Volume 2, no. 3 (April 1992)

JEAN VAN HEIJENOORT'S CONTRIBUTIONS TO PROOF THEORY AND ITS HISTORY*

IRVING H. ANELLIS

Modern Logic Publishing
Box 1036, Welch Avenue Station
Ames, Iowa 50010-1036, USA

Abstract. Jean van Heijenoort was best known for his editorial work in the history of mathematical logic. I survey van Heijenoort's contributions to model-theoretic proof theory, and in particular to the falsifiability tree method. This work of van Heijenoort's is not widely known, and much of it remains unpublished.

AMS (MOS) 1980 subject classifications (1985 revision):
Primary: 03B10, 03C07, 03B35, 03C35, 03F07
Secondary: 03-03, 01A60, 03F25

* Revised and corrected English version of the author's "La obra de Jean van Heijenoort en el campo de la lógica: sus aportaciones a la teoría de la demonstración," © Mathesis 5 (1989), 353-370.

§1. INTRODUCTION.

Jean van Heijenoort was best known as a historian of logic. He was famous for his editorial work, and especially for his anthology *From Frege to Gödel* (*1967*), which, as a representative documentary history of modern logic during the formative period of 1879-1931, has attained, in the words of Solomon Feferman (Feferman *1986*; Feferman & Feferman *1987*, p. 5), "the status of a classic." Not very well known, however, is van Heijenoort's non-historical work in logic, which consisted primarily of results in model-theoretic proof theory and which circulated in the narrow circle of his students and colleagues and remain largely unpublished. It is not known why these technical contributions remained unpublished, although van Heijenoort's well-known perfectionism may have played a prominent role in his decision that the writings in which these results appeared did not satisfy his standard of excellence. Many of the papers in this category circulated in manuscript form; and even those which circulated in typescript were frequently only drafts. It is the object of this paper to survey the technical content of van Heijenoort's contributions to model-theoretic proof theory. A physical description of the unpublished works in which these contributions were made was given in (Anellis *1988*).

In order to understand the choice of topics on which van Heijenoort worked, it is necessary to view his technical interests in the history of logic. Trained as a geometer and topologist (his doctoral thesis was in the field of convex sets), van Heijenoort traced questions in metamathematics and foundations of mathematics to problems of axiomatization of mathematics. He saw the development of quantification theory, beginning with the work of Frege, as the crucial factor in the history of mathematical logic. Van Heijenoort understood the primary goal of quantification theory to be the elucidation of the concepts of *consistency*, *completeness*, and (*being a*) *proof*. In this sense, van Heijenoort's technical work belongs to the tradition of Hilbertian metalogic, elaborated by Hilbert and Bernays as *Beweistheorie* as the logical study of the proofs of logic. Let us begin, then, by reviewing van Heijenoort's study of the relevant history.

§2. VAN HEIJENOORT'S VIEW OF THE HISTORY OF PROOF THEORY

In *El desarrollo de la teoría de la cuantificación* (*1976*) van Heijenoort gave an exposition and historical analysis of the theory of quantification from a metamathematical perspective. He argued that quantification theory is a "family of formal systems" ("*la teoría de la cuantificación es una familia de sistemas formales*;" (*1979*, p. 7)), the creator of which was Frege in the *Begriffsschrift*. The family members of quantification theory are the axiomatic method, Herbrand quantification, the Gentzen sequent calculus, and natural deduction as developed by Jaśkowski and Gentzen. The axiomatic method has two distinct branches, Frege-type systems and Hilbert-type systems. A Hilbert-type system is distinguished by being simply a set of well-formed formulae (wffs), including a list of axioms, a set of "rules of passage", that is derivation rules, for which a proof is a sequence of wffs, the last wff of the sequence being the formula which is proven. A Frege-type system is a formal language ("Begriffsschrift") containing an arbitrary set of axioms, a set of equivalence and inference rules, and in which nothing exists outside of proofs. These represent for van Heijenoort the four principal methods of approach to first-order predicate calculus. In *Desarrollo*, van Heijenoort traced the mutual relations among these four approaches, and traced their histories. He examined each in its classical, intuitionistic, and minimal versions, and pointed out the strengths and weaknesses of each.[1] Thus, for example, he noted that Herbrand's method is particularly suited for use with computers, but is not easily generalized to second-order logic. However, he showed in his paper on *Herbrand* (*1975b*, p. 6) that we can establish the constructive equivalence of a second-order Herbrand formula to a classical second-order formula, provided the formula $(\forall x)\ (\vdash_{Q} F \Leftrightarrow \vdash_{QH} F)$, properly gödelized, that is having an infinite list $\{\underline{0}, \underline{1}, \underline{2}, \ldots\}$ of variables of Q not occurring in F, can be shown to be provable in primitive

[1] As presented by Johansson (*1937*), a "minimal version" is a reduced intuitionistic formalism, in which only one of (1) $\vdash (\neg\, a \vee b) \rightarrow (a \rightarrow b)$ or (2) $\vdash (a \rightarrow b) \rightarrow (\neg\, a \vee b)$ holds.

recursive arithmetic, where Q is classical quantification theory and QH is Herbrand quantification theory.

In *Desarrollo*, van Heijenoort treated the principal methods of quantification theory proof-theoretically. The axiomatic method attains results based on the concept of *formal system* and provides an analysis of theorems, but is not yet itself a study of proofs. For Herbrand's system, quantifier-free formulae can be obtained effectively from quantified formulae, such that these quantifier-free formulae are sententially valid, by using Herbrand expansions. Thus, Herbrand helped introduce a new conception of *validity* into logic, where for Löwenheim the essential consideration was still *satisfiability*, or validity invariant with respect only to a particular model.

Gentzen's work in the sequent calculus rests on the results given by Herbrand. Herbrand's Fundamental Theorem, for example, can be understood to be a special case of Gentzen's *verschärter Hauptsatz*, and Gentzen's *Mittelsequenz* corresponds to Herbrand's valid disjunction D_k. Beyond that, Gentzen also gives an analysis of the sentential parts of the proof of validity. Thus, van Heijenoort was particularly interested in the ways in which proofs are carried out in the axiomatic, Herbrand, Gentzen sequential, and natural deduction methods. Indeed, it could be said that, for van Heijenoort, quantification theory is a family of methods of logical deduction.

The various methods for quantification theory, according to van Heijenoort, taken together, represent an evolution or development (*desarrollo*) of quantification theory, starting from the definition of the Hilbert program and Herbrand quantification, through the Gödel incompleteness results, to the Gentzen sequences and natural deduction, rather than provide an opportunity for possible conflict. The Hilbert program of metamathematical study of proofs arose just because the axiomatic method fails to study proofs even while it provides an analysis of theorems. It was Hilbert who undertook to define and carry out systematically the construction of mathematics within his system. This required a fully developed concept of proof, for the Hilbert program had two aspects: to define the technical apparatus which would permit a finitist construction of all of mathematics, and to ensure that the set of (mathematical) sentences derived within the axiomatic system was consistent. Frege's *Begriffsschrift*-

program, however, focussed almost exclusively on only part of the first aspect of the Hilbert program, namely the attempt to derive all of mathematics from the *Begriffsschrift*'s logical apparatus.

Consistency and completeness were raised as questions for quantification theory as soon as the universality of logic was proclaimed *and* deductive validity replaced satisfiability as the defining characteristic of *being a proof.* Universality, raised as an issue by Frege, required that all of mathematics should be constructed within the logical theory presented by the *Begriffsschrift*. It also required that the *Begriffsschrift* theory deal with what we now call "metamathematical" problems, such as completeness and consistency, since there is no extrasystematic or metasystematic apparatus to be distinguished from the system. Frege also required that every function in the system be defined for every argument in the universe of the system's syntax, that is, that every object in the semantics of the system be in the range of every function. Thus, there was no longer a question of restricting a theory to a select model. Thanks to Frege, the model was the universal domain – *the Universe* – rather than some arbitrary domains that acted as interpretations for submodels of the universal domain. This universal domain in effect contained only two objects, the True and the False. In practice, every object of the universe was, according to the assignment of truth-values, an element of either the True or the False. Now, as understood semantically by van Heijenoort, a formula (or theory) is satisfiable if there is some (at least one) assignment of truth-values which makes the formula (or theory) true, and valid if the formula (or theory) is true for every assignment of truth-values. These definitions by van Heijenoort of satisfiability and validity were based on Löwenheim's work, and in particular the realization that (a) a formula may be valid in some domain but false in another, and that (b) a formula may be valid in every finite domain but not valid in every domain.

Van Heijenoort's definition of proof was in essence model-theoretic. If Φ is a sequence of formulae $F_0, F_1, ..., Fn, Q$, then Φ is a proof of Q provided the deduction theorem holds, according to which $\Phi \vdash Q$ if and only if $\Phi \rightarrow Q$ and

$$(F_0 \wedge F_1 \wedge \ldots \wedge F_{n-1}) \rightarrow (F_n \rightarrow Q)$$

$$\vdots$$

$$(F_0 \rightarrow F_1 \rightarrow \ldots \rightarrow F) \rightarrow Q$$

The proof is *satisfiable* if there exists a model of $\Phi \vdash Q$ for which $\vDash \overline{\Phi} \rightarrow Q$ for at least one assignment of truth-values to Φ, and the proof is *valid* if $\Phi \rightarrow Q$ for every such assignment of truth-values to Φ. Defining a proof as an extended formula built up from a sequence of formulae, van Heijenoort's model-theoretic approach makes no distinction between deductive validity, that is the validity of proofs, and the validity of formulae. Extending these definitions, he was able to assert that *satisfiability* can be understood as *validity with respect to a specific model*, while *validity* can be understood as *satisfiability invariant with respect to any particular model.* The contrapositive gives us validity invariant with respect to a particular model as our definition of satisfiability.

In his comments on the work of Löwenheim and Herbrand, van Heijenoort stated (in *From Frege to Gödel* and elsewhere) that Herbrand's work on elucidating the concept of *proof* for Hilbert's axiomatic system was inspired by questions raised by the Löwenheim-Skolem theorem. To the two theses presented by van Heijenoort in *Desarrollo* that quantification theory is a family of formal systems, and that the four principal theories, rather than being in competition, represent a natural development, I add a third (introduced in *1978*; *1979* and detailed in *1991*), namely that the technical developments in Hilbert-type systems, including the development of *Beweistheorie* by Hilbert and Bernays, and the development of alternative theories of quantification, are primarily due to questions raised about the Löwenheim-Skolem theorem.

It is clear from Herbrand's own comments in his (*1930*) thesis *Recherches sur la théorie de la démonstration* that his investigations were undertaken to clarify the concept of *being a proof* for a Hilbert-type quantification system. In the introduction to his *Recherches*, Herbrand spoke of the recursive method to "prove that every true proposition has a given property *A*" (see the translation by Goldfarb *1971*, 49), and he immediately tied this to the finitist limit on recursive proofs enunciated by Hilbert. For Herbrand, this finitist limit challenges the transfinitist proofs of Löwenheim, in terms of $\aleph_0$-satisfiability, and requires that Löwenheim's infinite conjunction be reinterpreted as Herbrand expansion, the basis for Herbrand's method of quantification. Thus, as van Heijenoort stated (*1967*, 526) in his introduction to Herbrand's *Recherches*, "Herbrand's work can be viewed as a reinterpretation, from the point of view of Hilbert's program, of results of Löwenheim and Skolem," and that Herbrand's fundamental theorem is, as Herbrand himself stated (*1931*, 4) in his paper *Sur la non-contradiction de l'arithmétique*, "a more precise statement of the Löwenheim-Skolem theorem."[2]

It was, then, Herbrand, working with applications of Löwen-heim's concepts to Hilbert's system, who initiated the shift from satisfiability to validity, and Hilbert who explicitly made *Beweistheorie* a fundamental task for the logician.

§3. HERBRAND QUANTIFICATION

For van Heijenoort, Herbrand is a major figure in the history of logic, and he devoted much attention to the work of Herbrand, defending Herbrand against such critics as Fraïssé, whose criticisms of Herbrand's concept of validity had already been dealt with by van Heijenoort in his edition of Herbrand's *Écrits logiques* (van Heijenoort *1968*) and in the introduction to Herbrand's "Sur la non-contradiction de l'arithmétique" (in van Heijenoort *1967*; see Goldfarb also *1975*), for example.

In his "Préface" to Herbrand's *Écrits logiques*, van Heijenoort (*1968a*, 1-2) briefly traced the history of the development of quantification theory, with special emphasis on Herbrand's role as the focal point in that history. Van Heijenoort pointed out in particular that Herbrand studied Löwenheim's treatment of satisfiability for Hilbert's axiomatic system and

[2] For a more detailed consideration of the role of the Löwenheim-Skolem theorem on the work of Herbrand and on the rise of quantification theory from the proof-theoretic perspective, see (Anellis *1991*).

in *Recherches* generalized Löwenheim's (*1915*) results to validity by showing how to obtain, from the satisfiable quantified formulae of Hilbert's system, quantifier-free formulae that are sententially valid. Thus, if F is a formula in Hilbert's system which is satisfiable, then it is provable. Using methods developed by Löwenheim and strengthened by Skolem, F is rewritten in Skolem normal form as a new formula F'. Employing the method now known as Herbrand expansion, quantifiers are eliminated from F' to obtain a quantifier-free formula F_Q, where $F = \mathrm{Exp}[F', D]$, i.e. F_Q is the Herbrand expansion of F', and D is the domain containing the elements that are terms of Hilbert's quantification theory.

A quantifier of a classical formula F is *existentialoid* (*universaloid*) if it would become existential (universal) if F were put in prenex form. In the matrix of F, each existentialoid variable y is replaced by a functional term whose arguments are the universaloid variables that are superior to y, where a variable x is superior to a variable y in case the quantifier binding y is in the scope of the quantifier binding x.

The existential quantifiers are eliminated by *Herbrand disjunction*, so that

$$\exists xFx = (F(x/t_1),D) \vee (F(x/t_2),D) \vee \ldots \vee (F(x/t_k),D)$$

where D is a k-ary model and $t_1, t_2, \ldots, t_k$ are the terms of D that are arguments for the functions of the formula of Hilbert's system; and universal quantifiers are eliminated by *Herbrand conjunction*, so that

$$\forall xFx = (F(x/t_1),D) \wedge (F(x/t_2),D) \wedge \ldots \wedge (F(x/t_k),D)$$

for the k-ary model D and its terms.

When he came to develop the apparatus for falsifiability tree proofs, van Heijenoort used this distinction between satisfiability and validity and showed how these concepts can be applied both to formulae and to proofs.

Thus, in *Desarrollo*, van Heijenoort studied the members of the quantification theory family of formal systems as attempts to elucidate the concepts of *validity* and satisfiability and to develop the technical apparatus for carrying out valid proofs of logic. Thus, he not only dealt there with the historical development of quantification theory, but also made comparisons of the relative strengths and weaknesses of the various family members.

It was also in *Desarrollo* that van Heijenoort presented, in the context of his evaluations of the family members of quantification theory, a defense of the tree method. Indeed, nearly all of van Heijenoort's technical, that is to say non-historical and non-philosophical, writings were devoted to developing the tree method as a powerful method of logical deduction and validity checking. It is these papers that largely remain unpublished (although they were distributed to students and colleagues).

§4. Brief History of the Tree Method

It is probable that van Heijenoort first learned about Smullyan trees around 1964-1965, at precisely the same time that Smullyan was beginning his work developing the tree method.

Richard Jeffrey reported (*1987*) that he first encountered van Heijenoort in 1964-1965, at a time when he and Jeffrey both were in New York City, Jeffrey teaching at City College of New York and van Heijenoort at New York University and Columbia University. The two men met several times during this period as members of an informal group that convened occasionally to discuss logic and philosophy. It was also during this period, perhaps in 1964, that Jeffrey met Smullyan in New York and possibly also attended Smullyan's lectures at Princeton University. Jeffrey immediately became an enthusiastic supporter and proselytizer for the tableau method, and in particular of the so-called "Smullyan tree" as a one-sided Beth tableau.[3] It is safe to suppose then that this was when van Heijenoort, too, first learned about Smullyan trees, either directly from

[3] So-called because Zbigniew Lis developed a full-fledged analytic tableau or tree method by 1962 independently of Smullyan, and published his method in (*1960*) at a time when Smullyan was just beginning his work.

Smullyan or through Jeffrey.[4] In any event, the first important document we have available in van Heijenoort's hand on the tree method dates from 1968.

In the paper "On the Relation Between the Falsifiability Tree Method and the Herbrand Method in Quantification Theory", van Heijenoort (*1968b*) showed how the falsifiability tree method could easily be adapted to Herbrand expansion to test the validity of quantified formulae whether those formulae were in prenex form or not. Van Heijenoort (*1970*) is the published abstract of this result. Thereafter, van Heijenoort was a strong proponent of the falsifiability tree method, a method which, like its immediate precursors, united model theory with proof theory. Van Heijenoort's technical writings were all aimed at broadening and deepening the scope and capabilities of the tree method.

The tree method was first developed by Lis and presented by him in its first form in 1960, based directly upon Beth's deductive and semantic tableaux. However, Lis's paper, published in Polish with brief English and Russian summaries that gave no hint that a new and simpler method than that found in Beth's semantic tableaux was being presented, has largely been ignored. The method was reinvented by Hintikka and Smullyan from Beth's semantic tableaux. This reincarnation differed from Lis's development by being based on the method of model sets developed by Hintikka (*1955*; *1955a*) as well as Beth's semantic tableaux (*1955*; *1959*; *1962*). Hintikka's work on the model set method in propositional logic (Hintikka *1953*) already mentions the basic idea of the tree method.

Hintikka's model set is a set of formulae that can be interpreted as a partial description of a model in which all formulae are true. A proof of a formula F in Hintikka's theory is a failed attempt to build a countermodel $\sim F$ to F. Beth's tableau method does much the same thing, except that, for the tableau method, a proof of a formula $F \rightarrow G$ is a failed attempt to build a countermodel of $F \rightarrow G$ by describing a model in which F is true but G is not.

4 Smullyan (*1987*) states that he had met van Heijenoort several times, but does not state whether they ever discussed the tree method.

For Beth's method, it is necessary to keep track of both true formulae and non-true formulae. Formulae are listed in tabular form; all true formulae and their derivations are collected in the left-hand column of the table, all non-true formulae and their derivations in the right-hand column. A *tree* is a one-sided (left-sided) tableau in which all formulae are true. There are a small number of tree decomposition rules, one for each of the truth-functional connectives which we choose for our base, and one each for the universal and existential quantifiers.[5] Thus, for example, if we follow Jeffrey and provide decomposition rules for (disjunction, conjunction, material implication, material equivalence), along with the corresponding rules for the negation of these connectives, and double negation, we obtain the following rules:

```
1.  A ∨ B         2.  A ∧ B        3.  A ⊃ B        4.  A ≡ B
    /   \               |              /   \             /   \
   A     B              A            ~A     B           A    ~A
                                                        |     |
                                                        B    ~B

5.  ~~A
     |
     A

1′. ~(A ∨ B)      2′. ~(A ∧ B)     3′. ~(A ⊃ B)     4′. ~(A ≡ B)
        |               /   \             |              /     \
       ~A             ~A    ~B            A            ~A       A
        |                                 |             |       |
       ~B                                ~B             B      ~B
```

For the universal and existential quantifiers, we introduce the rules

[5] A *base* is the smallest set of connectives chosen for a deductive system and in terms of which the remaining connectives are defined. For example, the base in the first edition of the *Principia mathematica* is $\{\sim, \vee\}$, the Sheffer stroke in the second edition; and the base for Frege's *Begriffsschrift* is $\{\sim, \supset\}$.

$\underline{\forall x\ Gx}$
$G(x/\mu_0)$

.

.

.

$G(x/\mu_k)$, where $\mu_0, ..., \mu_k$ are mutants (permissible substitution instances) of the bound variable,

and

$\underline{\exists x\ Gx}$
$G(x/\nu)$, where ν is a mutant of the bound variable provided ν is new to the path in which it occurs.

Tree decomposition rules may be applied to any formula which is nonbasic. A formula is called *basic* if it is atomic or the negation of an atomic formula, that is, if it contains no subformulae, and hence no connectives to which decomposition rules can be applied.

Let Φ be a set of formulae at the initial node of a tree. By application of tree decomposition rules to the nonbasic formulae of Φ, we obtain successor nodes, each containing some subformulae of (one of) the nonbasic formulae of Φ. A path of a tree, or sequence of such nodes, is *terminated* or *finished* if tree decomposition rules have been applied to every nonbasic formula in the path. A path is *closed* if there appears a formula F at some node n in the path and its negation $\sim F$ occurs at some successor node n' of n in the same path; otherwise the path is *open* (*nonclosed*). A tree is closed if each of its paths is closed. The tree for Φ is a *proof* of each formula at a terminal node of an open path in the tree for Φ. The set of all formulae in the open paths of the tree for Φ is a *satisfiability model* for Φ. By downward induction on the tree, if each of the formulae of Φ at the initial node of the tree is true, then so are all of the subformulae at each of the successor nodes of the tree, and so are each of the formulae at the terminal nodes. Moreover, by upward induction on the tree, if each formula at the terminal nodes of the tree are true, then so are all formulae at their predecessor nodes, and so too are the formulae at the initial node of the tree. Thus, if we obtain a tree in which some path contains both a formula F and its negation $\sim F$, so that the path closes, then we have derived a contradiction. We will make use of this fact to consider falsifiability trees that allow us to determine whether a formula or set of formulae is valid.

A *falsifiability tree* is a tree or sequence of trees in which we attempt to find a falsifying assignment for (a set of formulae) F. Let $F_0, F_1, \ldots, F_n, Q$ be the formulae of Φ, and let the sequence $F_0, F_1, \ldots, F_n$ be a proof of Q. Construct a new tree for either $F_0, F_1, \ldots, F_n, \sim Q$ or the negation of the entire sequence $F_0, F_1, \ldots, F_n, Q$ such that we have either the formula $\Phi' = F_0 \wedge F_1 \wedge \ldots \wedge F_n \wedge \sim Q$ or the formula $\overline{\Phi} = \overline{F_0 \wedge F_1 \wedge \ldots \wedge F_n \wedge Q}$ at the initial node of the tree. (Thus, a proof can be understood as an "extended" formula obtained by the conjunction of each of the formulae, including the "Endformula" or conclusion, of the sequence; and a formula can be said to be valid or not, in this system, in precisely the same way that a proof is said to be valid or not.) If, after application of tree decomposition rules to each of these formulae (and any other of their decomposable subformulae), each path of this new tree closes, then Φ' or $\overline{\Phi}$ is *inconsistent* and Φ is *valid.*

An *assignment* for a set S of formulae is a a function which, when we are given a nonempty set U called the universe of the assignment, associates either (a) an element of U to some atomic term of S, (b) a k-ary function ($k > 0$) of S to some k-ary functional symbol of U; (c) an element of the set of truth-values {t, f} to some propositional symbol of S; or (d) a k-ary function ($k > 0$) of U in {t, f} to some k-ary predicate symbol. We say that [the value of] an assignment α of truth-values to Φ is true (written $v[\alpha, \Phi] = t$) is *valid* if $v[\alpha', \Phi] = f$ for each related assignment α' and $v[\alpha'', F_i] = t$ for each formula F_i of Φ.

The falsifiability tree, then, is an mechanization of proof by contradiction. It is likewise, as van Heijenoort (*1968b*) called it, "the dual of that [method] presented in (Jeffrey *1967*)." Moreover, it is a test for validity of proofs.

The falsifiability tree method is *sound* if each provable formula in the system is valid and can be proven to be valid by the falsifiability tree method, i.e. if a formula is provable, then it is valid. The falsifiability tree method is *complete* if each valid formula or set of formulae of the system is provable by the method, i.e. if a formula is valid, then it is provable. In his unpublished papers distributed to his students, van Heijenoort proved the completeness and soundness of the falsifiability tree method for classical quantification theory and for intuitionistic logic.

§5. VAN HEIJENOORT'S WORK ON THE FALSIFIABILITY TREE METHOD

In (*1973*), van Heijenoort gave a proof of the soundness and completeness of the falsifiability tree method for sentential logic, that is, for classical propositional calculus. This was followed by the (*1974*) paper "Falsifiability Trees", which gives a proof of the soundness and completeness of the falsifiability tree method for quantification theory, that is, specifically for classical first-order calculus (without identity). One consequence of van Heijenoort's completeness and soundness proof is the Löwenheim-Skolem theorem, for which van Heijenoort therefore was able to give a one-line proof. Van Heijenoort's proof makes use of König's lemma, and (*1974*) also contains a proof of this lemma. In connection with his proof of the soundness and completeness of the falsifiability tree method for quantification theory, van Heijenoort (*1974a*) also gave a proof of Quine's Law of Lesser Universes, according to which, if a and b are two cardinal numbers such that $0< a \leq b$, then if a formula is a-satisfiable, then it is b-satisfiable; and if a formula is b-valid, then it is a-valid. Several years earlier, in (*1972*), van Heijenoort had already given a proof of the soundness and completeness of the falsifiability tree method for the simple theory of types with extensionality.

In *1975*, van Heijenoort gave his first proof of the soundness and completeness of the tree method for intuitionistic logic. In the case of intuitionistic propositional logic, we are shown in his paper on "The Tree Method for Intuitionistic Sentential Logic" (*1975*) that a tree for an intuitionistic formula A is consistent if and only if A is classically provable; and that every nonconsistent ramified branch of a finished tree for an intuitionistic formula A yields a Kripke model which fails to satisfy A. The same reasoning is applied in the paper "The Tree Method for Intuitionistic

Quantification Theory" (*1975a*), in which a proof of the soundness and completeness of the tree method for first-order intuitionistic logic is carried out by adding to the proof for intuitionistic propositional logic the three cases of T∀ (true universal quantification), T∃ (true existential quantification), and F∃ (false existential quantification).[6] In his (*1979*) book *Introduction à la sémantique des logiques non-classiques*, van Heijenoort gave more elegant proofs of the soundness and completeness of the tree method for intuitionistic logic.

Also in 1975, van Heijenoort once again turned his attention to the work of Herbrand. We noted that in *Desarrollo* van Heijenoort (*1976*) considered Herbrand quantification primarily from an historical context, although he there also made comparisons of the relative strengths and weaknesses of the various members of the family of formal systems called quantification theory. Among the weaknesses of Herbrand quantification were that it was not easily generalized to second-order logic, and that there are no simple results which allow us to obtain an analogue of the Herbrand Fundamental Theorem for arbitrary formulae of intuitionistic quantification theory. One of the main strengths of Herbrand quantification was that it permitted reduction, through Herbrand expansion, of quantified formulae, whether in prenex form or not, to propositional formulae. We recall that a major result of van Heijenoort's (*1968b*) paper "On the Relation Between the Falsifiability Tree Method and the Herbrand Method in Quantification Theory" was that Herbrand quantification could readily be adapted to validity tests by the tree method precisely because quantified Herbrand formulae could be rendered quantifier-free. Now in 1975, van Heijenoort, in his paper on "Herbrand" (*1975b*) examined in detail the technical apparatus of Herbrand expansion and gave a proof of Herbrand's Fundamental Theorem.

The Fundamental Theorem states that: *given a formula F of classical quantification theory, we can effectively generate an infinite sequence of quantifier-free formulae* F_1, F_2, ..., *such that F is provable in (any standard system of) quantification theory if and only if there is a k such*

[6] In his lectures on foundations of mathematics, van Heijenoort declared that intuitionistic logic does not permit universal denial; therefore he did not have to consider the case of F∀ (false universal quantification).

that F_k is (sententially) valid; and moreover, F_k can be recovered from F through certain special rules.

Now if we analyze Herbrand's theorem, we notice that its main connective is the biconditional, so that it can therefore be reduced to two independent statements: that F is provable in standard quantification theory; and that the Herbrand expansion of a formula F to an infinite sequence of quantifier-free formulae is valid for each formula F_k of that sequence. Van Heijenoort uses this analysis to obtain his proof of the soundness and completeness of Herbrand quantification. Given the statements:

(1) *F is valid*

and

(2) *there is a k such that the k^{th} Herbrand expansion of F is (sententially) valid*

van Heijenoort shows that the implication from statement (2) to statement (1) is the soundness of Herbrand's proof procedure, and that the implication from statement (1) to statement (2) is its completeness. The conjunction of (1) and (2) is called the "semantic" Herbrand theorem.

Herbrand expansion, according to which we obtain a quantifier-free formula by obtaining a k-length conjunction from a universally quantified formula and a k-length disjunction from an existentially quantified formula, where we have a k-ary universe, is in fact just an enlargement of the Löwenheim-Skolem infinite conjunction presented by Skolem in his normalform translations of Hilbert's quantified formulae. What led Herbrand to develop his method was precisely his dissatisfaction with the Löwenheim-Skolem theorem, which asserts (if I may express it in its simplest terms) that if a formula of classical quantification theory is k-satisfiable for every finite k, then that formula is $\aleph_0$-satisfiable. What disturbed Herbrand, as we hinted earlier, was that this theorem was restricted to satisfiability. Herbrand's Fundamental Theorem was, in the words of Herbrand (*1931*, p. 4, quoted in English translation by van Heijenoort *1967*, p. 526 in his "Introduction" to Herbrand *1930*), a "more precise statement of the...Löwenheim-Skolem theorem," and thus can be viewed, as van Heijenoort (*1967*, 526) noted, "as a reinterpretation, from the point of view of the Hilbert program, of the results of Löwenheim and Skolem." In fact, what Herbrand did was to permit us to state that if a formula is $\aleph_0$-valid, then it is k-valid for every finite k, provided there exists no countermodel to that formula. Consequently, it is thanks to van Heijenoort's work on Herbrand that we are able to argue that the technical developments in Hilbert-type systems, including the development of proof theory by Hilbert and Bernays, as well as of the development of alternative theories of quantification, are due primarily to questions raised by Herbrand about the Löwenheim-Skolem theorem from the point of view of the Hilbert program.

Van Heijenoort's historical interests and his technical work in expanding and developing the tree method coincide, and focus on the need to define and explore the concepts of *satisfiability*, *validity*, and *being a proof*.

In 1978, van Heijenoort further extended his results of (*1975*) and (*1975a*) in which he applied the falsifiability tree method to intuitionistic logic and proved the soundness and completeness of the tree method for intuitionistic logic. To this he added an application of the tree method to propositional and first-order modal logic, together with a proof of the soundness and completeness of the tree method for modal logic. At the same time, he suggested, but did not carry out, the possibility of applying the tree method to two variants of three-valued logic. These new results on intuitionistic logic, modal logic, and three-valued logic were published in (*1979*) in van Heijenoort's booklet *Introduction à la sémantique des logiques non-classiques*. It represents the only formal publication of a proof by van Heijenoort of the soundness and completeness of the tree method. The first chapter of this work, which considers classical logic, contains, in much more sinewy form, parts of the same materials found in van Heijenoort's earlier, unpublished, technical papers, in particular from the (*1974*) paper on "Falsifiability Trees", although the material and their presentation are far from identical.

§6. SKETCH OF VAN HEIJENOORT'S PROOFS OF THE SOUNDNESS AND COMPLETENESS OF THE FALSIFIABILITY TREE METHOD

Van Heijenoort's proofs of the soundness and completeness of the tree method are far more rigorous than the intuitive, informal proofs given in (Jeffrey *1967*). Van Heijenoort's proofs employ the same concepts and follow the same patterns as do the proofs presented by (Bell & Machover *1977*), although van Heijenoort's proofs are somewhat longer and require more bookkeeping, in part because van Heijenoort proofs, unlike those of (Bell & Machover *1977*), do not make explicit use of Hintikka sets. Of course, van Heijenoort's proofs, although for the most part unpublished, predate by several years the published proofs found in (Bell & Machover *1977*) or van Heijenoort's own published proofs in his *Introduction à la sémantique des logiques non-classiques*.

By way of example, we can give a simplified sketch of van Heijenoort's (*1973*) proof of the soundness and completeness of the falsifiability tree method for propositional logic.

Assume that we have both downward and upward induction on the tree. Suppose the following theorem has already been proven.

Theorem 1. *If T is a falsifiability tree for a formula F and there is an assignment α of truth-values to the sentential variables of F such that $v[\alpha, \sim F] = t$, then there is a formula G occurring at a node of a branch β of T such that $v[\alpha, G] = t$.*

We now prove *soundness.*

Theorem 2. *If there is a closed falsifiability tree for a formula, then that formula is valid.*

Proof. Let T be a closed falsifiability tree for a formula F. Assume that there is an assignment α such that $v[\alpha, F] = f$ or $v[\alpha, \sim F] = t$. By the previous theorem, there is then a branch β of T such that, if formula G is at a node of β, then $v[\alpha, G] = t$. Since T is closed, so is β, and there is a formula H such that both H and $\sim H$ are at nodes of β, so that we have obtained the contradiction $v[\alpha, H] = t$ and $v[\alpha, \sim H] = t$. ▌

Now we prove *completeness.*

Consider the following theorem.

Theorem 3. *If there is a nonclosed finished falsifiability tree for a formula F, then there is an assignment* α *such that* $v[\alpha, F] = f$.

Proof. The nonclosed finished tree for F has a nonclosed branch β. Let α be an assignment of truth-values to the sentential variables of F such that if p is a basic formula occurring at a node of β, then $v[\alpha, p] = t$, and if $\sim q$ is a basic formula occurring at a node of β, then $v[\alpha, q] = f$. By upward induction on the tree, we then have either $v[\alpha, \sim F] = t$ or $v[\alpha, F] = f$.

If there is a node n of β at which the formula is not basic but T is finished, then we apply the appropriate tree decomposition rule to the formula at n. By doing so, we obtain one or two successor nodes of n containing subformulae of the formula at n. By induction on the tree, these subformulae are true for assignment α. ▌

Now by contraposition on Theorem 3, we obtain

Theorem 4. *If a formula F is valid, then there is a closed tree for F,*

which is the completeness theorem for trees. ▌

§7. Conclusion

It is to be hoped that a collection of van Heijenoort's technical papers on quantification theory and the tree method, whether unpublished or merely distributed to students and a few colleagues, can be published in the near future, so that his work will become better known and receive the attention which it deserves and which has already been accorded to his historical writings. Brief evaluations of his work, especially of his work on quantification theory and the falsifiability tree method, have already been given in (Anellis *1987*; *1987a*).

References.[7]

ANELLIS, I.H. 1978. *Review of Jean van Heijenoort, El desarrollo de la teoría de la cuantificación*, ts., September 14; privately printed in I.H. Anellis, *Introduction to proof theory: papers in metamathematics* (Itta Bena, Mississippi Valley State University, 1980), 38-42. Reprinted THIS ISSUE, 338-341.

—. 1979. *The Löwenheim-Skolem theorem, theories of quantification, and Beweistheorie*, abstract #763-02-4, Notices of the American Mathematical Society **26**, A-22.

—. 1987. *Jean van Heijenoort as a logician*, S.D. Comer (editor), *Third Southeastern Logic Symposium, Abstracts, March 7, 1987*, 3-4.

—. 1987a. *Jean van Heijenoort as a logician – in memory of Jean van Heijenoort, 1912-1986*, in *Societé canadienne d'histoire et de philosophies des mathématiques/Canadian Society for History and Philosophy of Mathematics, 13th Annual Meeting, McMaster University, Hamilton, Ontario, May 25-27, 1987, Programme*, abstract #2, p. 4.

—. 1988. *Some unpublished papers of Jean van Heijenoort: sources*, Historia Mathematica **15**, 270-274.

—. 1990. *From semantic tableaux to Smullyan trees: the history of the falsifiability tree method*, Modern Logic **1**, 36-69.

[7] Van Heijenoort's materials are available in the Jean van Heijenoort Papers, 1946-1983, Archives of American Mathematics, University Archives, University of Texas at Austin.

—. 1991. *The Löwenheim-Skolem theorem, theories of quantification, and proof theory*, in T. Drucker (editor), *Perspectives on the history of mathematical logic*, (Boston/Basel/Berlin, Birkhäuser), 71-82.

BELL, J.L. and MACHOVER, M. 1977. *A course in mathematical logic*, Amsterdam/New York/Oxford, North-Holland.

BETH, E.W. 1955. *Semantic entailment and formal derivability*, Koninklijke Nederlandse Akad. Wetenschappen (n.s.) **18**, no. 13, 309-342.

—. 1959. *The foundations of mathematics*, Amsterdam, North-Holland.

—. 1962. *Formal methods*, Dordrecht, Reidel & New York, Gordon & Breach.

FEFERMAN, A.B and FEFERMAN, S. 1987. *Jean van Heijenoort (1912-1986)*, The Paris Logic Group (editors), *Logic Colloquium '85* (Amsterdam, North-Holland), 1-7. Reprinted, with corrections and additions, THIS ISSUE, 231-238.

FEFERMAN, S. 1986. Letter to I.H. Anellis, October 15.

GOLDFARB, W.D. (editor). 1971. *Jacques Herbrand, Logical writings*, Cambridge, Mass., Harvard University Press. English translation of (van Heijenoort *1968*)

—. 1975. *Review of J.-P. Bénéjam, "Application du théorème de Herbrand à la présentation de thèses tératologiques du calcul des prédicats élémentaire" and R. Fraïssé, "Réflexions sur la complétude selon Herbrand"*, Journal of Symbolic Logic **40**, 238-239.

HERBRAND, J. 1930. *Recherches sur la théorie de la démonstration*, Ph.D. thesis, University of Paris. Reprinted in Prace Towarzystwa Naukowego Warszawkiego, Wydział III, no. 33 (1930). Reprinted in

(Herbrand *1968*), 35-353; English translation in (Herbrand *1971*), 44-202; English translation of Chapter 5 in (van Heijenoort *1967*), 525-581.

—. 1931. *Sur la non-contradiction de l'arithmétique*, Journal für die reine und angewandte Mathematik **166**, 1-8. English translations in (Herbrand *1971*), 282-298 and (van Heijenoort *1967*), 618-628.

—. 1968. *Écrits logiques*, J. van Heijenoort (ed.), Paris, Presses Universitaire de France.

—. 1971. *Logical writings*. See (Goldfarb *1971*).

HINTIKKA, J. 1953. *Distributive normal forms in the calculus of predicates*, Acta Philosophica Fennica **6** (Helsinki).

—. 1955. *Form and content in quantification theory*, in J. Hintikka, *Two papers on symbolic logic*, Acta Philosophica Fennica **8** (Helsinki), 7-55.

—. 1955a. *Notes on quantification theory*, Societas Scienarum Fennica Comment. Phys.-Math. **17**, no. 12 (Helsinki), 1-13.

JEFFREY, R.C. 1967. *Formal logic: its scope and limits*, New York, McGraw-Hill.

—. 1987. Letter to I.H. Anellis, May 6.

JOHANSSON, I. 1937. *Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus*, Compositio Mathematica **4**, 119-136.

LIS, Z. 1960. *Wynikanie semantyczne a wykanie formalne*, Studia Logica **10**, 39-60.

LÖWENHEIM, L. 1915. *Über Möglichkeiten im Relativkalkül*, Mathematische Annalen **76**, 447-470. English translation in (van Heijenoort *1967*), 228-251.

SMULLYAN, R.M. 1968. *First-order logic*, Berlin/Heidelberg/ New York, Springer-Verlag.

—. 1987. Letter to I.H. Anellis (no date; posted 15 May), 4pp., ms.

VAN HEIJENOORT, J. 1967. (editor) *From Frege to Gödel: a source book in mathematical logic, 1879-1931*, Cambridge, Mass., Harvard University Press.

— (editor). 1968. See (Herbrand *1968*).

—. 1968a. *Préface*, in (Herbrand *1968*), 1-2.

—. 1968b. *On the relation between the falsifiability tree method and the Herbrand method in quantification theory*, 12pp. ts., 20 November.

—. 1970. Abstract: *On the relation between the falsifiability tree method and the Herbrand method in quantification theory*, Journal of Symbolic Logic **35**, 358.

—. 1972. *The falsifiability-tree method for the simple theory of types with extensionality*, 23pp. ts., 23 July.

—. 1973. *Soundness and completeness of the falsifiability tree method for sentential logic*, 9pp. ms., 23 September.

—. 1974. *Falsifiability trees*, 25pp. ts., 30 September.

—. 1974a. <*Proof of the law of lesser universes*>, 4pp. ts.

—. 1975. *The tree method for intuitionistic sentential logic*, 5pp. ms., 5 May.

—. 1975a. *The tree method for intuitionistic quantification theory*, 3pp. ms, 9 May.

—. 1975b. *Herbrand*, 25pp. ts., 18 May.

—. 1976. *El desarrollo de la teoría de la cuantificación*, Mexico City, Universidad Nacional Autónoma de México. (Copyright 1975.)

—. 1979. *Introduction à la sémantique des logiques non-classiques*, Paris, Collection de l'École Normale Supérieur des Jeunes Filles, no. 16.

—. 1982. *L'oeuvre logique de Jacques Herbrand et son contexte historique*, in J. Stern (ed.), *Proceedings of the Herbrand symposium, Logic Colloquium '81, Marseilles, France, July 1981* (Amsterdam/New York/Oxford, North-Holland), 57-85. Revised English translation: (van Heijenoort *1986a*).

—. 1986. *Selected essays*, Naples, Bibliopolis. (Copyright 1985.)

—. 1986a. *Jacques Herbrand's work in logic and its historical context*, in (van Heijenoort *1986*), 99-121. Revised English translation of (van Heijenoort *1982*).