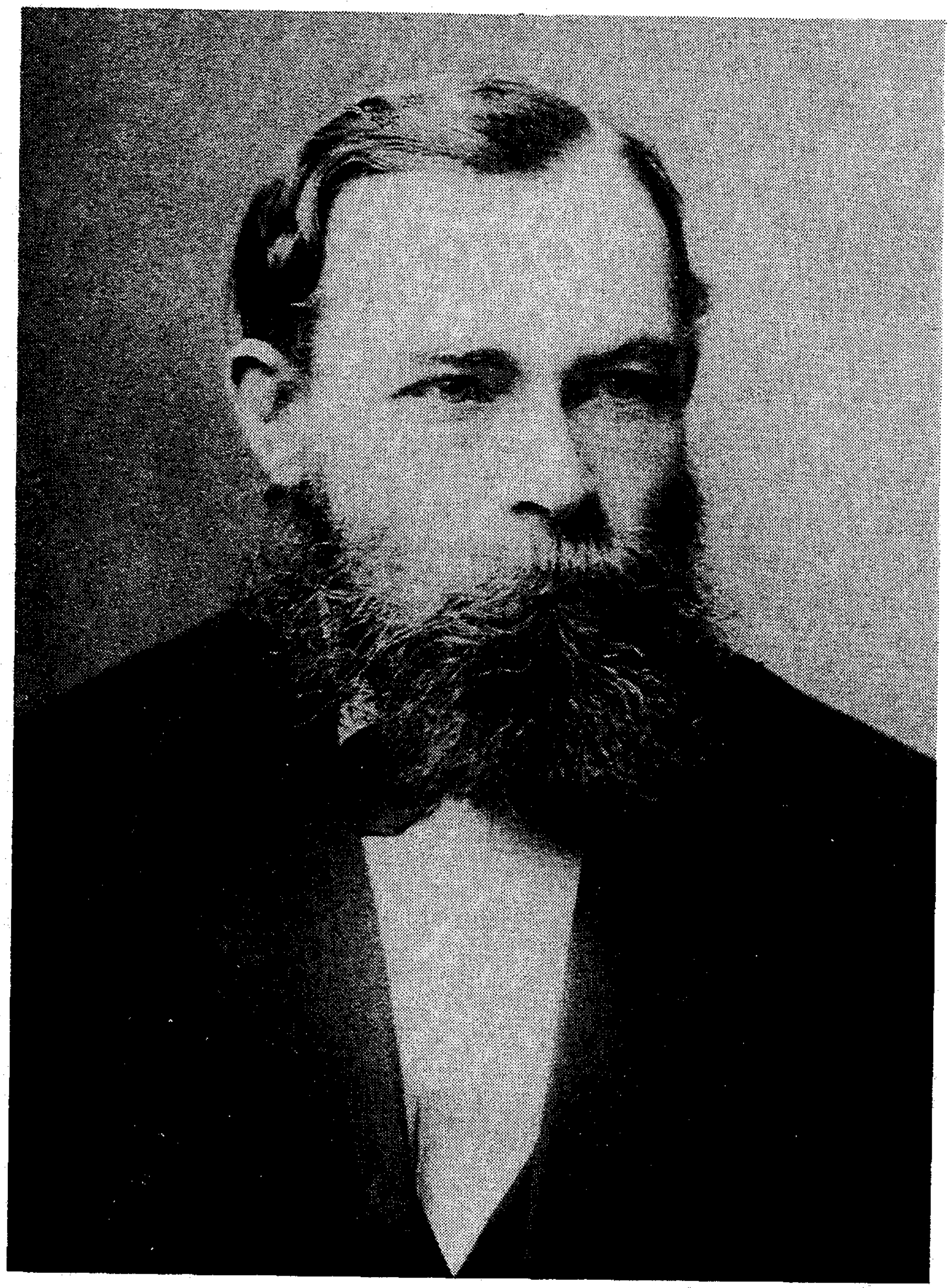

Gottlob FREGE
(1848 – 1925)

EDITORIAL

This issue of *Modern Logic*, guest-edited by Christian Thiel, is dedicated to the publication, one hundred years ago, of volume 1 of Gottlob Frege's *Grundgesetze der Arithmetik.* Frege regarded this work as the culmination of the work of his lifetime up to that time. More importantly, he originally intended the book to be a rigorous and detailed presentation of his efforts, first begun in the *Begriffsschrift* of 1879, to provide a fully worked-out system of mathematics (especially of arithmetic and analysis) founded strictly and exclusively upon the propositions of logic. Thus, Frege opens the volume with the introductory words: "In this book there are to be found theorems upon which arithmetic is based, proved by the use of symbols which taken together I call Begriffsschrift [concept-script]." In fact, much of the arithmetic is lacking in the *Grundgesetze*. Instead, we are presented with the logical apparatus Frege that deemed requisite to defining the concept "Number" — meaning natural number, along with that definition.

Frege's *Begriffsschrift* (*1879*) relied on iteration of applications of the proper ancestral relation to build up the sequence of natural numbers starting from 0 and then, in the *Grundlagen* (*1884*, 93)) used that concept to justify mathematical induction In particular, consider *Nx* ("*x* is a natural number"). The proper ancestral relation is defined (using the more familiar Peano-Russell notation rather Frege's own Begriffsschrift-notation) as $R^{**}xy \equiv R^{*}xy \cdot x \neq y$, where R^{*}xy is the ancestral relation and given by

$$R^{*}xy \equiv \forall S\,(\mathrm{x} \in \mathrm{S} \cdot \forall u\,(u \in \mathrm{S} \cdot \ Ruv \supset v \in S) \supset y \in S)\,.$$

We begin with the definitions

(1) $Nx \equiv \forall S\,(S \text{ is inductive} \supset x \in S)$
(1′) $Nx \equiv \forall F\,(F \text{ is inductive} \supset F(x))$ (giving us $F(0)$)
(1″) $Nx \equiv \forall F\,(F(0) \cdot \forall y\,(F(y) \supset F(y')) \supset F(x))$ (giving us $F(n)$, $F(n')$, ...)

We can prove induction with definition (1″).

In §26 of the *Begriffsschrift*, the proper ancestral is given as $R^{*}xy \equiv R^{**}xy \vee x = y$. Having determined that $Nx \equiv \forall F(F_0 \cdot \forall x\,(Fx \supset Fx') \supset Fx)$, we now obtain mathematical induction:

$$\forall F(F_0 \cdot \forall x\,(Fx \supset Fx') \supset Fx) \supset \forall x\,(Nx \supset Fx)).$$

Replacing *Nx* with its equivalent, we then get

$$\forall F(F_0 \cdot \forall x\,(Fx \supset Fx') \supset Fx) \supset \forall x\,(\forall F(F_0 \cdot \forall x\,(Fx \supset Fx') \supset Fx) \supset Fx))$$

for mathematical induction.

There is no definition of *Nx* in the *Begriffsschrift*,, but it does appear in Part II of the *Grundlagen.* There, *E*(*F*,*G*) are *gelichzahlig* (equinumerate) where φ is one-one and

$$E(F,G) \equiv \exists\varphi\forall x\,(Fx \supset \exists y\,(Gy \cdot y = \varphi(x)).$$

Starting then from $N_0(F)$ as the set of objects falling under [the concept] F as the number 0, we are able to obtain

$$\begin{aligned} N_0(F) &\equiv \forall x \sim Fx \\ N_1(F) &\equiv \exists x\,(Fx \cdot \forall y\,(Fy \supset y = x)) \\ &\;\;\vdots \\ N_{n+1}(F) &\equiv \exists x\,(Fx \cdot N_n(\lambda y(Fy \cdot y \neq x)), \end{aligned}$$

where $N_{n+1}(F) \equiv \exists x\,(Fx \cdot N_n(\lambda y(Fy \cdot y \neq x))$ is a recursive definition of the successor of n for any natural number n and $N_n(F)$ means that the concept F is n-numerous. (Notice that number here is defined by a second-order predicate. That will soon lead to very well-known problems.)

From the philosophical point of view, *Grundgesetze* was intended in large measure to be a defense of the position, later called logicism, according to which mathematics is reduced to, or built up from, logic. For this reason, it is difficult, if not impossible, to fully separate the mathematical and philosophical aspects of Frege's work. In the *Grundgesetze* we saw the difficult notion of "falling under a concept" deployed to define the concept of Number and to help generate the sequence of natural numbers. Here too, the issue of the distinction between sense (*Sinn*) and denotation (*Bedeutung*) are brought to bear in developing the notion of a truth-value. Hence philosophy of language and mathematics stand side-by-side in what Frege intended as his *magnum opus* and as the fulfillment of Leibniz's program to develop logic as a *mathesis universalis*. The issues which Frege raised in the *Grundgesetze* (as well as such of his philosophical essays as "Über Sinn und Bedeutung" and "Über Begriff und Gegenstand" published in 1892) have been an active part of philosophy ever since Bertrand Russell brought them to the attention of the scholarly community in 1903 in his *Principles of Mathematics*, and remain so today, as reflected in many of the articles appearing in this issue.

In the paper "The background of Frege's Identifiability Thesis" which he was preparing for this special issue (to appear in a future issue), Christian Thiel takes up a topic from *Grundgesetze* itself. For more than twenty years, §10 of this work, including the line of argument that Dummett has called "Frege's permutation argument", has proved a first-rate teaser. Praised as a skillful thought-experiment in the beginning, its validity was denied by Peter Schroeder-Heister in 1984 and 1987, and defended by Adrian W. Moore in 1983 and by Moore and Andrew Rein in 1986 and 1987. Little attention has been given to the purpose the argument was to fulfil. Frege devised it in support of his subsequent identification of "the True" and "the False" with two particular courses-of-values (*Wertverläufe*). Closer inspection of Frege's procedure shows that his steps follow exactly the construction pattern of an extension of a field as developed in nineteenth century algebra. Although this discovery does not terminate the controversy mentioned, it divests the unbeloved paragraph of much of its mysteriousness, and last but not least shows Frege —

perhaps to the surprise of some of his recent critics — as handling what was then a hypermodern mathematical technique in a masterly manner.

The reception of Frege's work itself makes an interesting history. Russell opened his exposition of Frege's work (in "Appendix A. The Logical and Arithmetical Doctrines of Frege" of the *Principles of Mathematics*) by stating that: "The work of Frege...appears to be far less known that it deserves...." He then explains that "Frege's work abounds in subtle distinctions, and avoids all the usual fallacies which beset writers on Logic. His symbolism, though unfortunately so cumbrous as to be very difficult to employ in practice, is based upon an analysis of logical notions much more profound than Peano's, and is philosophically very superior to its more convenient rival." The reviews of the 1879 *Begriffsschrift* at the hands of Schröder (*1881*) and others were generally unfavorable. The Begriffsschrift-notation by itself was enough to hamper acceptance. Its unwieldiness was one of Ernst Schröder's central complaints, and the style of mathematical logic represented by the function-theoretic approach as presented by Frege and Russell was not widely accepted until Russell replaced Frege's notation with an adaptation of Giuseppe Peano's more compact and typographically aesthetic notation. But Schröder and others had more contentual complaints as well. Schröder, for example, concluded his review of the *Begriffsschrift* by expressing the opinion that Hermann Grassmann had already completed much of what Frege had explicitly stated he had set out to do in the *Begriffsschrift*. Paul Tannery (*1879*) was even more critical, asserting that the *Begriffsschrift* consisted of little more than an explanation of the symbols used. The problem of the reception of Frege's work by his contemporaries is represented in this issue by Benjamin S. Hawkins, Jr.'s article.

When, a full decade after its appearance, Russell brought the *Grundgesetze* and the rest of Frege's published scientific corpus to the renewed public attention of the mathematical and philosophical communities, he focussed concern on the *Grundgesetze*'s Basic Rule V. This is the famous — or infamous — rule that allows a function to serve as the indeterminate argument of another, higher-order, function, and in which Russell therefore first publicly located the paradox which now bears his name. Prior to publicly announcing the paradox in the famous "Appendix A" of the *Principles of Mathematics* (along with his first effort at its solution, "Appendix B. The Doctrine of Logical Types"), Russell notified Frege privately, in the now famous letter of 16 June 1902. "For a year and a half," Russell wrote in the letter, "I have been acquainted with your *Grundgesetze der Arithmetik*," after which he expresses agreement with Frege's logicist program and exposes the paradox.[1] In this letter, too, Russell announces the near-completion to his *Principles* and his intention to provide a complete exposition of Frege's work in the *Principles*. In his reply of 22 June 1902, Frege generously thanked Russell, expressing surprise and consternation at Russell's discovery, and he immediately undertakes to look for the cause of the difficulty which Russell spotted, finding the culprit to be in the transformation of the generalization of an equality into an equality of courses-of-values, while adding that Russell's discovery of a contradiction within his system shakes the very foundation on which he attempted to build arithmetic. A reproduction of the whole of Frege's letter appears in this issue, courtesy of

[1]The origin, and even the chronology of the origin, of the Russell paradox has lately been a matter of dispute (see, e.g., Anellis (*1991*) and Garciadiego (*1992*)).

the Insitut für mathematische Logik und Grundlagenforschung of the Westfählische Wilhelms-Universität in Münster.

Recent scholarship, however, has begun to seriously examine, if not universally adopt, the position that Basic Rule V is not crucial to Frege's enterprise, and that his program for developing arithmetic axiomatically can be carried in the *Grundgesetze* without appealing to the rule. Thus, for example, Richard G. Heck, Jr. calls for a reassessment of Frege's treatment of arithmetic and argues that arithmetic can be developed in the *Grundgesetze* from Hume's Principle and need not depend upon Basic Rule V, but that Frege failed to develop arithmetic axiomatically in the *Grundgesetze*. Thus, the work whose hundredth anniversary we celebrate in the current issue can be expected to continue to play its role the philosophical and mathematical history of logic for perhaps another hundred years.

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