	-0.5007	-0.1156	0.2664
0.5	0.5223	-0.1968	0.1968	0.5327
0.6	0:2664	0.1156	0.5007	0.7895
0.7	0.0286	0.4561	0.8140	1.0553
0.8	0.3922	0.8558	1.1624	1.3526
0.9	0.9492	1.3904	1.6068	1.7362

TABLE 1. Percentage points of $F_{sc}(x, 1, \lambda)$

$$\begin{aligned}
\alpha_m &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos^m\theta(1+\rho\cos\theta)(1+\lambda\sin\theta)d\theta \\
&= \frac{1}{2\pi}2^m B(\frac{m+1}{2}, \frac{m+1}{2}) + \frac{\rho}{2\pi}2^{m+1}B(\frac{m+2}{2}, \frac{m+2}{2}) \\
&= \frac{2^{m-1}}{\pi}(B(\frac{m+1}{2}, \frac{m+1}{2}) + 2\rho B(\frac{m+2}{2}, \frac{m+2}{2}))
\end{aligned}$$

and

$$\begin{aligned}
\beta_m &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \sin^m\theta(1+\rho\cos\theta)(1+\lambda\sin\theta)d\theta \\
&= \frac{1}{2\pi}2^m B(\frac{m+1}{2}, \frac{m+1}{2}) + \frac{\rho}{2\pi}\frac{2}{m+1} + \frac{\lambda}{2\pi}2^{m+1}B(\frac{m+2}{2}, \frac{m+2}{2}) + \frac{\rho\lambda}{2\pi}\frac{2}{m+2} \\
&= \frac{1}{2\pi}(2^m B(\frac{m+1}{2}, \frac{m+1}{2}) + 2^{m+1}\lambda B(\frac{m+2}{2}, \frac{m+2}{2}) + \frac{2\rho}{m+1} + \frac{2\rho\lambda}{m+2}).
\end{aligned}$$

We will use the following two lemmas to prove the characterizations of this distribution. Towards this, let us introduce the :

Assumptions A : (i) X is an absolutely continuous random variable with cdf $F(x)$ and pdf $f(x)$. (2) $E(X)$ exists and $f(x)$ is differentiable. $\Diamond$

Define

$$\alpha = \inf\{x|f(x) > 0\} \quad and \quad \beta = \sup\{x|f(x) < 1\}.$$

LEMMA 8. *If $E(X/(X \leq x)) = g(x)\frac{f(x)}{F(x)}$, where $g(x)$ is a continuous differentiable function in (α, β), then*

$$f(x) = ce^{\int \frac{x-g'(x)}{g(x)}dx},$$

where c is determined by the condition $\int_\alpha^\beta f(x)dx = 1$.

Proof of lemma 8. we have

$$g(x) = \frac{\int_\alpha^x uf(u)du}{f(x)}.$$

Thus

$$\int_\alpha^x uf(u)du = f(x)g(x).$$

By differentiating both sides of the above equation, we obtain

$$xf(x) = f'(x)g(x) + f(x)g'(x).$$

On simplification, we get

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)}.$$

On integrating both sides of the above equation, we obtain

$$f(x) = ce^{\int \frac{x-g'(x)}{g(x)}dx},$$

where c is determined by the condition $\int_\alpha^\beta f(x)dx = 1$. □

LEMMA 9. . *Let us assume that the assumptions **A** hold. If*

$$E(X/(X \geq x)) = h(x)\frac{f(x)}{1 - F(x)}$$

where $h(x)$ is a continuous differentiable function in (α, β), then

$$f(x) = ce^{\int = \frac{x+h'(x)}{h(x)}dx},$$

where c is determined by the condition $\int_\alpha^\beta f(x)dx = 1$.

Proof of lemma 9. Let us set

$$h(x) = \frac{\int_x^\infty uf(u)du}{f(x)}.$$

Thus

$$\int_x^\infty uf(u)du = f(x)g(x)$$

By differentiating both sides of the above equation, we obtain

$$-xf(x) = f'(x)h(x) + f(x)h'(x).$$

On simplification, we get

$$\frac{f'(x)}{f(x)} = -\frac{x+h'(x)}{h(x)}.$$

On integrating both sides of the above equation, we obtain

$$f(x) = ce^{\int -\frac{x+h'(x)}{h(x)}dx},$$

where c is determined by the condition $\int_\alpha^\beta f(x)dx = 1$. □

THEOREM 13. *Suppose that X is an absolutely continuous random variable with cdf $F(x)$ with pdf $f(x)$ for $-\pi < x < \pi$. We assume that $f'(x)$ exists for all x, -$\pi < x < \pi$ and $E(X)$ exists. Then the following two propositions are equivalent.*

(a) $E(X/(X \leq x) = g(x)\tau(x)$ where $\tau(x) = \frac{f(x)}{F(x)}$,

$$g(x) = \frac{2\pi p(x)}{(1+\rho\cos x)(1+\lambda\sin x)}$$

and

$$\begin{aligned} p(x) &= \frac{\lambda}{2} + \frac{1}{16\pi}(8\lambda\sin x - 2\pi\lambda\rho\lambda\rho + \rho\lambda\sin 2x - 8x\lambda\cos x - 2x\rho\lambda\cos 2x \\ &+ 4x^2 - 4\pi^2 + 4\rho + 4\rho\cos x + 4x\rho\sin x). \end{aligned}$$

(b) For $-\pi < \theta < \pi$ *and* $0 < |\rho| \leq 1,\ 0 \leq \lambda \leq 1,$

$$f_{sc}(\theta, \rho, \lambda) = \frac{1}{2\pi}(1 + \rho\cos\theta)(1 + \lambda\sin\theta)$$

Proof of Theorem 13. Suppose that for $-\pi < \theta < \pi . 0 < |\rho| \leq 1,\ 0 \leq \lambda \leq 1,$

$$f_{sc}(\theta, \rho, \lambda) = \frac{1}{2\pi}(1 + \rho\cos\theta)(1 + \lambda\sin\theta).$$

then we have

$$\begin{aligned} f_{sc}(x)g(x) &= \int_{-\pi}^{x} \frac{u}{2\pi}(1 + \rho\cos u)(1 + \lambda\sin u)du \\ &= \frac{\lambda}{2} + \frac{1}{16\pi}\Big(8\lambda\sin x - 2\pi\lambda\rho\lambda\rho + \rho\lambda\sin 2x - 8x\lambda\cos x - 2x\rho\lambda\cos 2x \\ &+ 4x^2 - 4\pi^2 + 4\rho + 4\rho\cos x + 4x\rho\sin x\Big) =: p(x). \end{aligned}$$

We get

$$g(x) = \frac{2\pi p(x)}{(1 + \rho\cos x)(1 + \lambda\sin x)}.$$

Now suppose that

$$g(x) = \frac{2\pi p(x)}{(1 + \rho\cos x)(1 + \lambda\sin x)},$$

Thus

$$g\prime(x) = x - \frac{2\pi p(x)}{(1 + \rho\cos u)(1 + \lambda\sin u)}\left(\frac{-\rho\sin x}{1 + \rho\cos x)} + \frac{\lambda\cos x}{1 + \lambda\sin x}\right).$$

Hence,

$$\frac{x - g'(x)}{g(x)} = \frac{-\rho\sin x}{1 + \rho\cos x)} + \frac{\lambda\cos x}{1 + \lambda\sin x}.$$

By Lemma 8, we have

$$\frac{f'(x)}{f(x)} = \frac{-\rho\sin x}{1+\rho\cos x)} + \frac{\lambda\cos x}{1+\lambda\sin x}.$$

On integrating both sides of the equation with respect to x, we obtain

$$f(x) = c(1+\rho\cos x)(1+\lambda\sin x).$$

Using the condition

$$\int_{-\pi}^{\pi} f(x)dx = 1,$$

we obtain

$$f(x) = \frac{1}{2\pi}(1+\rho\cos x)(1+\lambda\sin x). \blacksquare$$

THEOREM 14. *Suppose that X is an absolutely continuous random variable with cdf $F(x)$ with pdf $f(x)$ for -$\pi < x < \pi$. We assume that $f'(x)$ exists for all x, -$\pi < x < \pi$. and $E(X)$ exists. Then the following assertions are equivalent.*

(a) $E(X/(X \leq x)) = h(x)r(x)$ *where* $r(x) = \frac{f(x)}{1-F(x)}$ *and*

$$h(x) = \frac{2\pi q(x)}{(1+\rho\cos x)(1+\lambda\sin x)}$$

with and

$$q(x) = \lambda(1-\frac{\rho}{4}) - p(x).$$

(b) For $-\pi < \theta < \pi$ *and* $0 < |\rho| \leq 1$

$$f_{sc}(\theta,\rho,\lambda) = \frac{1}{2\pi}(1+\rho\cos\theta)(1+\lambda\sin\theta),$$

Proof of Theorem 14. Suppose that we have, for $-\pi < \theta < \pi .0 < |\rho| \leq 1$ and $0 \leq \lambda \leq 1$,

$$f_{sc}(\theta, \rho, \lambda) = \frac{1}{2\pi}(1 + \rho\cos\theta)(1 + \lambda\sin\theta).$$

Then we get

$$f_{sc}(x)q(x) = E(X/(X \geq x)) = E(X) - E(X)/(X \leq x)) = \lambda(1 - \frac{\rho}{4}) - p(x).$$

Hence

$$q(x) = \frac{2\pi(\lambda(1 - \frac{\rho}{4}) - p(x))}{(1 + \rho\cos x)(1 + \lambda\sin x)}.$$

Now, suppose that

$$q(x) = \frac{2\pi(\lambda(1 - \frac{\rho}{4}) - p(x))}{(1 + \rho\cos x)(1 + \lambda\sin x)}.$$

Thus,

$$\begin{aligned} q\prime(x) &= -x - \frac{2\pi(\lambda(1 - \frac{\rho}{4}) - p(x))}{(1 + \rho\cos x)(1 + \lambda\sin x)}(\frac{-\rho\sin x}{1 + \rho\cos x)} + \frac{\lambda\cos x}{1 + \lambda\sin x}) \\ &= -x - q(x)(\frac{-\rho\sin x}{1 + \rho\cos x)} + \frac{\lambda\cos x}{1 + \lambda\sin x}). \end{aligned}$$

Now

$$\frac{x + q'(x)}{q(x)} = -(\frac{-\rho\sin x}{1 + \rho\cos x)} + \frac{\lambda\cos x}{1 + \lambda\sin x}).$$

By Lemma 9, we have

$$\frac{f'(x)}{f(x)} = \frac{-\rho\sin x}{1 + \rho\cos x)} + \frac{\lambda\cos x}{1 + \lambda\sin x}.$$

On integrating both sides of the equation with respect to x, we obtain

$$f(x) = c(1 + \rho\cos x)(1 + \lambda\sin x).$$

Using the condition

$$\int_{-\pi}^{\pi} f(x)dx = 1,$$

we obtain

$$f(x) = f_{sc}(x) = \frac{1}{2\pi}(1 + \rho\cos x)(1 + \lambda\sin x). \blacksquare$$

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