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Logic and Foundations of Mathematics in Peano's School*

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SUMMARY. It is generally held that Peano and his school played an important role at the end of the last century in the development of logic and foundations of mathematics. This paper surveys the results achieved by Peano and his school and examines the motivation behind their work, thus demonstrating the validity of their reputation.

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1. Introduction and Background. Peano and his school provide an important landmark for historians of logic and the foundations of mathematics. At the same time they pose a problem that is not easy to solve. It is generally acknowledged that Peano's school played a very important role at the end of the last century in developing a modern approach to logic and the foundations of mathematics. Yet it is by no means easy to ascertain how much their work really influenced further research in these areas. Nor is it easy to explain the rapid decline of Peano's school, a decline that began already in the first years of this century, despite very promising beginnings.

This paper will survey the results actually obtained by Peano's school and will examine the motivation behind their work. In a future paper we will explore the relation of Peano's school to later developments in logic and foundations of mathematics, and

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we will explain why they did not join new and more modern fields of foundational research.

Before proceeding, we will briefly examine the state of logical research and the foundations of mathematics prior to the time of Peano, particularly those phases of the history that are significant for our analysis.

It is well known that the logicians of the so-called algebraic tradition (Boole, De Morgan, Schröder, et al.) thought that mathematical logic was essentially an algebra of logic. For them, logic was characterized by the application of algebraic techniques to traditional logic; there was an almost total lack of applications of logic to mathematics. (This is due in part to the fact that the algebraic logicians' aims were different from those of later logicians. Boole's primary purpose, for example, was to present a mathematical system that captured the "laws of thought" generally accepted by logicians. But it is also due to the fact that the Boolean calculus is inadequate for symbolizing mathematical reasoning.) Peano and Frege, on the other hand, deemed it useful to apply logic to the study of mathematical reasoning and the foundations of mathematics. While Peano meant to reach the highest possible level of rigour, Frege had in mind the more ambitious goal of securing a firm foundation for mathematics by defining its fundamental concepts in logical terms and by proving its primitive propositions as theorems of logic.

This happened at the turn of the century, but we are still far from the goals presently attributed to mathematical logic. Following the substantial failure of the logicist attempt to reduce mathematics to logic (Frege, Whitehead and Russell) and the gradual disappearance of that trend of thought which conceived of mathematical logic as the study and rigorous justification of mathematical arguments, mathematical logic has today in principle become an "autonomous" mathematical discipline. It is increasingly detached from the foundations of mathematics conceived in traditional terms while, at the same time, it interacts with mathematics in quite a different sense, i.e., with algebraic geometry through model theory or, recently, with information science. (This does not mean, however, that it has completely lost contact with its origins in the philosophical framework; cf., e.g., [Agazzi *1986*]).

The evolution of mathematical logic along the lines just outlined makes it difficult to evaluate the contributions of Peano and his students in this field. On the one hand, we ought not to judge the production of the Peano school *a posteriori*; on the other hand, we must examine the extent to which their results influenced new research or can be deemed substantial anticipations of later developments.

In considering the historical development of the foundations of mathematics, we are bound to adopt a similar approach. The contributions of Peano's school to arithmetic and geometry, together with their remarks on the axiomatic method and their many investigations aimed at reducing to the lowest possible number the primitive concepts and axioms of mathematical theories undoubtedly are a part of "foundational" research. But here we are a long ways from the criticisms of the foundations of mathematics as intended by logicism; and we are still farther from various reactions to the crisis in foundations, with respect to which, as we shall see later, Peano's school always adopted a rather detached attitude.

2. Logic. According to the current interpretation, the first significant application of logic to mathematics is due to Peano and was obtained by means of the logical symbolism for which he is rightly famous, a symbolism which allows us to give mathematical propositions and proofs a symbolic form.

In the *[1888]* booklet *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann, preceduto dalla operazioni delle logica deduttiva*, in which Peano began his logical studies, we find an innovation that is important for the application of logic to mathematics, namely the distinction between logical and mathematical symbols. In order to avoid any ambiguity that might be caused by the use of the same symbols both in the logical context and in mathematical formulas, Schröder's logical symbols are replaced by new ones. Thus, Peano writes [*1888*; see *1957-59*, vol. II, 18]** :

> I deemed it useful to replace the signs $\cap$, $\cup$, $-A$, $\circ$, $\bullet$ with the logical symbols $\times$, $+$, A_1, 0, 1, used by Schröder, so as to prevent any possible confusion between logical and mathematical symbols (a danger which was noted by Schröder himself).

(The use of punctuation instead of brackets, which will constantly appear beginning in 1889, must be considered an aspect, although marginal, of this requirement. Peano himself wrote: "To separate the various propositions among them, we could use brackets as in algebra. We arrive at the same result, with greater simplicity and without producing equivocals with the brackets in algebraic formulas, using an opportune punctuation" (Peano [*1891b*; see *1957-59*, II, 93]).

Furthermore, we find what surely is the most important part of the work, the distinction between "categorical" and "conditional" (containing variables) propositions, and in particular, the introduction of the abstraction operator, which allows us to pass from a conditional proposition to a class. Thus, if α is a conditional proposition containing the variable x, then "$x{:}\alpha$" denotes the class of all those entities for which the proposition α is true. (This notation will be modified several times in Peano's later

** The quotations from Peano's work have been drawn, whenever possible, from the reprint of the same in *Opere Scelte* [1957-1959], edited by U. Cassina. Similarly, for Pieri's contributions, we refer to [1980].

work. It becomes "$(x\epsilon)\alpha$" in *Arithmetices principia* [*1889a*], "$\overline{x\,\epsilon}\,\alpha$" in "Démonstration de l'intégrabilité des équations différentielles ordinaires" [*1890d*], and afterwards "$x\ni\alpha$" (for instance in the third edition of the *Formulario*)). The abstraction operator can be considered a device that allows us to define logical operators on conditional propositions as well as on classes. The procedure that Peano follows is essentially the following: writing "$x{:}\alpha < x{:}\beta$ " means, according to Peano's notation, that the class of entities for which α is true is included in the class of entities for which β is true. If we agree to write "$\alpha < \beta$ " instead of "$x{:}\alpha < x{:}\beta$", we can interpret this new expression to mean "the proposition α has as its consequence the proposition β", or "if α is true, then β is also true" (notice that the symbol "$<$ " will be replaced by "$\supset$" in Peano's later writings).

In *Arithmetices principia, nova methodo exposita* [*1889a*], where we find the first version of "Peano's axioms" for arithmetic, the relation that today we call the relation of membership is introduced. Peano denotes it with "ϵ" and explicitly stresses its difference from class inclusion. (In this paper, however, Peano still confuses the two relations in the case of a class containing just one element. This is corrected the following year, in "Démonstration de l'intégrabilité des équations différentielles ordinaires" [*1890d*].) As Russell [*1903*, 31] rightly observes, "It is one of the greatest of Peano's merits to have clearly distinguished the relation of the individual to its class from the relation of inclusion between classes." Furthermore, we find a notation for universal quantification: "$a \supset_{x,y} b$" means "for every x, y, from a one can deduce [*deducitur*] b".

Peano identifies as the main aspects of his symbolism the abstraction operator and the membership relation, which give him the possibility of writing the propositions of arithmetic in symbolic form. Thus, for instance, in the review of Frege's *Grundgesetze der Arithmetik* published in *Rivista di Matematica* in [*1895b*; see *1957-59*, II, 190], he writes:

> A few years ago, considering the class determined by a condition [...], I showed that the entire logical calculus of classes was transformed into a calculus of propositions (*Calcolo geometrico*, 1888); [...] And then, it was sufficient to introduce a convention for denoting individual propositions (ϵ sign) in order to develop completely a theory in symbolic form, i.e. in *Arithmetices principia* (1889).

In 1894, in *Notations de logique mathématique* (*Introduction au Formulaire de mathématiques*), the distinction between real and apparent variables is introduced, although these terms will be used only beginning in 1897, in the second edition of the

Formulario. Peano's distinction corresponds to the one we now make between free and bound variables.

Starting in 1897, we can find a systematic use of the symbol "∃" to denote existential quantification. This symbol was used for the first time in "Studii di logica matematica," and Peano attributes its introduction to his assistants (see [*1957-59*, II, 215]):

> The proposition $a \sim= \Lambda$, where a is a class, means therefore "the a's exist." Since this relation occurs quite frequently, some collaborators deem it useful to denote it with only one notation instead of the group $\sim= \Lambda$. Those who hold this opinion can write for instance: $a \in \mathrm{K} \,.\supset :\, \exists\, a \,.=.\, a \sim= \Lambda$ Def.

(This is read as follows: if a is a class, then the a's exist if and only if a is not empty.)

While the creation of an adequate logical apparatus certainly represents one of the merits still attributed to Peano (van Heijenoort [*1967*, 84] writes, for instance: "The ease with which we read Peano's booklet today shows how much of his notation has found its way, either directly or in a somewhat modified form, into contemporary logic"), an enlargement of perspective is essential in order to understand the deepest motivations out of which the research carried out by Peano's school originated. In particular, it is useful, in our opinion, to consider the scientific production of the Peano school in a broader sense, without limiting ourselves exclusively to their contributions to logic.

Actually, Peano's first works were concerned with mathematical analysis and might, at first sight, appear to be unrelated to the work devoted to logic that followed. In our opinion, however, these early works give us some keys to reflect upon the entire Peanesque corpus, provided emphasis is given to their methodological aspects. An inspection of these early works, in fact, reveals that these non-logical contributions, although having a strictly technical character (they concern, for instance, differential equations and the integrability of functions), have in common with the succeeding work in logic the pursuit of precision in definitions and proofs, the attempt to eliminate unnecessary hypotheses, and the use of counterexamples in order to show the incorrectness of certain assertions. (In the work "Sull'integrabilità delle funzioni" of 1883 the definition of *definite integral* is introduced in terms of the least upper bound and of the greatest lower bound of the set of circumscribed and inscribed rectangular areas, and the definition of the measure (today called the Peano-Jordan measure) of a plane surface is sketched. The latter will be developed in the book *Applicazioni geometriche del calcolo infinitesimale* of 1887, and extended to the surfaces of Euclidean space in the paper "Sulla definizione dell'area d'una superficie" of 1890. The definition of the famous "Peano curve," a continuous curve which completely fills an area, passing through each point in a square (found in the paper "Sur une courbe qui remplit toute une aire plane"), dates from the same year. Clarifications and extensions of many topics in infinitesimal analysis are presented in the book *Calcolo differenziale e principii di calcolo integrale, pubblicato con aggiunte dal Dr. Giuseppe Peano* of 1884, which, although the title page lists A. Genocchi (one of Peano's teachers) as the author, was actually written entirely by Peano. Other important results concern the independence of functions ("Sur les wronskiens" of 1889 and "Sul determinante wronskiano" of 1897), the theorem, called the Peano-Schwarz theorem, on second-order mixed partial derivatives ("Sur l'intervertion des dérivations partielles" of 1890), and on a form of the remainder in Taylor's formula ("Une nouvelle forme du reste dans la formule de Taylor" of 1889). In the works "Sull'integrabilità delle equazioni differenziali del primo ordine" of 1886 and "Integrazione per serie delle equazioni differenziali lineari" of 1887, Peano proves the existence and the (local) unicity of the solution of a differential equation of the form $y' = f(x,y)$ under the hypothesis only of the continuity of f, and finds the (local) solution of a system of differential equations as the sum of a series of iterated integrals. Peano's results on differential equations and systems are extended and systematically expounded in the work "Démonstration de l'intégrabilité des équations différentielles ordinaires" of 1890.) Even though the tendency toward rigour that characterized most research in analysis during the last century was already under way, we believe that, in Peano's case, we can speak of it being a matter of permeating attitude and mentality. The debates – for instance the famous one with C. Segre – on the importance of rigour and on ways to pursue it, in our view confirm the fact that Peano was more interested in specifying, in a rigorous way, concepts and proofs concerning widely developed theories than in venturing into unexplored areas of research. (Segre, in a paper in Peano's *Rivista di Matematica*, wrote that sometimes, in the initial exploration of a new field of mathematics, it is necessary to sacrifice rigour and resort to incomplete procedures (see Segre [*1891a*, 53]). Peano answered [*1891d*, 66]: "the lack of rigour in mathematical work cannot, in our opinion, by any means be defended or excused;" Peano's conviction was that "a result cannot be considered as *obtained* if it is not rigourously proved, even if exceptions are not known." Farther on, Peano [*1891d*, 67] added: "Absolute rigour, if it is a necessary condition for a work's being scientific, is not yet a sufficient condition. Another condition concerns the hypotheses from which one starts. If an author starts from hypotheses which are in conflict with experience [...], he can, actually, deduce some wonderful theory, such that one may exclaim: what an advantage, if the author had applied his reasoning to practical hypotheses!" Segre [*1891b*, 55] replied: "In my opinion, if a theory is wonderful, and therefore reaches the main aim of science, the honour of the human mind, I cannot ask anything else" and argued that "in most cases the period of discovery preceded the period of rigour." Peano objected with harsh criticism to some of Segre's work and wrote [*1891d*, 158], among other things, that "I have never seen in any branch of mathematics, from arithmetic to theoretical mechanics, that a period of discovery precedes the period of rigour. In mathematics, a theorem is discovered when it is proved. The progress of mathematics always consists in adding new truths to old ones. Nor do these two periods occur at different times to individual researchers: there is, instead, the period of research which precedes the moment of discovery.")

Peano's love for rigour, and the sensibility he always showed for mathematics education, were the most likely motivations underlying the project of the *Formulario* – which undoubtedly is the most original, but also the most questionable work carried out by Peano and his school. (Peano's sensitivity for mathematics education was witnessed, among other things, by the fact that *Rivista di Matematica* was founded by Peano with openly declared didactic intent. On the second title page of the first number, for instance, we read: "*Rivista di Matematica* has an essentially didactic aim, since it is especially interested in improving teaching methods. It will also include papers and discussions concerning the fundamental principles of science and the history of mathematics; the review of treatises and of all publications concerning teaching will play an important role.")

The *Formulario* had five later editions. These were not mere reprints, but contain improvements and even remarkable extensions. The first edition was published in 1895 with the title *Formulaire de mathématiques* and included, in addition to *Notations de logique mathématique* (*Introduction au Formulaire de mathématiques*) of 1894, many papers that were published in *Rivista di Matematica* since 1892. The second edition was published in three parts in the years 1897, 1898, and 1899 respectively; to this one must add the "Introduction au tome II du Formulaire de mathématiques," published in *Rivista di Matematica* in 1896. The third edition, again with the same title, was published in 1901. In the fourth edition, of 1903, the title was changed to *Formulaire mathématique* and the French was retained for the explanations and notes. In the last edition, of 1908 (reprinted in facsimile, with introduction and notes by U. Cassina, published by Edizioni Cremonese in Rome in 1960), the title was changed to *Formulario mathematico*, and the work was written in Latin "sine flexione" (a language created by Peano in 1903 and intended to serve as the universal scientific language, playing the role assumed by Latin in previous centuries).

The aim of the *Formulario*, whose structure makes it look like a sort of mathematical encyclopedia (in some respects comparable to Bourbaki's project), and whose elaboration took the best energies of Peano and his collaborators for about fifteen years, was to present all of the different mathematical theories with both the highest degree of synthesis and precision and with didactic purposes. As Peano explained [*1896a*; see *1957-59*, II, 199]:

> Or il est possible de publier un *Formulaire de mathématiques* qui contienne toutes les propositions connues dans les sciences mathématiques, toutes les démonstrations, toutes les méthodes. Elles, écrites en symboles, occupent peu de place, beaucoup moins qu'on ne pourrait croire. Il y a une infinité de livres et Mémoires inutiles, erronés, ou dans lesquels on répète ce que d'autres auteurs ont déjà dit. Ce qui reste à écrire dans le Formulaire n'est qu'une partie infiniment petite de ce qu'on a publié jusqu'à présent. [...] Chaque professeur pourra adopter pour texte ce Formulaire, car il doit contenir toutes les propositions et toutes les méthodes.

The birth date of the *Formulario* project must be fixed in 1891, when, in a passage from the article "Sul concetto di numero" [*1891e*; reprinted: *1957-59*, III, 109], Peano expressed himself in the following terms:

> It would be very useful to assemble all known propositions concerning certain points of mathematics and to publish those collections. Limiting ourselves to arithmetical propositions, I do not think it would be difficult to express them in logical symbols; and then they would gain conciseness as well as precision. And it is likely that the propositions regarding certain arguments of mathematics can be contained within a number of pages no larger than the one required by their bibliography. [...] *Rivista di Matematica* will manage, next year, to publish some collections of this kind. We therefore invite readers to write some of them and kindly send them to us.

It seems reasonable to think, on the other hand, that some contributions to arithmetic and geometry, for instance Peano's famous axioms for arithmetic, or the axiomatization of (a part of) elementary geometry, also given in 1889, can be considered to be anticipations of the *Formulario* project. And already in these anticipatory works we find the formulation for which Peano is so rightly famous – the use of mathematical logic. Both *Arithmetices principia* and *I principii di geometria logicamente esposti* are preceded by an introductory logical part, an essential use of which is made in the rest of these two works. For instance, Peano writes in *I principii di geometria logicamente esposti* [*1889c*; reprinted: *1957-59*, II, 57]:

> We thus have the means to express the propositions of geometry in a rigorous way, which cannot be obtained in the common language, and the solution of the proposed problems is therefore much easier.

Logic is essentially understood as an *artificial language* that is able to express with the highest possible clarity mathematical concepts and proofs, which up to then had been expressed principally by means of ordinary language. As Peano himself underlines on several occasions, the principles of logic are nothing other than the symbolic transcriptions of the correct modes of reasoning in mathematics. Thus, for example, he wrote [*1900*; reprinted in *1957-59*, II, 320]: "Le lois de logique, contenues dans la suite, on été en général trouvées en énonçant, sous form de règles, les déductions qu'on rencontre dans les démonstrations mathématiques."

This is the main role logic plays in Peano's school. The creation of the artificial language mentioned above is, as we said before, the kind of contribution to which many rightly refer when they stress the most original aspects of Peano's production in logic.

Mathematical logic, then, is fundamentally intended as an *instrument* for realizing the *Formulario* project, and not, as we are used to considering it nowadays, as an autonomous mathematical discipline.

There was, to be sure, a stage in the development of Peanesque logic, starting with the "Formole di logica matematica" of 1891 and ending with the "Formules de logique mathématique" of 1900, in which logic was also studied as a mathematical discipline, capable as such of being given an adequate axiomatic treatment. Peano himself, in fact, wrote, as far back as his *I principii di geometria logicamente esposti* of 1889 [reprinted in *1957-59*, II, 81], that:

> The list of logical identities we are using had already been made in the booklet of mine mentioned above [the booklet in question is *Arithmetices principia* of 1889]; many of them had been brought together by Boole. Their number is large; it would be an interesting study, which we will not undertake here, to distinguish the fundamental ones – those which must be admitted without any doubt – from the remaining identities which are included along with the fundamental ones. This research would produce a study concerning logic that would be similar to the ones made in the present case for geometry and made in the previous booklet for arithmetic.

In previous works, logical laws were simply *listed*, but in this new treatment a distinction is made between logical axioms and theorems, in the sense that some logical laws are *derived* from other laws. We also find a distinction being made between primitive and defined logical symbols. Regarding the concept of *proof*, an explicit characterization is given that remains at a rather intuitive level. Peano, for example, writes in the "Formole di logica matematica" [*1891c*; reprinted in *1957-59*, II, 103-104] that: "Proving a proposition means getting it by suitably combining the propositions which have already been admitted." Moreover, Peano's proof procedures allow us to only partially determine the precise nature of the notion of proof under which he operates. For example, starting from the propositions

3. $ab \supset a$
4. $ab \supset ba$
8. $a \supset b \,.\, b \supset c : \supset . \, a \supset c$

(for which we maintain Peano's numbering), the proposition $ab \supset b$, which for Peano is P10, is proved in the following way:

$$\mathrm{P4} \,.\, \begin{pmatrix} b, a \\ a, b \end{pmatrix} \mathrm{P3} \;\, . \, \mathrm{P8} : \supset . \, \mathrm{P10}$$

(the second proposition is obtained by replacing a with b and b with a in P3, that is, $ba \supset b$).

Some criticisms of Peano's procedure have been set forth by J. van Heijenoort, who, in presenting the English translation of the *Arithmetices principia* in the well-known collection *From Frege to Gödel*, writes [van Heijenoort *1967*, 84], among other things, that:

> There is, however, a grave defect. The formulas are simply listed, not derived; and they could not be derived, because no rules of inference are given. [...] What is far more important, he [Peano] does not have any rule that would play the role of the rule of detachment. The result is that, for all his meticulousness in the writing of the formulas, he has no logic that he can use. [...] when ultimately he does detach, it is a move totally unjustified in his system.

This criticism hits the mark inasmuch as it points out the missing distinction between axioms and rules of inference that is the basis of modern logical calculi and allows one to develop proofs in a purely formal way. It can be argued, however, that the logical laws *also* play the role of rules. (We read, for instance, in *Notations de logique mathématique* that "Les règles du raisonnement sont les formules mêmes de logique" [*1957-59*, II, 174]. See also the quotation, already given, from [Peano *1900*, reprinted in *1957-59*, II, 320].) The step that, according to van Heijenoort, is "totally unjustified" would then appear to be justified in case Peano explicitly referred to the formula $a \,.\, a \supset b : \supset . \, b$. This formula, however, does not appear in the list of logical laws proposed in *Arithmetices principia* (most likely because it does not readily admit of translation within class theory), although it will systematically appear in all of the later works by Peano, beginning in 1891 with his "Formole di logica matematica". Nevertheless, it does not play the role that today we assign to the rule of detachment (it is not, for instance, used explicitly in the proof quoted above).

While this interpretation, in our opinion, weakens van Heijenoort's criticism, we should not forget that the failure to distinguish between logical laws and inference rules is an aspect of the failure to distinguish between theory and metatheory. Symbolic propositions are not understood, as we say today, as strings of symbols, but rather as translations of the corresponding propositions of the natural language; and we always find in them a semantic interpretation in addition to the syntactic representation of the logical form. In other words, syntactic and semantic aspects are simultaneously present, so that we do not find the distinction that will characterize the subsequent development of logical calculi. (Frege [cf. *1897*] emphasizes that while the calculus of Boole is a *calculus ratiocinator*, but not a *lingua characteristica*, and the mathematical logic of Peano is mainly a *lingua characteristica* and only secondarily a *calculus ratiocinator*, his own ideography is both things with equal strength. Frege is right, and his contributions are actually substantial anticipations of modern logical calculi, although, as we know, his work had no perceptible influence, whereas Peano's contributions gained recognition from the outset, mainly from Russell. For instance, Blanché [*1970*, 323] writes: "Dans les dernières années du XIX^{e} siècle, ce n'est pas vers Frege que se tournent les regards de ceux qui s'intéressent à la philosophie des mathématiques et à la symbolisation de leur langage; c'est vers Giuseppe Peano (1858-1932) et vers l'équipe de mathématiciens italiens qui travaille en group avec lui.")

On the other hand, the language proposed by Peano is perfectly in line with the aim pursued by Italian mathematicians to express mathematical propositions without the prolixities and ambiguities found in ordinary language, thereby obtaining a highly synthetic and rigorous exposition.

In order to stress further the instrumental character attributed to logic, we quote the following passage included in Peano's review of Whitehead and Russell's *Principia Mathematica* [*1913*; reprinted in *1957-59*, II, 391]:

> In *Formulario*, mathematical logic is only an instrument to express and deal with propositions of ordinary mathematics; it does not have an autonomous aim. Mathematical logic is expressed in 16 pages; one hour of study is sufficient to know what is necessary for applying this new science to mathematics.

Strangely enough, this happened just when, abroad, the studies which led to the maturation of logic as a really autonomous discipline, and, above all, to metalogical studies, began to flourish.

This was the prevailing attitude within the school too. Most of the logical work carried out by Peano's collaborators (Vailati, Padoa, Pieri, Burali-Forti) can in fact be considered to be either the incorporation of new material into the *Formulario* or expositions that have a didactic or informative character (sometimes they simply aim at "praising" Peano's merits). We can recall, for instance, Vailati's contribution to the section on mathematical logic – and in particular to the historical notes – of the *Formulario*'s first edition. Peano himself, in his *Notations de logique mathématique*, quotes Vailati's work "Le prorietà fondamentali delle operazioni della logica deduttiva" of 1891, where an interesting study of relations can be found (among other things, this article introduced the term "reflexivity" for the first time in its usual contemporary meaning). The second edition of the *Formulario* adopted the procedure of accompanying formulas, including logical formulas, with the names of the people who first enunciated them, and several formulas are attributed to Burali-Forti (e.g. from his "Sul quelques propriétés des ensembles d'ensembles et leurs application à la limite d'une ensemble variable" of 1896), while Padoa, Pieri, and Vailati are also mentioned. In the case of the logic section of the third edition of the *Formulario*, Peano points out that some changes were due to "important remarks" by Padoa (from Padoa's "Note di logica matematica" of 1899).

Similarly, the first edition of Burali-Forti's volume *Logica matematica* of 1894 gives a very detailed account of Peano's logic. There are some points in this work – concerning, for instance the concept of proof – in which explanations are given that cannot be found in Peano's work. The second edition of 1919 is far more complete, containing in fact 500 pages as compared with the 150 pages of the first edition. It is nevertheless disappointing to note that, despite the natural enrichment of the subject, we still find the same approach so far as the role and aims of mathematical logic are concerned.

Peano and his school continued to maintain their approach as time went on. In 1930, for instance, on the eve of the publication of Gödel's theorem, Padoa again proposed some themes and problems that faithfully reflect the ones found in a work he wrote in 1912; this earlier work, in turn, arose from previous contributions of his and Peano's. Padoa [*1930*, 79] writes, for instance:

> May the reading of these pages urge some young student to devote himself to logic and give it new contributions, especially concerning the choice of a suitable system of postulates, in order to overcome the paradoxical phase of this science which, while supplying other sciences with the instrument for their deductive reconstruction, has not yet been able to reconstruct itself and thus to satisfy the methodological requirements which it has imposed upon other disciplines.

Nevertheless, Bertrand Russell's well-known remarks about Peano's stimulus upon his own development leave no doubt about the actual value of Peano's research and its cultural impact in this seminal period of history. (See, e.g., [Russell *1967*, 217-218], where Russell writes: "In July 1900 [in fact it was in August] there was an International Congress of Philosophy in Paris [...]. The Congress was a turning point in my intellectual life, because there I met Peano. I already knew him by name and had seen some of his work, but had not taken the trouble to master his notation. In discussion at the Congress I observed that he was always more precise than anyone else, and that he invariably got the better in any argument upon which he embarked. As the day went by, I decided that this must be owing to his mathematical logic. I therefore got him to give me all his works, and as soon as the Congress was over I retired to Fernhurst to study quietly every word written by him and his disciples. It became clear to me that his notation afforded an instrument of logical analysis such as I had been seeking for years, and that by studying him I was acquiring a new and powerful technique for the work that I had long wanted to do.") Whatever their shortcomings over against the later development of logic, Peano's school exerted a significant influence on logic at the end of the nineteenth century.

3. Foundations of Mathematics. The foundational research in which logic is applied to mathematics and the contributions that can be considered to be anticipations of more recent metatheoretical investigations deserve separate treatment.

3.1. Foundations of geometry and the new axiomatics. The contributions of Peano's school to the axiomatization of geometry are very important, and the writings in this field, especially those of Peano and Pieri, are significant milestones for those who study the historical development of the foundations of geometry.

The writings of Peano that explicitly concern the foundations of geometry are *I principii di geometria logicamente esposti* of 1889 and "Sui fondamenti della geometria" of 1894, both of which are chronologically located between Pasch's *Vorlesungen über neuere Geometrie* of 1882 and Hilbert's *Grundlagen der Geometrie* of 1899. In the first of these works, Peano's intention is to use deduction to obtain the main theorems of positional geometry, namely the ones concerning the relations of membership and order, starting from the smallest number of primitive geometrical entities and axioms. Peano takes *point* and *segment* as primitive concepts, even if, in fact, the latter is the ternary relation $c \in ab$, which means "c is an inner point of the segment ab" (or "c is between a and b"). Peano is able to use this relation to define the concepts of *straight line* and *plane* and to develop a positional geometry which is improved significantly in comparison with Pasch's.

In the second of the writings mentioned, Peano's treatment also includes contributions to metric geometry based upon the notion of congruence, which is defined in terms of the concept of motion (which in turn is characterized as a special affinity satisfying appropriate axioms).

Some of Peano's other contributions to geometry must be mentioned even though they are only indirectly connected with the aims of the present work, because, in our opinion, they are important for an exhaustive evaluation of the global results achieved in this field. In the *Calcolo geometrico* of 1888 (which was inspired both by Grassmann's work and the work of Möbius and Schlegel), Peano considers vectors, on which he bases his geometrical calculus (vector algebra). In this work, we find, among other things, the first axiomatic definition of the (abstract) structure of vector space (including a vector space with an infinite basis). These topics were then developed in the booklet *Gli elementi di calcolo geometrico* of 1891 and the paper "Saggio di calcolo geometrico" of 1896.

For some aspects of Peano's work along these lines, the next essay, "Analisi della teoria dei vettori" of 1898 is more interesting. In this paper, there is a conceptual reversal: whereas in the previous works vector theory was developed within the framework of elementary geometry, in this new work vector theory is presented axiomatically and is used to derive elementary geometry. Peano takes as primitives the concepts of *point* and the quaternary relation of *equidifference* among four points a, b, c, d, denoted by $a - b = c - d$ (which can be interpreted in different ways; e.g. the segments ab and cd have the same length, direction and are parallel, or the figure $abcd$ is a parallelogram). In order to introduce additional metric considerations, he also takes the *inner product* to be a primitive concept. Thus Peano has anticipated the modern way of founding elementary geometry on linear algebra, and it is essentially different from the traditional way of founding elementary geometry.

Pieri's attention in the field of geometry was directed to the axiomatization of projective geometry. In the work "Sui principii che reggono la geometria di posizione" of 1895, the concepts of *projective point*, *projective straight line*, and *projective segment* are taken as primitive, and the theory is developed up to the proof of Staudt's theorem. In a subsequent essay, "Sugli enti primitivi della geometria projettiva astratta" of 1897, the primitive concepts are reduced to two, *projective point* and *straight line joining two projective points* (the segment can, in fact, be defined by means of the notion of a *harmonic group*). Later, in the paper "Nuovo modo di svolgere deduttivamente la geometria projettiva" of 1898, Pieri introduces a new axiomatization of projective geometry in accordance with the suggestion of the Klein's Erlanger program, taking *point* and the *homographic transformation* as the primitive concepts. In the paper "I principii della geometria di posizione composti in sistema logico deduttivo", also of 1898, a new exposition of projective geometry is presented that is still based upon the concepts of *point* and *straight line joining two points*. It is with regard to this work that Russell wrote [*1903*, 382*n*.]: "This is, in my opinion, the best work on the present subject." The work "Sui principii che reggono la geometria delle rette" of 1901 presents a new treatment of projective geometry, based upon the concepts of *ray* and *incidence between rays*, in which *point* is not taken as primitive.

While working on the axiomatics of projective geometry, Pieri also devoted himself to a rigorous treatment of Euclidean geometry, improving on Peano's exposition. In the important essay "Della geometria elementare come sistema ipotetico deduttivo" of 1899, elementary geometry is developed assuming only the two primitive notions of *point* and *motion*. This work anticipates one of the most remarkable contributions to the foundations of geometry by Peano's school. The contribution is the foundation of all of elementary geometry on the two primitive notions of *point* and the ternary relation "C belongs to the sphere with center A and passing through B". The contribution will be made by Pieri himself several years later in his work "La geometria elementare istituita sulle nozioni di 'punto' e 'sfera'" of 1908, and it is the first actual example of an exposition of elementary geometry founded on the properties of the group of isometries.

The work of Padoa and Vailati must also be included in any survey of the contributions of Peano's school to geometry. In Padoa's paper "Un nouveau système de définitions pour la géometrie euclidienne" of 1900, elementary geometry is founded on the primitive notions of *point* and *overlapping of points*. In the essay "Sui principii fondamentali della geometria della retta" of 1892, Vailati considers the points of a given straight line with the binary relation $b\mathrm{S}a$ ("the point b follows the point a"), by means of which he defines Peano's ternary relation ($c \in ab$) and, assuming only three primitive propositions concerning the relation S, proves all of the properties that are given by the axioms of Peano's system. In two works of 1895, "Sulle relazioni di posizione tra punti d'una linea chiusa" and "Sulle proprietà caratteristiche delle varietà a una dimensione," Vailati characterizes the fundamental properties of the quaternary relation of *separation* (of points of a closed line), which are even today assumed as axioms of order for elliptic and projective geometry.

In nearly all of the work of Peano's school, the main goal (besides presenting the specific geometrical content) is to find the smallest number of primitive concepts and axioms required for the different theories. This aim is also shared by their contributions to arithmetic and is motivated by the need to characterize the "fundamental ideas" and "principles" of every science within the framework of a program that will be realized in the *Formulario*. On many occasions, Peano and his collaborators will explicitly state that they carried out the Leibnizian project of developing the *characteristica universalis*, even if their success in fact seems questionable.

In the works considered, we find a conception of the axiomatic method that displays many interesting and innovative features deserving of a more detailed analysis. In particular, we owe to Pieri the clearest formulation of many of the characteristics of the so-called "modern" conception of the axiomatic method, according to which mathematical theories are understood, in the terminology we owe to Pieri, as hypothetico-deductive systems. (For instance, in [Pieri *1898a*; see *1980*, 106], we read: "The most peculiar characteristic of primitive entities of any hypothetico-deductive system is that they can be given arbitrary interpretations within certain restrictions that are imposed by the *primitive propositions* (*axioms* or *postulates*). In other words, the ideal content of propositions and of the symbols that denote some primitive subject is determined only by the primitive propositions concerning it, and the reader can assign an *ad libitum* meaning to those words and to those symbols, provided that this is consistent with the general properties demanded for this entity by the primitive propositions." Furthermore, Peano had already anticipated the conceptual turning point usually ascribed to Hilbert when he explicitly stated that the primitive entities of geometry are totally undefined and that it is possible to assign arbitrary meanings to them provided those meanings satisfy the axioms. Peano writes in [*1889c*; reprinted in *1957-59*, II, 61, 77]:

> The sign **1** is read point. [...] If a and b are points, then by ab we intend the class of points inside the segment ab. Therefore, the formula $c \in ab$ means "c is an inner point of the segment ab". [...] We thus have a category of entities called points. These entities are not defined. Moreover, given three points, we consider a relation among them, represented by $c \in ab$, and this relation is likewise undefined. The reader may understand by the sign **1** any category of entities whatever, and by $c \in ab$ any relation among three entities of that category. All of the definitions (of §2) will always have meaning, and all of the propositions of §3 will hold. Depending upon the meaning given to the undefined signs **1** and $c \in ab$, the axioms can be satisfied or not. If a certain group of axioms is verified, then all of the propositions deduced from them will be true, since these propositions are transformations of those axioms and of the definitions.

Peano is in many respects, however, still tied to the intuitive content of mathematical theories, and thinks that the axioms must reflect the simplest observations of the physical world. Thus, for example, in the "Sui fondamenti della geometria" [*1894*; reprinted in *1957-59*, II, 119, 141], he writes:

> [...] it will be necessary to determine the properties of the entity p and of the relation $c \in ab$ by means of axioms or postulates. The most

> elementary observation shows us a long list of properties of these entities. We must only collect these common notions, order them, and assume as postulates only those that cannot be deduced from more simple ones. [...] Certainly it is permitted to anyone to put forward whatever hypotheses he wishes and to develop the logical consequences contained in those hypotheses. But in order for this work to deserve the name of Geometry, it is necessary that these hypotheses or postulates express the result of the more simple and elementary observations of physical figures.

Pieri, on the other hand, is very explicit in taking a step beyond Pasch and Peano in the direction of freeing geometry from experience, and it seems more reasonable, therefore, to compare his approach to Hilbert's. In an essay of 1896 dealing with projective geometry, for instance, he proposes to adopt a "purely deductive and abstract approach," which he specifies as follows [Pieri *1896*; see *1980*, 84]:

> *Abstract*, as it leaves out any *physical* interpretation of the premises, and therefore also their evidence and geometrical intuition, unlike another trend (which I shall call *physical-geometrical*) according to which the primitive entities and the axioms are to be deduced from direct observation of the external world and identified with the ideas we get by means of experimental induction from some particular objects and physical facts (Pasch, Peano, ...).

An idea usually attributed to Hilbert but anticipated by Gergonne is the conception of axioms as "implicit definitions". The question of the treatment of axioms as "implicit definitions" is in fact complex, and in our opinion careful investigation shows that its usual historical interpretation is misleading (cf. [Borga *1981*]). If Gergonne used the term "implicit definitions" in the context of his theory of definitions, then the meaning attributed to it is hardly adaptable to the case of the axioms of a theory. The correct conception of axioms as implicit definitions must instead be attributed to the members of Peano's school. We owe to Pieri in particular a very explicit formulation of this conception, which he spoke of as "definitions by postulates," when he wrote [*1900*; see *1980*, 264*n*.]:

> Si par définition on entende une pure et simple imposition de nom à des choses déjà connues ou acquises à la science, les idées premières seront les concepts non définis. Mais on entend encore la "définition" en un sense plus large: c'est ainsi qu'on dira, par exemple, que les concepts primitifs ne sont pas définies autrement que par les postulats. En effet,

ces dernières, attribuent à ces concepts certaines propriétés qui suffisent à les caractériser en vue des finis déductives qu'on se propose.

This conception was often applied by Peano's school before it was popularized as a result of Hilbert's work. Moreover, it is worth noting that Hilbert's attitude, in its turn, is not easy to interpret. On the one hand, one may say that his *Grundlagen der Geometrie* was in fact inspired by this conception. On the other hand, one may maintain that Hilbert's attribution to the axioms of a definitional character was, at least in the beginning, unclear to the point of raising a number of objections by Frege in a short but intensive correspondence.

3.2. Foundations of arithmetic. Peano's first contribution to the axiomatic treatment of arithmetic, as we have already mentioned, is found in the essay *Arithmetices principia, nova methodo exposita* of 1889. Here, Peano assumes the four primitive concepts of number (N), successor ($a+1$ denotes the successor of a), identity (=), and the following nine axioms [Peano *1889a*; reprinted in *1957-59*, II, 34]:

1. $1 \in N.$
2. $a \in N. \supset . a = a.$
3. $a, b \in N. \supset . a = b . = . b = a.$
4. $a, b, c \in N. \supset \therefore a = b . b = c : \supset . a = c.$
5. $a = b . b \in N : \supset . a \in N.$
6. $a \in N. \supset . a+1 \in N.$
7. $a, b \in N. \supset : a = b . = . a+1 = b+1.$
8. $a \in N. \supset . a+1 -= 1.$
9. $k \in K \therefore 1 \in k \therefore x \in k : \supset_x . x+1 \in k :: \supset . N \supset k.$

(Notice, incidentally, that the symbol "=" is also used in axioms 3 and 7 to denote logical equivalence between propositions.) With the aid of the logical symbols introduced in the first part of his work, Peano is able in a few pages to summarize many arithmetical results, and then to also extend his treatment to the rational and real numbers.

In the introduction to this work, Peano explicitly points out that he was inspired by H. Grassmann's results, and he adds that Dedekind's book *Was sind und was sollen die Zahlen?*, published in 1888, had also been useful to him. Some people have seen in Peano's acknowledgement of Dedekind an indication that Dedekind's work had directly influenced Peano's axiomatization of arithmetic. On the contrary, however, Peano observes in his paper "Sul §2 del Formulario, t. II: Arithmetica" appearing in

Rivista di Matematica in 1898 that the *Arithmetices principia* had been completed independently of Dedekind's work. Peano read Dedekind's booklet after his own had already been sent to press, and he simply drew from Dedekind's work the "moral proof" (as he wrote) of the independence of the propositions that he had assumed to be primitive. Moreover, beyond the strict formal analogy between the conditions that Dedekind assumes to characterize the natural numbers and Peano's axioms, there is a substantial difference between the two approaches. Thus, while Peano takes the concept of *number* as primitive and hence irreducible to simpler notions, Dedekind's aim was to start from more general "set-theoretical" notions.

Peano makes his ideas about the value of his approach to foundational studies more explicit in the paper "Sul concetto di numero" of 1891. In this paper, only five primitive propositions are assumed, since identity is no longer taken to be a primitive concept of arithmetic but instead is now included among the logical concepts. When we refer to Peano's axioms for arithmetic, we usually mean the formulation of the axioms which is presented in this paper. In non-symbolic form, the axioms are [Peano, *1891e*; reprinted in *1957-59*, III, 185.]:

1. "One is a number".
2. "The symbol + after a number produces a number".
3. "If a and b are numbers, and if their successors are equal, they are equal well".
4. "One does not follow any number".
5. "If s is a class including one and if the class formed by the successors of s is included in s, then every number is included in s".

In the fifth edition of the *Formulario* [Peano *1908*, 27], finally, there are six axioms of arithmetic, with the primitive symbol "0" replacing "1":

0. $N_0 \in Cls.$
1. $0 \in N_0 .$
2. $a \in N_0 . \supset . a+ \in N_0 .$
3. $s \in Cls . 0 \in s : a \in s . \supset . a+ \in s :. \supset . N_0 . \supset s .$
4. $a, b \in N_0 . a+ = b+ . \supset . a = b .$
5. $a \in N_0 . \supset . a+ -= 0 .$

The idea for introducing axiom 0 is due to Padoa, who in his "Note di logica matematica" of 1899 noticed that the proposition "$N_0 \in Cls$," although used in the proofs, is not provable from the other axioms. Later, in the paper "Théorie des nombres entiers absolus" of 1902, Padoa was able to reduce the number of arithmetical axioms to five by assuming that the proposition "$x \in a . \supset . a \in Cls$" is a primitive proposition belonging to logic. With this new logical proposition and from arithmetical axiom 1, one immediately obtains axiom 0. In this same work, moreover, Padoa presents a different axiomatization of arithmetic, based on two primitive concepts, namely *number* and *successor*, and four axioms. In place of Peano's axioms 1 and 5, he assumes that "there exists at least one number which is not the successor of any number." After proving the uniqueness of the number that is not the successor of any number, he can define it to be zero.

Pieri in turn gives another axiomatization of arithmetic in his paper "Sopra gli assiomi aritmetici" of 1908. Pieri's axiomatization is based on two primitive concepts and four axioms, and it differs from Padoa's in that it replaces the principle of induction used by Padoa with the equivalent principle of the least integer ("in an non-illusory class of numbers there exists at least one number which is not the successor of any number of the class"). Members of Peano's school thought that the "arithmetical" character of the principle of the least integer confirms the "mathematical" nature of induction, although contemporary opponents of mathematical logic, among them H. Poincaré, considered it to be of purely logical character.

3.3. Metatheoretical problems. The Peano school's reduction of the number of primitive concepts and axioms of mathematical theories to the smallest possible number quite naturally led them to consider the problem of independence. Thus, for example, Peano gave a proof of the independence of his axioms for arithmetic as early as 1891 in his paper "Sul concetto di numero". This is obtained by means of suitably constructed interpretations that make one axiom of the set false and all the others true.

Padoa's contributions to the question of independence likewise merit attention for the novelty of his approach. The novelty resides in the fact that Padoa also deals with the independence of primitive concepts. The results of his analysis, given during the International Congress of Philosophy in Paris in 1900, include in particular a technique – later to be known as "Padoa's method" – capable of proving the independence (understood as mutual undefinability) of primitive symbols. In order to prove that a primitive symbol x cannot be defined in terms of other primitive symbols, two interpretations of the theory are to be built that must both satisfy the axioms and differ only with respect to their interpretation of x.

This method, although clearly analogous to the technique used to prove the independence of axioms, is justified by Padoa only by an appeal to intuitive considerations. Furthermore, Padoa deems his method to be not only a sufficient condition, but even a necessary condition, for independence. Today we know that accepting it as a necessary condition amounts to Beth's theorem (1953), whose proof is not at all trivial. While we may be inclined to regard this insight in the theory of definitions as an "intuition" of Padoa's, we cannot help noting at the same time that there was only a small amount of meaningful research following up on Padoa's work on the theory of definitions that would subsequently allow Tarski and others to deal with these problems in a rigorous way.

While the problems of independence were widely studied within Peano's school, the same cannot be said of the problem of consistency that will later come to play a central role in studies of logic. Even as late as 1906, after the statement of the Russell antinomy and after Hilbert had outlined his foundational program in 1904, Peano still continued to display a certain indifference to the issue of consistency, to the extent that he claimed in his paper "Super theorema de Cantor-Bernstein" [Peano *1906*; reprinted in *1957-59*, I, 343]:

> A consistency proof for arithmetic, or for geometry, is in my opinion not necessary. In fact, we do not create the axioms arbitrarily, but assume instead as axioms very simple propositions which appear explicitly or implicitly in every book of arithmetic or geometry. The axiom systems of arithmetic and geometry are satisfied by the ideas of number and point that every author in arithmetic or geometry knows. We think of numbers, and therefore numbers exist. A consistency proof can be useful if the axioms are intended as hypotheses which do not necessarily correspond to real facts.

In a paper on "Le problème n. 2 de M. David Hilbert", Padoa [*1903*] attacks Hilbert, believing that Hilbert failed to realize that Padoa had solved the problem of the consistency of analysis (Hilbert's second problem) in the [*1902*] paper "Théorie des nombres absolus" delivered at the same International Congress of Mathematicians in Paris in 1900 during which Hilbert had posed his famous list of open problems. (Today we would say that Padoa's "solution," which is proposed for the theory of integers, is based on the existence of the standard model. This solution is, of course, very different from what Hilbert had in mind.)

A more critical attitude was taken by Pieri, who diverged sharply from Padoa's views and sought to obtain a proof of the consistency of arithmetic (see in particular [Pieri *1904*, 307*n.*]). In today's terminology, we would say that Pieri [*1906a*] essentially constructs a model of arithmetic within set theory.

Some attention should also be given to the so-called "Burali-Forti paradox." Burali-Forti is usually credited with the discovery, or at least the publication, in 1897, of the first antinomy in Cantorian set theory. The Burali-Forti paradox is now more commonly referred to as the antinomy of the greatest ordinal number, according to which the class of all ordinals has an ordinal number greater than each number in the class itself and is therefore also greater than every ordinal number and in particular is greater than itself.

Although this current interpretation seems to link Burali-Forti's contribution directly to the problem of consistency, Burali-Forti's actual purpose was quite different. He meant, in fact, to prove by *reductio ad absurdum* that the trichotomy law does not hold for ordinal numbers, writing [*1897a*, 154] that:

> The main purpose of this paper is to prove that there indeed exist *transfinite* numbers (or *order-types*) *a*, *b* such that *a* is neither equal to, nor less than, nor greater than, *b*.

Using reasoning that involves the class of all ordinal numbers in an essential way, Burali-Forti indeed obtains a contradiction. But this is intended as the natural conclusion of an indirect proof. And when Cantor proved in 1897, contrary to Burali-Forti's conclusion, that the law of trichotomy does in fact hold, Burali-Forti misunderstood Cantor's result, while recognizing, in his short note "Sulle classi bene ordinate", that he had appealed to a different definition of ordinal number. (Burali-Forti misunderstood the Cantorian notion of ordinal number. He thought of ordinal numbers as order types of those classes he called "perfectly" ordered, while Cantor identified ordinal numbers with order types of well ordered classes.) Burali-Forti did not realize that his argument could also be applied to ordinal numbers as conceived by Cantor, and that the crucial point, after all, was the assignment of an ordinal number to the class of all ordinal numbers. Only after Russell's antinomy had gained currency did a modified form of Burali-Forti's argument attain the character of an antinomy. (For a deeper historical analysis of Burali-Forti's paradox, see Moore & Garciadiego [*1981*] and Garciadiego [*1985*].)

4. Conclusion. In the light of our historical analysis, we claim that the foundational contributions of Peano and his school show beyond a doubt that the origin of the "modern" conception of the axiomatic method is more properly ascribed to Peano's school, and in particular to Pieri, than to Hilbert. For Peano and his school, however, the axiomatic method retained its traditional purpose of providing a rigorous treatment of mathematical theories and was not understood, as it will be by the formalists, as an instrument of research for the creation of new mathematical theories.

The Peano school also articulates and tackles metatheoretical problems, especially the problem of the independence of the axioms. Once again, it must be recognized, however, that they were still far from the modern spirit of metatheoretical investigation, not so much with respect to the techniques employed, but with respect to the meaning attributed to those investigations. The foundations of mathematics attains full maturation, in modern terms, when metatheoretical problems are thought to possess intrinsic value. Thus, for example, the primary interest in an independence proof, beginning from the time of Hilbert's earliest contributions, resides in the possibility of creating a new theory, namely a theory that includes among its axioms the negation of the proposition whose independence is being proved. In Peano's school, on the other hand, the principal attitude was to consider metatheoretical properties as conditions for the "purity" and "logical perfection" of deductive theories. Padoa writes, for instance, in his contribution to the 1900 International Congress of Mathematicians in Paris (see [Padoa *1902*, 309]):

> In the *Introduction logique à une théorie déductive quelconque* that precedes our *Essai d'une théorie algèbrique des nombres entiers*, we have analyzed the formal structure of a deductive theory in order to state the main conditions for its logical perfection and the practical rules to see if these conditions are satisfied by a given theory.

Similar views were also expressed by Pieri in a lecture delivered in 1906, when he said [*1906b*; see *1980*, 423]:

> Besides the consistency or compatibility of the hypotheses, logicians also prove the relative independence of the primitive notions (the fact that none can be defined in terms of others) and the relative independence of the postulates (that is, none is a consequence of the others). These conditions ought to be satisfied in a perfect hypothetico-deductive system.

A similar situation, as has already been pointed out, also holds for logic. The conception of a rigorous logical symbolism for mathematics and the attainment of the project of the *Formulario* without doubt constitute Peano's major achievements. The symbolism has survived, and it led Russell to attain remarkable improvements in logic.

Nevertheless it is difficult to maintain that the work actually performed by Peano and his school, taken in its totality, constitutes an anticipation of subsequent developments. It seems more appropriate, instead, to characterize their work as a synthesis of and capstone to the contributions of the nineteenth century. We believe, moreover, that Peano's school may be asserted to have accomplished its purpose of using the instrument of logic in order to reach the highest possible level of rigor in mathematics. This fact may perhaps also provide the most satisfactory explanation for the Peano school's lack of interest in new trends of research. The philosophical atmosphere and the hostility from some mathematicians surely played a role as well in the decline of Peano's school. But the primary reason appears to be that Peano and the members of his school thought they had completely carried out their program. The impact of the philosophical climate on the direction of the research of Peano and his school and on its decline will be considered in more detail in a future paper. In that paper we will also examine the influence exerted by Peano's school on other logicians (such as Russell) and the relationships between the contributions of Peano's school and that of other logicians.

LIST OF THE MENTIONED WORKS BY PEANO AND HIS SCHOOL

(The complete list of Peano's works can be found in [Kennedy *1980*]. Pieri's works are listed in [Pieri *1980*], those by Vailati in [Vailati *1911*].)

Giuseppe PEANO

[1883] *Sull'integrabilità delle funzioni*, Atti della Reale Accademia delle Scienze di Torino **18**, 439-446; reprinted: [*1957-59*], I, 25-32.

[1884] A. Genocchi, *Calcolo differenziale e principii di calcolo integrale, pubblicato con aggiunte dal Dr. Giuseppe Peano*, Torino, Bocca; (partially) reprinted: [*1957-59*], I, 47-73.

[1886] *Sull'integrabilità delle equazioni differenziali del primo ordine*, Atti della Reale Accademia delle Scienze di Torino **21**, 677-685; reprinted: [*1957-59*], I, 74-81.

[1887a] *Integrazione per serie delle equazioni differenziali lineari*, Atti della Reale Accademia delle Scienze di Torino **22**, 437-446.

[1887b] *Applicazioni geometriche del calcolo infinitesimale*, Torino, Bocca.

[1888] *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann, preceduto dalle operazioni della logica deduttiva*, Torino, Bocca; (partially) reprinted: [*1957-59*], II, 3-19.

[1889a] *Arithmetices principia, nova methodo exposita*, Torino, Bocca; reprinted: [*1957-59*], II, 20-55.

[1889b] *Sur les wronskiens*, Mathesis **9**, 110-112; reprinted: [*1957-59*], I, 93-94.

[1889c] *I principii di geometria logicamente esposti*, Torino, Bocca; reprinted: [*1957-59*], II, 56-91.

[1889d] *Une nouvelle forme du reste dans la formule de Taylor*, Mathesis **9**, 182-183; reprinted: [*1957-59*], I, 95-96.

[1890a] *Sulla definizione dell'area d'una superficie*, Rendiconti della Reale Accademia dei Lincei (serie 4), **6**, 54-57; reprinted: [*1957-59*], I, 102-106.

[1890b] *Sur une courbe qui remplit toute une aire plane*, Mathematische Annalen **36**, 157-160; reprinted: [*1957-59*], I, 110-114.

[1890c] *Sur l'interversion des dérivations partielles*, Mathesis **10**, 153-154; re-printed: [*1957-59*], I, 117-118.

[1890d] *Démonstration de l'intégrabilité des équations différentielles ordinaires*, Mathematische Annalen **37**, 182-228; reprinted: [*1957-59*], I, 119-142.

[1891a] *Gli elementi di calcolo geometrico*, Torino, Candeletti; reprinted: [*1957-59*], III, 41-71.

[1891b] *Principii di logica matematica*, Rivista di Matematica **1**, 1-10; reprinted: [*1957-59*], II, 92-101.

[1891c] *Formole di logica matematica*, Rivista di Matematica 1, 24-31, 182-184; reprinted: [*1957-59*], II, 102-113.

[1891d] *Osservazioni del direttore sull'articolo precedente* [by C. Segre] and *Risposta*, Rivista di Matematica **1**, 66-69, 156-159.

[1891e] *Sul concetto di numero*, Rivista di Matematica **1**, 87-102, 256-267; reprinted: [*1957-59*], III, 80-109.

[1894a] *Sui fondamenti della geometria*, Rivista di Matematica **4**, 51-90; reprinted: [*1957-59*], III, 115-157.

[1894b] *Notations de logique mathématique (Introduction au Formulaire de mathématiques)*, Torino, Guadagnini; reprinted: [*1957-59*], II, 123-176.

[1895a] *Formulaire de mathématiques*, t. 1, Torino, Bocca.

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