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George Tourlakis
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REVIEW

RICHARD E. HODEL

Set theory and the construction of the axiom system ZFC is one of the great success stories of twentieth century mathematics. Moreover, the history of this development is quite interesting and can be briefly outlined as follows. Zermelo, in 1904, used the Axiom of Choice [power set version] to prove that every set can be well-ordered. Cantor originally felt that this was an obvious fundamental property of sets, but later he realized the need for a precise proof and it became a major unsolved problem. After Zermelo gave his proof, the question then arose: Can the Axiom of Choice be derived from the other axioms of set theory? But at that time, set theory had not been axiomatized! So in 1908 Zermelo presented his list of seven axioms for set theory; these are: extensionality, separation, null set, pair, union, power set, and infinity. These axioms were selected with two considerations in mind: provide an axiomatic framework for a proof of the well-ordering theorem; avoid contradictions. The other axioms of ZF, replacement and foundation, were not added until much later. The axiom system $ZFC = ZF + AC$ is now the most widely used axiom system for modern mathematics.

The book under review is the second volume of a two-volume work that, in the words of the author, "bridges the gap between introductory expositions of logic or set theory on the one hand, and the research literature on the other." He then goes on to say that volume two on set theory is a thorough development of Zermelo-Frankel set theory and is situated between two opposite poles: "elementary texts that familiarize the reader with the vocabulary of set theory and build set-theoretic tools for use in courses in analysis, topology, or algebra—but do not get into metamathematical issues. On the other hand are those texts that explore issues of current research interest, developing and applying tools (constructibility, absoluteness, forcing, *etc.*) that are aimed to analyze the inability of the axioms to settle certain set-theoretic questions." (See p. xi of the Preface.) Although the author does not list books in either of these two categories, it seems reasonable that among introductory texts in set theory, we might list the books by Devlin, Hrbacek and Jech, Moschovakis, and Roitman (see the references); books in the research category are those of Jech, Kanamori, and Kunen.

The philosophy of the book is further summarized by the author as follows: (1) all of the fundamental tools of set theory as needed elsewhere in mathematics are included in detailed exposition; (2) due to space considerations, applications of set theory to other branches of mathematics are not included. (I will say more about (2) later in this review.) For further emphasis, the author states that the text under review just "does set theory" and is aimed at the reader who wants an elaboration on the more subtle results of set theory. The text is suitable for an advanced undergraduate or beginning graduate course in mathematics, computer science, or philosophy.

Since set theory forms the foundations of mathematics, it is of interest not only to mathematicians but also to philosophers; this naturally leads to philosophical discussions on how and why the axioms of set theory are selected. There are a number of principles that can be used, the most important of which are outlined below. For more detail, see the survey in [10].

• **Cantorian finiteness** Infinite sets are not so different from finite sets. So, if a statement is "obviously true" of finite sets, then it is reasonable to formulate it as an axiom for infinite sets. Note that by this principle the axiom that every set can be well-ordered is automatically adopted.

• **Iterative concept of set** This principle originated with Zermelo in the 1930's and has been discussed in considerable detail by Shoenfield [13] and Boolos [1]. The principle claims that sets are constructed in stages. Thus, to construct set x at a certain stage, one must first construct all of the elements of x at some earlier stage. For example, this principle makes it easy to justify the axiom of foundation.

• **Limitation of size** The paradoxes of set theory are caused by sets that are "too large." So the axioms should be selected so that new sets are not too much larger than the set(s) from which they are constructed.

• **One step back from diaster** The axioms of set theory should be chosen so that we can construct as many sets as possible without actually introducing a contradiction.

• **Maximum principle** Anything that can be a set, is a set. A variation of this idea, suggested by von Neumann, states: a collection fails to be a set if and only if it can be put in a one-to-one correspondence with the universe V of all sets. This is a strong version of the *limitation of size principle* and is closely related to the *one step back from disaster* principle.

We note that, despite the existence of various philosophical points of view for selecting axioms, most mathematical books on set theory spend little or no time discussing these ideas. The book under review is a notable exception.

Now let us turn to a detailed discussion of the book itself, which consists of eight chapters organized as follows:

Chapter I, A Bit of Logic, is a condensed version of volume 1 of this series and treats the following topics: Tarski semantics for first-order logic; axioms and rules of inference for first-order logic; outline of the main ideas needed to prove the Completeness Theorems of first-order logic; statement of Gödel's Incompleteness Theorem. The author uses this background material later in the text; for example, there are formal proofs of the following first-order formulas:

(1) converse of extensionality: $\vdash (x = y) \to \forall z(z \in x \leftrightarrow z \in y)$;

(2) Russell's paradox: $\vdash \neg\exists x\forall y[R(y, x) \leftrightarrow \neg R(y, x)]$.

Chapter II is rather short but sets the stage for Chapter III on the axioms of ZF. A somewhat unique feature of the book is the inclusion of urelements; the reasons for this decision are presented in Chapter II. There is also a lengthy discussion of the *principle of stages*; this principle will be used in Chapter III when the axioms of ZF are given. The principle is also used to justify a weak version of foundation: $x \notin x$ for every set x; this weak version of foundation is then used to explain away Russell's paradox.

Chapter III is a careful discussion of the axioms of ZF, in the following order: extensionality, separation (and unrestricted comprehension), pairing, union, foundation, replacement, and power set; infinity is omitted for now. The chapter ends with the development of ordered pairs, Cartesian products, relations, and functions using this list of axioms.

Chapter IV is devoted to the axiom of choice and various equivalent forms. The author uses skepticism that AC applies to all possible (infinite) collections of sets to motivate the concept of a "well defined" set. This leads to an informal version of Gödel's constructible universe L and a plausibility argument that AC holds in this restricted setting.

Chapter V completes the description of the ZF axioms by adding the axiom of infinity. This axiom is then used to construct the natural number system and to prove the existence of the transitive closure of a set.

Chapter VI, entitled "Order", covers a wide range of topics, beginning with well-ordered sets, recursive definitions, Mostowski's Collapsing Theorem, ordinals, recursion over ordinals, and Zermelo's Theorem that AC implies that every set can be well-ordered. The cumulative hierarchy is constructed, and there are proofs of (1) foundation $\Rightarrow V = \bigcup\{V_\alpha : \alpha \in ON\}$; (2) von Neumann's result that if ZFC without foundation is consistent, then ZFC is consistent. Absoluteness is introduced, and there is a construction of Gödel's constructible universe and a proof that if ZF is consistent, then so is ZFC. The chapter ends with a discussion of ordinal arithmetic.

Chapter VII is entitled "Cardinality" and covers a wide range of topics, ranging from Cantor's early results on equipotent sets to Gödel's proof of the consistency of GCH. Cardinal numbers, cardinal arithmetic, and the $\aleph$ function are discussed. There are proofs of the Schröder-Bernstein Theorem and Hartogs's Theorem; also, of the fact that the existence of a strongly inaccessible cardinal gives a set model of ZFC. The chapter ends with a proof that if ZF is consistent, then $ZFC + GCH$ is also consistent.

Chapter VIII is an introduction to forcing. There is a clear explanation of why the method works, despite the fact that there is no countable transitive model of set theory. Given a countable transitive model M of ZFC and a partially ordered set P in M, the set $M[G]$ is constructed and there is a detailed proof that $M \subseteq M[G]$, $G \in M[G]$, and that $M[G]$ is a countable transitive model of ZFC. Finally, the author proves Cohen's result that if ZFC is consistent, then $ZFC + \neg CH$ is also consistent.

The author clearly achieves his stated goal. The book covers the essential ideas of ZFC set theory in a logical, as opposed to a historical, order. There are extensive comments on metamathematical issues, and some topics are dealt with in great detail, for example recursive definitions and recursive constructions over a wide variety of structures. All in all, this book is well-suited as a reference for set theory and the subtle philosophical issues that can arise. However, as a text for classroom use, it may not be the best possible choice, especially for a class consisting of mathematics or computer scientist majors who do not already have a fairly detailed knowledge of Cantor's original interest in set theory, namely cardinality, equipotent sets, and well-ordered sets.

However, my main reservation about using the book as a text is the author's viewpoint that the book just does set theory and forgoes applications. I missed a discussion of the measure problem, Ulam's Theorem, measurable cardinals, the number of strongly inaccessible cardinals below the first measurable cardinal, and various topics in infinite combinatorics (for example, stationary sets, club sets, and their applications). I would prefer less emphasis on the various principles of recursion and more on applications, even if the applications are to set theory itself! But this criticism is clearly based on personal preferences, and many readers, especially those with a philosophical bent, will find this book a good choice for the classroom.

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Duke University, Durham, N.C. 27708
E-mail address: hodel@math.duke.edu