Shaughan Lavine, *Understanding the Infinite*, Harvard University Press, Cambridge, MA and London, 1994. ix + 372 pp.

Reviewed by

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This book is a defense of contemporary set theory as a positive and unproblematical addition to mathematics. The author addresses what he considers to be the two main arguments against the set theorist's claim of self-evidence regarding infinite collections: the historical and the epistemological. The former is, according to Lavine, due to a misunderstanding of the historical development of the axiomatizations of set theory. Contrary to the widely accepted picture, the axioms were not a series of nearly desperate measures to safeguard set theory against the paradoxes. This is argued in the first half of the book. The epistemological argument, on the other hand, is the rejection of any possible claim for intuition about the infinite on the basis that no set theorist has direct experience of the infinite. Conceding this lack of direct experience, the author argues that we nonetheless do have intuition of the infinite, which he thinks is an extrapolation (made formally precise) from our experience of the indefinitely large. The second half of the book includes a survey of various philosophies of mathematics, the exposition of Mycielski's finitistic mathematics, and the author's adaptation of the latter to a theory of the indefinitely large which extrapolates to infinitary set theory.

The interwoven strands of history, mathematics, and philosophy throughout the book rule out a detailed description of each chapter. Instead, only some of the strands will be exposed as we go through the chapters.

Chapter I is an introduction to the project and intentions of the author. Chapters II through V deal primarily with the first theme of setting the historical record straight regarding the axioms of set theory *vis-à-vis* the paradoxes: the latter may worry Frege and Russell but do not affect the mathematics of Cantor's set theory. Among other things, the author describes the history of the very slow acceptance of irrational numbers, of the debate over what a function could be, of Cantor's set theory, of Russell's paradox and his 1908 theory of types, and of the successive axiomatizations of set theory since Zermelo's first. The author argues persuasively that the different motivations of Frege, Russell, and Cantor led to different conceptions of collections which in turn led to different reactions to the paradoxes. Using Maddy's distinction between mathematical (which Lavine calls combinatorial) and logical collections (the former are those whose terms can be enumerated, without Maddy's iterative idea of set; the latter are the extensions of predicates [Maddy *1990*, 102–103]), the author argues that the paradoxes, as well as the axiom of choice, were problematic primarily to those who took collections to be logical (as did Frege and Russell). Cantor held collections to be combinatorial: they arose as mathematical objects from his investigations in analysis. According to Lavine, a combinatorial collection is to a logical one as the general (arbitrary) idea of a function is to the rule-governed idea of a function (and the former concept of collection arose in opposition to the latter). In fact, the real disagreement about the axiom of choice around 1904, the author claims, was one about logical versus combinatorial collections being employed.Some strands come together: combinatorial collections, arbitrary functions, the axiom of choice are just as natural as the irrational numbers. Lavine then gives a history of Cantor's work on sets beginning with his need for a system of notation for the indices of the sequence of successively derived sets

$$P^{(0)}, \mathrm{P}^{(1)}, P^{(2)}, \ldots, P^{(\infty)}, P^{(\infty+1)}, P^{(\infty+2)}, \ldots,$$
$$P^{(\infty\cdot 2)}, \ldots, P^{(\infty\cdot 3)}, \ldots, P^{(\infty 2)}, \ldots$$

where $P^{(0)} = P$, the derived set P' of P is the set of limit points of P, $P^{(k+1)} = (P^{(k)\prime}$, $P^{(\infty)} = \bigcap_{k=0}^{P^{(k)}} P^{(k)}$, etc. Although Cantor did not work axiomatically, the author reconstructs from the 1883-1891 writings an axiomatization in which he claims ordinals and well-orderings have primacy (in contrast with Frege, for whom, since infinite logical collections could not be "counted", cardinals had primacy). By 1895, the realization by Cantor that certain previously obvious principles (in particular, the idea that any set could be well-ordered) were in need of proof led to the loss of the above primacy with the arrival of the Power Set Axiom. "Cantor's theory was in trouble, but it was not trouble caused by the paradoxes. It was trouble caused by trying to fit the Power Set Axiom into a theory that took well-orderings to be primary." (p. 97) The history of the successive axiomatizations of set theory by Zermelo *et alii* end with a theory of classes proposed by Lavine. The axioms of this last theory include an axiom of Limitation of Size ("A class is of the same power as the universe of sets if and only if it is not a set.") and an axiom of Limitation of Comprehensiveness ("A class is a set if and only if its union is a set."). The latter was proposed by von Neumann, and the former, by Fraenkel.

Although the author is convincing in showing that the paradoxes were of less concern to set theorists than to say Russell or Frege, I'm not as sure as he is that Zermelo's axioms were not in some way motivated by the paradoxes. In a note to Cantor's 1899 letter to Dedekind (see [van Heijenoort *1967*, 117]), Zermelo writes: "It is precisely doubts of this kind [that a proof might involve 'inconsistent multiplicities'] that impelled the editor [i.e. Zermelo] a few years later to base his own proof of the well-ordering theorem (1904) purely upon the axiom of choice without using inconsistent multiplicities." The distinction between consistent and inconsistent multiplicities made by Cantor in this letter (an inconsistent multiplicity is a collection that leads to contradiction if one assumes that all its elements are together) as well as a definition of consistent set by Schröder in 1890 certainly preceded Russell's paradoxes. But although Zermelo wanted to make clear what his assumptions were in the 1904 proof and, argues Lavine, only wanted to avoid the use of these multiplicities in the logic of the proof, is it not also possible that a program to eliminate these incon-sistent multiplicities was being simultaneously carried out in the creation and refinement of axioms? Lavine does not rule out the possibility that Zermelo had strong intuitions about which collections were to be excluded from a mathematical theory of sets (not just from the background logic).

The remaining chapters VI through IX are devoted to the second main theme of the book: the problems of self-evidence for, and knowledge of, the mathematical infinite. The author believes that mathematics, as it is practiced, is committed to the infinite: actual infinite sets and properties regarding them are routinely used by mathematicians. But what can we, *qua* finite beings, know about the infinite?

In Chapters VI and VII Lavine surveys various epistemological stances (including several variants of phenomenological finitism), philosophies of mathematics (among them Hilbert's finitary mathematics as well the author's modification of R. L. Goodstein's primitive recursive arithmetic), and several attempts to characterize what we do in fact know about the infinite (the Quine-Putnam and Maddy indispensibility arguments, Hellman's possible structures, and Shapiro's second order logic with first order set theory). Along the latter lines, the author proposes a second order *schematic* set theory to serve as our background set theory (the second order schema occurs in the Axiom of Replacement, with a substitution rule that admits function symbols from any expansions of the language).

Chapters VIII and IX, the last two chapters, are the most original of the book. To make formal the idea of indefinitely large (finite) size, which Lavine believes to be the source of our intuitions about the infinite, the author applies Jan Mycielski's locally finite theories. I will necessarily simplify the author's exposition. Given a theory T in a first order language L, expand L to L^* by adjoining unary relation symbols Ω_p (index p is a rational number). These Ω_p's will represent indefinitely large finite sets of decreasing degrees of availability (with increasing index p). To any formula φ of L corresponds a "regular relativization" φ * gotten by relativizing all the quantifiers in φ to the Ω_p's in such a way that if, in φ, Ω_q bounds a quantifier within the scope of a quantifier bounded by Ω_r, then $r < q$. The associated theory Fin(T) is then given by the regular relativizations of the axioms of T, some axioms governing the Ω_p's (axioms of indefinitely large size), and axioms of equality. A key step in showing a natural connection between ZFC (Zermelo-Fraenkel set theory with the Axiom of Choice) and Fin(ZFC) is a theorem of Mycielski and Janusz Pawlikowski: for any first order theory T, T is consistent if and only if every finite subset of Fin(T) has a finite model (the last statement is the definition of Fin(T) as a locally finite theory). The proof can be carried out in what amounts to primitive recursive arithmetic. Lavine's explanation of how it comes to be that we have intuitions about ZFC is that we already have experience of indefinitely large finite sets, which is formalized by Fin(ZFC), and we 'extrapolate' to ZFC. "We extrapolate from Fin(ZFC) to ZFC by setting the bounds on the quantifiers — the Ωs — equal to one another to form a single domain of quantification V that is so large that it can never require enlargement in any context" (p. 316). The question of availability (which can be increased) of large sets is a context-dependent idea. The infinite is arrived at by removing the context-dependence. After some concrete examples from finitistic real analysis, the author then motivates Fin(ZFC) within finite mathematics as a theory of hereditarily finite sets (i.e., members of V_ω in the cumulative hierarchy of sets without urelements), including indefinitely large ones, with intended membership relation and successor operation. The regular relativizations of the axioms of ZFC are carefully stated and explained in this context. The regular relativization of the Axiom of Infinity is called the Axiom of a Zillion and requires the Ω_p's to be intransitive — in which case it guarantees an indefinitely large finite set with no available predecessor. All the proofs and definitions of Fin(ZFC) can be encoded and carried out in primitive recursive arithmetic, and the author argues that Fin(ZFC) can be described in a metalanguage that does not involve infinite sets (he wishes to have no hidden infinitary ideas in the process of extrapolation). Thus Fin(ZFC) is naturally motivated with finite sets and indefinitely large ones with no infinite presuppositions. The author illustrates how extrapolation explains two different conceptions of infinitary set theory — the idea of a single maximal domain arises by extrapolation from finite set theory, while the idea of a progression of domains arises by extrapolation from the model theory of finite set theory. Extrapolation of the finite theory of classes proposed earlier by the author (which does not follow automatically from the previous discussion, as it is not a first order theory), on the other hand, explains how Limitation of Comprehensiveness fits within the picture given by Limitation of Size (what was a "shotgun marriage" between the Power Set Axiom and Cantor's transfinite symbols is a harmonious fit at the finitary level).

If we accept that we do have intuition about the mathematical infinite, then Lavine's theory of indefinitely large size is plausible in explaining the finitary source of this intuition. What remains a mystery to this reader is then the question of how we might have intuition about indefinitely large size: the work in explaining the ellipses in the sequence of derived sets above seems to have been relegated to the extrapolation from the finite which can be embraced by consciousness all at once to the indefinitely large finite. This question of whether the author's theory formalizes ordinary experience of the indefinitely large together with the question of whether it can be taken to capture the intuitions that played an actual historical role cannot be answered until further historical and psychological evidence has been gathered.

This is a very engaging and interesting book. The style is sometimes a bit too conversational: several times this reader felt distracted by an aside which could have been consigned to a footnote or to a note in an appendix. But this is a minor quibble, and I would highly recommend the book to anyone interested in the history and/or the metamathematics of set theory.

References

MADDY, P. 1990. *Realism in mathematics*, Oxford, Clarendon Press of Oxford University Press.

VAN HEIJENOORT, J. (editor). 1967. *From Frege to Gödel: A source book in mathematical logic, 1879 – 1931*, Cambridge, MA, Harvard University Press.