Leon Henkin, "The Discovery of My Completeness Proofs," *The Bulletin of Symbolic Logic* **2** (1996), 127–158.

Reviewed by

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It is sometimes said, by no one in particular and by many in general, that research mathematicians develop an interest in the history of mathematics only after they are no longer active in creative original research, at the end of their careers, when they are prepared to take a long reflective look back at their accomplishments and assess how these fit with the general development of their field. The specific stimulation for Henkin's writing of this article was an invitation which he received to participate in a symposium devoted to "Histoire de la théorie des modèles". In any event, we assuredly owe much both to Henkin for penning this paper and to Hourya Sinaceur, the organizer of the symposium on history of model theory that led to Henkin's participation and the article which it bred.

This is the sort of article that anyone with a serious interest in the history of logic should welcome. It is the sort of writing that any historian of logic would be excited to read and, were it not already published, love to discover. It is a detailed personal account of the creation of a seminal part of a pioneering aspect of the recent history of modern logic. Henkin's paper is a blend of intellectual biography, careful and detailed exposition, and history. Along the way, we learn about Henkin's education, in particular what he learned about logic as a student. This in turn provides us the bonus of an insight into the status of knowledge of logic during Henkin's student days, from the late 1930s through the early 1940s, and into the pedagogy of logic during the period in question.

The central focus of this article is on the background to, preparation for, and writing of, Henkin's doctoral thesis, *The Completeness of Formal Systems*, which he submitted to Princeton University in June 1947. The thesis itself has never been published, but out of it grew several well known papers appearing in the *Journal of Symbolic Logic*, in particular Henkin's "The Completeness of the First-order Functional Calculus" (**14** (1949), 159–166), and "Completeness in the Theory of Types" (JSL **15** (1950), 81–91), and the paper "Some Interconnections Between Modern Algebra and Mathematical Logic" (in *Transactions of the American Mathematical Society* **74** (1953), 410–427). The notation which Henkin used in his dissertation and the first two of the three papers that grew out of the dissertation, bearing the influence of Church's notation, was rather more cumbersome than necessary. In the dissertation and the two papers appearing in the JSL, Henklin clung to the use of only a binary relation symbol for equality and two ternary relation symbols to formalize addition and multiplication, and used no operation symbols. Only in the third paper, published in the *Transactions*, did he abandon this "anachronistic feature" (see p. 137, fn. 16) in favor of employing a first-order calculus having operation symbols.

After a brief introduction (§1, pp. 127–128) reflecting on the nature of mathematical discovery and outlining the contents of the remainder of the paper, Henkin discusses the extent of the knowledge which he gained as a student from 1938 to 1942 (§2, pp. 128–132). This is followed by an exposition of the contents of the doctoral thesis in the version accepted by Alonzo Church (§3, pp. 135–142), and then by a recounting of the specifics and technical details of Henkin's efforts in writing the thesis (§4, pp. 142–157).

Henkin's education in logic began inauspiciously enough, with a philosophy department introductory logic course taught by Ernest Nagel in the autumn of 1938, when Henkin was a sophomore mathematics major at Columbia University. The course, fortunately, awoke Henkin's interest in logic, and led him to peruse Russell's *Principles of Mathematics*. There he discovered the principle of choice, and he notes that Russell's example, of picking out one shoe from each of infinitely many pairs in a shoe store stocked with infinitely many pairs of shoes and socks, and of choosing one sock from each pair, is the same type of difficulty that he faced in the work on his thesis. The reading of the *Principles* led in turn to a stab at reading the *Principia*. Henkin managed only the introductory material at first, but came away with an appreciation for the theory of types, which would also come to play its rôle in the work on his thesis.

The following autumn, Henkin enrolled in Nagel's advanced logic course, which focused on systems of propositional and first-order logic as presented in Hilbert and Ackermann's textbook *Grundzüge der theoretischen Logik* (1928). Although completeness was not considered in the course itself, Nagel set Henkin on an independent study of Quine's proof of the completeness of propositional logic through a reading of Quine's recently published *Journal of Symbolic Logic* article "Completeness of the Propositional Calculus" (3 (1938), 37–40). Although he could not yet comprehend the *direction* of Quine's arguments and had not yet grasped the concept of a mathematical proof, Quine's paper gave Henkin his first inkling that research was being published in logic, that mathematics instructors at universities could do research as well as teach, and also led to his recognition that he could, albeit with effort, read and comprehend the principal ideas being discussed in contemporary research in logic. One historical lesson that Henkin's remarks evoke but which Henkin neither hints at nor makes explicit, which can be drawn from Henkin's discussion thus far is that there was a time, not so long ago, when, with a good advanced logic course behind one's belt and strenuous exertions, most, if not all, logicians could read and comprehend most, if not all, of the articles published in the JSL and many could read and comprehend most of its contents. It was a time, also, when the JSL was considerably less bulky than it is today, before it was necessary to be a narrow specialist in one of the several subfields of logic in order to easily follow the literature of the branch in which one specialized.

Soon after Henkin entered Nagel's advanced class, Tarski stopped at Columbia on his trek across America from Harvard to Berkeley, and delivered a lecture on Gödel's work on incompleteness which Henkin attended. The themes on which Tarski dwelt in his talk were on undecidable propositions in the theory of types and on decision procedures that had been found for some formal systems and shown not to exist for others. The undergraduate student Henkin asked Tarski whether there could be a decision procedure to determine whether a sentence of the system studied by Gödel was unprovable.

Of the students and faculty in the mathematics department at Columbia at that time, Henkin was the only one to evince an interest in logic. When, therefore, one faculty member, F. J. Murray, who had collaborated with von Neumann, learned of the publication by Princeton University Press of Gödel's monograph *The Consistency of the Axiom of Choice and the Generalized Continuum-Hypothesis with the Axioms of Set Theory* (1940), he proposed to Henkin that they study it together. Henkin, however, was left mostly to his own devices in studying this monograph because, due to the press of his other obligations, Murray was unable to read the work and offer Henkin his guidance. What Henkin learned from his study of what

Gödel wrote in this little booklet, and what he didn't learn from what Gödel did not write about choice functions, but worked out for himself through his study of it, provided him with the tools and techniques that he used in his thesis. "This event," Henkin opines (p. 131), "was probably my most important learning experience as an undergraduate."

In the autumn of 1941, Henkin entered the doctoral program in the mathematics department at Princeton University, having chosen Princeton largely because of the presence in the department of Alonzo Church, who Henkin was given to understand was a well known logician. Church's one-year logic course, on which *Church's Introduction to Mathematical Logic* is partially built, was the entire extent of Henkin's formal graduate education before his studies were abruptly condensed by America's entry into the Second World War. One can gain, as Henkin tells us, an adequate picture of the contents of the first semester of the course by rummaging through the first four chapters of the published first volume of Church's *Introduction to Mathematical Logic* and of the second semester's contents by looking at the titles of the first three chapters of the proposed second volume as it appears in the "Tentative Table of Contents of Volume Two" in Volume I of the *Introduction*. Thus, the course covered propositional logic, first-order functional calculi and pure first-order functional calculi, and Gödel's completness theorem in the first semester, and higher-order functional calculi, second-order arithmetic (especially Peano's system), and Gödel's incompleteness theorems in the second semester. Henkin tells us (p. 123) that Church often reminded his students that the approach taken in the course is to use "'the logistic method" to study "logistic systems."' In respect to its content, it was roughly equivalent to a one-semester introductory graduate course in mathematical logic as taught today in mathematics departments or, to a lesser extent, to an advanced graduate course in symbolic logic as taught today in a philosophy department.

In May 1942, after completing the reading for his qualifying examination for his M.A, Henkin left Princeton to take up war work, first on radar and then on research relating to the development of the atomic bomb. When in March 1946 he returned to Princeton, he began a year of work on his doctoral thesis. He also joined Church's course, already a month in progress when Henkin arrived back in Princeton, on Frege's theory of sense and denotation, work that led Church to the λ-calculus. We shall momentarily see that it transpired that the use of λ-conversion broke a one-year impasse in Henkin's efforts to solve the principal difficulty he had to overcome in designing his proofs. Here is another vivid, if tacit, lesson on the state of logic in Henkin's student days and on the rapid and multifarious development of the discipline over the last half-century. Few doctoral candidates today would not marvel at, not to mention exude envy for, the fact that with only a year of undergraduate logic and a year and a half of formal graduate study in logic, Henkin could produce such solid, deep and significant results as he did, never mind even begin to *think* about writing with that spare, by todays's standards barebones, formal academic background in logic.

In §3, Henkin describes and sketches the main points and results of his thesis. The first part of the thesis contains his proof of the completeness of first-order logic and gives some applications of the completeness theorem to several first-order systems. §4 of the article under review is an account of the discovery of the completeness proof as stated in Part I of the thesis. In Part II of the thesis, entitled "Applications to algebra", Henkin began to make his contributions to what came to be known as model theory. Here, for example, he establishes an applied first-order calculus for ring theory, and applies the compactness property for first-order logic to a new proof of Stone's Boolean representation theorem. This result is then generalized to algebraic structures. Part III of the thesis, called "The calculi of higher order", introduces structures called "general models" which are used to interpret the pure functional second-order calculus $\mathfrak{L}_2$ obtained from pure first-order functional calculus by permitting propositional and predicate variables to appear in quantifiers. The aim of this section, seen as the apex of the thesis, was to shed "new light" on Gödel's incompleteness results. In particular (as Henkin says, p. 140), the generalized completeness theorem for $\mathfrak{L}_2$ shows, in light of Gödel's incompleteness results for $\mathfrak{L}_2$, that there certainly must be non-standard general models for $\mathfrak{L}_2$. "The proof of that completeness theorem," he adds (p. 140), "gives a general method for constructing such non-standard models."

Part IV of the thesis, called "Applied systems of logic", begins by noting that the principle of compactness for higher-order formal languages is unlikely to find applications in various parts of mathematics like the applications which it found for first-order languages, because formal definitions for higher-order concepts, e.g. the concept of *topological space*, change their meaning when interpreted with respect to a general model (see p. 141). The thesis does not provide presentations of new results about structures that are considered in mathematics. Most of the remainder of the thesis is instead given over to descriptions of applied formal languages for set theory and for number theory and notes the possibility of finding non-standard models for axiom systems used in these areas, giving examples of how, and under what conditions, non-standard models of number theory arise (see p. 142). The thesis concludes with a philosophical note concerning non-standard models suggesting "that Gödel's incompleteness results can be considered as stating a fundamental inability to communicate the kind of mathematical systems we are examining, rather than an inability to establish facts about such a system" (p. 142).

In §4 of the article (especially pp. 142–153), Henkin recounts the tribulations and triumphs that he experienced in working on and obtaining the results that were presented in his thesis. One of these centered around the use of λ-conversion to break the afore-mentioned one-year impasse in his efforts to show that there are choice functions which select an element from each nonempty set of reals, an impasse that caused him terror and had him on the verge of quitting school (see pp. 149–150). The technical details of how this led to the remainder of the puzzle falling into place and of how the completeness results were proven forms the backbone of §4 (pp. 150–153).

The article concludes with three observations:

(A) Several points which Gödel added to the second printing (1951) of *The Consistency of the Axiom of Choice and the Generalized Continuum-Hypothesis with the Axioms of Set Theory* would have probably inhibited Henkin, had he known about them in 1946 when he began work on his thesis, from starting work on the problem that led to his discoveries. This is the conjecture that existence of a nameable choice function for nonempty sets of reals is consistent with the axioms of Gödel-Bernays set theory. Paul Cohen's proof of the independence of the axiom of choice by forcing led Solomon Feferman to show that it is consistent with the axioms of set theory, including the axiom of choice, that there is no formula of Gödel-Bernays set theory which defines a well-ordering of the reals. It follows from this that the conjecture that Henkin "fruitlessly tried to prove" is at least consistent (p. 154);

(B) The structure and presentation of the results of the dissertation hide, rather than reveal, the history of Henkin's discoveries. This leads Henkin to wonder about the nature of the task of the historian of mathematics when they have no account to rely upon, such as the one presented here, in their efforts to reconstruct the work of logicians. It is consequently suggested that the path of mathematical discovery is not a directed graph (a point, be it noted, that was made, in different words, by the reviewer — see his "Distortions and Discontinuities in Mathematical Progress: A Matter of Style, A Matter of Luck, A Matter of Time, . . . A Matter of Fact," *Philosophica* **43** (1989), 163–196), and that it is sometimes crucial for the historian to get inside the "black box" and behind the visible public, not to say published, documentation, in order to ascertain the actual path of discovery. (pp. 154–156);

(C) The difference between Henkin's completeness proof for first-order logc and Gödel's is briefly noted and explained, and Henkin notes a simplification of the procedure that he would in hindsight come to make (p. 157).

This reviewer has only two sets of contradictory "complaints" in connection with the article under review. First: the reviewer wishes that every logician who has made a seminal contribution to the development of his subject would have the foresight, before age caught up with him or her, to write such an account as this so that we can understand the work, influences, and thoughts behind their achievements; at the same time, the reviewer worries that, were everyone to write such articles about their work, there would soon be little left for the historian to do. Second: the reviewer regrets that this article was not published in *Modern Logic*, while yet being extremely pleased that it did appear in a general logic journal such as the *Bulletin*, accessible to all logicians, including research logicians, in the sincere hope and expectation that its appearance will increase, promote, and stimulate an interest in and appreciation for the history of the subject which can only be to the general good and not merely to the benefit of *Modern Logic*. To conclude: in all, we owe to Henkin a new gratitude for writing on his discovery of his completeness proofs to add to the equal debt already owed him for the discoveries themselves.