AUTOMORPHISMS ON THE
ALTERNATIVE DIVISION RING

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ABSTRACT. In this work, we prove that, if $\mathfrak{R}$ is an arbitrary alternative division ring, then, under a mild condition, every semi-automorphism $\varphi : \mathfrak{R} \to \mathfrak{R}$ is either an automorphism or an anti-automorphism. We extend Hua's result [7] for an alternative division ring.

1. Alternative rings and semi-automorphism maps. Let $\mathfrak{R}$ be a ring which is not necessarily associative or commutative, and consider the following convention for its multiplication operation: $xy \cdot z = (xy)z$ and $x \cdot yz = x(yz)$ for $x, y, z \in \mathfrak{R}$, for short. We denote the *associator* of $\mathfrak{R}$ by $(x, y, z) = xy \cdot z - x \cdot yz$ for $x, y, z \in \mathfrak{R}$. A ring $\mathfrak{R}$ is said to be *alternative* if $(x, x, y) = 0 = (y, x, x)$ for all $x, y \in \mathfrak{R}$. It is well known that every alternative ring satisfies

$$(x, y, x) = 0,$$

for all $x, y \in \mathfrak{R}$, as well as satisfying the Moufang identities

(1.1) $$(xax)y = x[a(xy)],$$

(1.2) $$y(xax) = [(yx)a]x,$$

(1.3) $$(xy)(ax) = x(ya)x,$$

for all $x, y, a \in \mathfrak{R}$. It can easily be seen that any associative ring is an alternative ring. A map $\varphi : \mathfrak{R} \to \mathfrak{R}$ is called a *semi-automorphism* from $\mathfrak{R}$ onto itself if φ is an additive map such that

$$\varphi(xyx) = \varphi(x)\varphi(y)\varphi(x),$$

for all $x, y \in \mathfrak{R}$, and satisfies the condition $\varphi(1) = 1$.

Semi-automorphisms between two rings were first introduced by Ancochea and studied in quaternion division algebras in [1]. It was extended to algebras with finite order by Ancochea [2] and Kaplansky [8], and to sfields by Hua [7] in 1949. It was shown, in all of those cases, that a semi-automorphism is either an automorphism or an anti-automorphism. Semi-automorphism and automorphism has been studied in other fields of mathematics; the interested reader is referred to some related papers, such as [3, 4, 5, 9].

2. Semi-automorphism on alternative division rings. In any ring $\mathfrak{R}$ with 1, an element x is said to have an *inverse* x^{-1} in the case where there is an x^{-1} in $\mathfrak{R}$ satisfying $xx^{-1} = x^{-1}x = 1$. In an alternative ring, if x has an inverse, it is unique and also

$$(2.1) \qquad (x, x^{-1}, y) = (x^{-1}, x, y) = (x^{-1}, y, x) = 0,$$

with $y \in \mathfrak{R}$. If every nonzero element in an alternative ring $\mathfrak{R}$ with 1 has an inverse, then $\mathfrak{R}$ is called a *division ring*, and conversely.

According to [6],

> Since Hamilton's first example of non-commutative division algebra, the quaternion algebra and division algebra have received a great deal of attention. By comparison, infinite dimensional division algebras and sfields were neglected. Hua came onto the scenearound 1950 and proved several theorems in this area by direct and elementary methods. The well known examples of semi-automorphisms are automorphisms, which satisfy $\sigma(ab) = \sigma(a)\sigma(b)$, and anti-automorphisms which satisfy $\sigma(ab) = \sigma(b)\sigma(a)$. An outstanding problem was whether there exists a semi-automorphism which is neither an automorphism nor an anti-automorphism. Hua [7] settled this problem in 1949 by proving that every semi-automorphism is either an automorphism or an anti-automorphism. The fundamental theorem of projective geometry on a line over a sfield of characteristics of 2, namely, any one-to-one mapping carrying the projective line over a sfield of characteristics of 2 onto itself and keeping harmonic relations invariant is a semi-linear transformation induced by an automorphism or an anti-automorphism, was thus derived.

This motivated us to consider the same problem in the case of alternative rings. We investigate the problem of when a semi-automorphism on an alternative division ring must be a homomorphism or an anti-homomorphism. We extended Hua's result as follows.

Theorem 2.1 ([7]). *Let $\mathcal{K}$ be an sfield. Every semi-automorphism of $\mathcal{K}$, that is, a mapping $\sigma : \mathcal{K} \to \mathcal{K}$ onto itself, satisfies*

$$\sigma(a + b) = \sigma(a) + \sigma(b),$$
$$\sigma(aba) = \sigma(a)\sigma(b)\sigma(a),$$

and $\sigma(1) = 1$ is either an automorphism or an anti-automorphism.

The main purpose of this paper is to extend Hua's result for an alternative division ring. We will prove the following main result.

Theorem 2.2. *Let $\mathfrak{R}$ be an alternative division ring. Consider a semi-automorphism $\varphi : \mathfrak{R} \to \mathfrak{R}$, satisfying:*

$$(*) \qquad \varphi(ab \cdot c + c \cdot ba) = \varphi(a)\varphi(b) \cdot \varphi(c) + \varphi(c) \cdot \varphi(b)\varphi(a).$$

Then, $\varphi : \mathfrak{R} \to \mathfrak{R}$ is either an automorphism or an anti-automorphism.

We will prove this theorem through the use of some lemmas. These lemmas have the same hypotheses as Theorem 2.2, and they are generalizations of Hua's results for the class of alternative division rings.

Lemma 2.3. *We have $\varphi(x^{-1}) = \varphi(x)^{-1}$ for all $x \in \mathfrak{R} \setminus \{0\}$.*

Proof. For all $x \in \mathfrak{R}$, we get $\varphi(x) = \varphi(1x) = \varphi(xx^{-1}x) = \varphi(x)\varphi(x^{-1})\varphi(x)$. Now, multiplying this equality on the left side by $\varphi(x)^{-1}$, we obtain

$$\begin{aligned}
1 &= \varphi(x)^{-1}\varphi(x) \\
&= \varphi(x)^{-1} \cdot (\varphi(x)\varphi(x^{-1}))\varphi(x) \\
&= \varphi(x)^{-1}(\varphi(x)\varphi(x^{-1})) \cdot \varphi(x) \\
&= (\varphi(x)^{-1}\varphi(x))\varphi(x^{-1}) \cdot \varphi(x) \\
&= \varphi(x^{-1})\varphi(x),
\end{aligned}$$

where we use (2.1). Analogously, we have $1 = \varphi(x)\varphi(x^{-1})$. Therefore, $\varphi(x^{-1}) = \varphi(x)^{-1}$. $\qquad\square$

Lemma 2.4. *For all $x, y \in \mathfrak{R}$, we obtain $\varphi(xy + yx) = \varphi(x)\varphi(y) + \varphi(y)\varphi(x)$.*

Proof. Replacing x by $x + y$ and y by 1 in $\varphi(xyx) = \varphi(x)\varphi(y)\varphi(x)$, we have

$$\varphi((x + y)^2) = \varphi((x + y)1(x + y)) = \varphi(x + y)1\varphi(x + y) = (\varphi(x + y))^2,$$

which implies by additivity of φ that $\varphi(xy + yx) = \varphi(x)\varphi(y) + \varphi(y)\varphi(x)$. $\qquad\square$

Lemma 2.5. *For all $x, y \in \mathfrak{R}$ we have $\varphi(xy) = \varphi(x)\varphi(y)$ or $\varphi(xy) = \varphi(y)\varphi(x)$.*

Proof. We show that $[\varphi(xy) - \varphi(x)\varphi(y)][\varphi(xy) - \varphi(y)\varphi(x)] = 0$. In fact,

$$
\begin{aligned}
[\varphi(xy) &- \varphi(x)\varphi(y)][\varphi(xy) - \varphi(y)\varphi(x)] \\
&= (\varphi(xy))^2 + (\varphi(x)\varphi(y))(\varphi(y)\varphi(x)) \\
&\quad - \varphi(xy) \cdot \varphi(y)\varphi(x) - \varphi(x)\varphi(y) \cdot \varphi(xy) \\
&= \varphi((xy)^2) + \varphi(x)(\varphi(y)^2)\varphi(x) \\
&\quad - \varphi(xy \cdot yx + xy \cdot xy) \\
&= \varphi((xy)^2) + \varphi(x)\varphi(y^2)\varphi(x) - \varphi(xy^2x + (xy)^2) \\
&= \varphi((xy)^2 + xy^2x - (xy^2x + (xy)^2)) \\
&= \varphi(0) \\
&= 0,
\end{aligned}
$$

where we use the condition $(*)$ of Theorem 2.2 and the Moufang identity. Since $\mathfrak{R}$ is an alternative division ring, it follows that $\varphi(xy) = \varphi(x)\varphi(y)$ or $\varphi(xy) = \varphi(y)\varphi(x)$. $\qquad\square$

Lemma 2.6. *If we have a pair of elements $a, b \in \mathfrak{R}$ such that $\varphi(ab) = \varphi(b)\varphi(a) \neq \varphi(a)\varphi(b)$, then $\varphi(ac) = \varphi(c)\varphi(a)$ for all $c \in \mathfrak{R}$, and $\varphi(db) = \varphi(b)\varphi(d)$ for all $d \in \mathfrak{R}$.*

Proof. The idea of the proof of this lemma is the same as that in [**7**]; however, for better clarity, we provide it here. If $c = b \in \mathfrak{R}$, then the result is obvious. Let $c \in \mathfrak{R}$ be an arbitrary element with $c \neq b$. From Lemma 2.5, we have

(i) $\varphi(ac) = \varphi(a)\varphi(c)$;
(ii) $\varphi(ac) = \varphi(c)\varphi(a)$.

Suppose that $\varphi(ac) = \varphi(a)\varphi(c) \neq \varphi(c)\varphi(a)$. We obtain the following identity

$$\varphi(a)\varphi(c) + \varphi(b)\varphi(a) = \varphi(ac) + \varphi(ab) = \varphi(a(c+b)).$$

Now, by Lemma 2.5 and additivity of φ, we have

$$\varphi(a(c+b)) = \varphi(a)\varphi(c+b) = \varphi(a)\varphi(c) + \varphi(a)\varphi(b)$$

or

$$\varphi(a(c+b)) = \varphi(c+b)\varphi(a) = \varphi(c)\varphi(a) + \varphi(b)\varphi(a);$$

however, this implies that $\varphi(b)\varphi(a) = \varphi(a)\varphi(b)$ or $\varphi(a)\varphi(c) = \varphi(c)\varphi(a)$, which is a contradiction. Therefore, $\varphi(ac) = \varphi(c)\varphi(a)$, for all $c \in \mathfrak{R}$. Similarly, we proved that, for any d, $\varphi(db) = \varphi(b)\varphi(d)$. $\square$

We are ready to prove our main theorem.

Suppose that φ is not an automorphism. We show that φ is an anti-automorphism. Since φ is not an automorphism, there are $a, b \in \mathfrak{R}$ such that $\varphi(ab) = \varphi(b)\varphi(a) \neq \varphi(a)\varphi(b)$. We want to show that $\varphi(dc) = \varphi(c)\varphi(d)$ for all $c, d \in \mathfrak{R}$. We suppose, by contradiction, that $\varphi(dc) = \varphi(d)\varphi(c) \neq \varphi(c)\varphi(d)$, by Lemma 2.5. By the same argument as that used in the proof of Lemma 2.6, we get

$$\varphi(ac) = \varphi(a)\varphi(c) \quad \text{and} \quad \varphi(db) = \varphi(d)\varphi(b).$$

Now, as in [**7**], we have the following identity:

$$\varphi(b)\varphi(a) + \varphi(ac) + \varphi(db) + \varphi(d)\varphi(c) = \varphi((a+d)(b+c)).$$

However,

$$\varphi((a+d)(b+c)) = \varphi(a+d)\varphi(b+c) \quad \text{or} \quad \varphi((a+d)(b+c)) = \varphi(b+c)\varphi(a+d);$$

by Lemma 2.5, this contradicts $\varphi(b)\varphi(a) \neq \varphi(a)\varphi(b)$ or $\varphi(d)\varphi(c) \neq \varphi(c)\varphi(d)$, by additivity of φ. This proves our Theorem 2.2, which follows as a consequence of Hua's result [**7**].

Corollary 2.7. *Let $\mathcal{K}$ be an sfield. Every semi-automorphism of $\mathcal{K}$, that is, a mapping of $\sigma : \mathcal{K} \to \mathcal{K}$ onto itself satisfies*:

$$\sigma(a + b) = \sigma(a) + \sigma(b),$$
$$\sigma(aba) = \sigma(a)\sigma(b)\sigma(a),$$

and $\sigma(1) = 1$ is either an automorphism or an anti-automorphism.

Proof. Merely observe that σ satisfies the condition $(*)$ of Theorem 2.2. In fact, linearizing $\sigma(aba) = \sigma(a)\sigma(b)\sigma(a)$ we obtain

$$\sigma(ab \cdot c + c \cdot ba) = \sigma(ab) \cdot \sigma(c) + \sigma(c) \cdot \sigma(ba).$$

Therefore, by Theorem 2.2, σ is either an automorphism or an anti-automorphism. $\square$

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