ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 5, 2018

ON NONLOCAL FRACTIONAL LAPLACIAN PROBLEMS WITH OSCILLATING POTENTIALS

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ABSTRACT. In this paper, we deal with the following fractional nonlocal p-Laplacian problem:

$$\begin{cases} (-\Delta)_p^s u = \lambda\beta(x)u^q + f(u) & \text{in } \Omega, \\ u \geq 0, \ u \not\equiv 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary of $\mathbb{R}^N$, $s \in (0,1)$, $p \in (1,\infty)$, $N > sp$, λ is a real parameter, $\beta \in L^\infty(\Omega)$ is allowed to be indefinite in sign, $q > 0$ and $f : [0,+\infty) \to \mathbb{R}$ is a continuous function oscillating near the origin or at infinity. By using variational and topological methods, we obtain the existence of infinitely many solutions for the problem under consideration. The main results obtained here represent some new interesting phenomena in the nonlocal setting.

1. Introduction. Recently, following the seminal paper of Caffarelli and Silvestre [**9**], a large number of contributions have appeared on problems which involve fractional nonlocal operators. Here, the two different notions are emphasized of fractional Laplacian operators on bounded domains which were considered in the literature, namely the *spectral* Laplacian operator (see, among others, Cabré and Tan [**7**], Tan [**42**] and Barrios, Colorado, de Pablo and Sánchez [**3**]) and the *integral* operator with some of its generalizations (see, for instance, [**27**] and the references therein). In [**41,** Theorem 1], the authors compared

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2010 AMS *Mathematics subject classification.* Primary 35J20, 35J62, 35J92, Secondary 35J15, 47J30.

Keywords and phrases. p-fractional Laplacian operator, infinitely many solutions, variational methods.

This research was supported by the INdAM-GNAMPA Project 2015, *Modelli ed equazioni non-locali di tipo frazionario*, and INdAM-GNAMPA Project 2017, *Teoria e modelli non locali.* The second author was supported by *Sostegno alla Ricerca Individuale* Universitá degli Studi di Napoli "Parthenope."

Received by the editors on May 31, 2017, and in revised form on August 29, 2017.

DOI:10.1216/RMJ-2018-48-5-1399

these two operators by studying their spectral properties and obtained, as a consequence of this careful analysis, that the two operators are different. Also, see **[29]** for an exhaustive study of this comparison.

A recent trend in the fractional framework is to consider a new nonlocal and nonlinear operator, the so-called *fractional p-Laplacian* $(-\Delta)^s_p$. See, for instance, Di Castro, Kuusi and Palatucci **[13, 14]**, Franzina and Palatucci **[17]**, Kuusi, Mingione and Sire **[21]**, Lindgren and Lindqvist **[22]**, and the famous work of Caffarelli **[8]**. Also, see the papers of Pucci, et al., **[32]**–**[36]**, where some existence and multiplicity results for fractional problems involving the p-Laplacian operator were obtained via variational methods.

In this direction, the aim of the present paper is to deal with the following problem:

$$\begin{cases} (-\Delta)^s_p u = \lambda\beta(x)u^q + f(u) & \text{in } \Omega, \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $s \in (0,1)$, $p \in (1,\infty)$, $N > sp$, $q > 0$ and $\lambda \in \mathbb{R}$ are parameters, while $\beta \in L^\infty(\Omega)$ and $f : [0,+\infty) \to \mathbb{R}$ is a continuous function.

More precisely, our aim here is to study the number and the behavior of solutions of problem (1.1), where f *oscillates near the origin* or *at infinity.* This analysis will be carried out using variational and topological techniques. In the sequel, we state our main results, treating the two cases separately, that is, when the nonlinearity f oscillates near the origin or at infinity, respectively.

A special case involving the classical fractional Laplacian operator $(-\Delta)^s$ is as follows:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$, with $N > 2s$ and $s \in (0,1)$. Furthermore, let $f \in C([0,+\infty); \mathbb{R})$, and suppose that*

$$-\infty < \liminf_{t\to 0^+} \frac{\int_0^t f(z)\,dz}{t^2} \leq \limsup_{t\to 0^+} \frac{\int_0^t f(z)\,dz}{t^2} = +\infty,$$

in addition to

$$-\infty < \liminf_{t\to 0^+} \frac{f(t)}{t} < 0.$$

Then, there exists an interval $\Lambda \subset (0,\infty)$ such that, for any $\lambda \in \Lambda$, the following nonlocal problem

$$\begin{cases} (-\Delta)^s u = \lambda u + f(u) & in\ \Omega, \\ u \geq 0, u \not\equiv 0 & in\ \Omega, \\ u = 0 & in\ \mathbb{R}^N \setminus \Omega, \end{cases}$$

admits a sequence of weak solutions $\{u_j\}_j \subset H^s(\mathbb{R}^N) \cap L^\infty(\Omega)$, with $u_j = 0$ in $\mathbb{R}^N \setminus \Omega$, and such that

$$\lim_{j\to+\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_j(x)-u_j(y)|^2}{|x-y|^{N+2s}}\,dx\,dy = \lim_{j\to+\infty} \|u_j\|_{L^\infty(\Omega)} = 0,$$

where $\|u_j\|_{L^\infty(\Omega)} := \max_{x\in\Omega} u_j(x)$.

A similar multiplicity result may be proven requiring similar asymptotic behavior of the potential at infinity. More precisely, the next theorem holds.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$, with $N > 2s$ and $s \in (0,1)$. Furthermore, let $f \in C([0,+\infty); \mathbb{R})$, and suppose that*

$$-\infty < \liminf_{t\to+\infty} \frac{\int_0^t f(z)\,dz}{t^2} \leq \limsup_{t\to+\infty} \frac{\int_0^t f(z)\,dz}{t^2} = +\infty,$$

in addition to

$$-\infty < \liminf_{t\to+\infty} \frac{f(t)}{t} < 0.$$

Then, there exists an interval $\Lambda \subset (0,\infty)$ such that, for any $\lambda \in \Lambda$, the problem

$$\begin{cases} (-\Delta)^s u = \lambda u + f(u) & in\ \Omega, \\ u \geq 0, u \not\equiv 0 & in\ \Omega, \\ u = 0 & in\ \mathbb{R}^N \setminus \Omega, \end{cases}$$

admits a sequence of weak solutions $\{u_j\}_j \subset H^s(\mathbb{R}^N)$, with $u_j = 0$ in $\mathbb{R}^N \setminus \Omega$, such that

$$\lim_{j\to+\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_j(x)-u_j(y)|^2}{|x-y|^{N+2s}}\,dx\,dy = +\infty.$$

Finally, we should emphasize that the coefficient $\beta \in L^\infty(\Omega)$ in problem (1.1) is allowed to be *indefinite* in sign, as suggested in several well-known works (see, for instance, **[1, 2, 4, 11, 12]** and the references therein).

Our results are in connection with the existence of infinitely many weak solutions of the following Dirichlet problem:

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u \geq 0, \ u \not\equiv 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, which has been extensively studied in the literature, assuming that f is odd in order to apply a variant of the classical Lusternik-Schnirelmann theory.

To the contrary, few papers deal with nonlinearities having no symmetry properties. For instance, in **[31]**, Omari and Zanolin proved that, if

$$\liminf_{t\to 0^+} \frac{\int_0^t f(z)\,dz}{t^p} = 0 \quad \text{and} \quad \limsup_{t\to 0^+} \frac{\int_0^t f(z)\,dz}{t^p} = +\infty,$$

then, for every $\lambda > 0$, problem (1.2) has a sequence of weak solutions in $W_0^{1,p}(\Omega)$ satisfying that $\|u_j\|_{L^\infty(\Omega)} \to 0$ as $j \to +\infty$ (also see **[31, 39]** for related topics). Successively, in **[30,** Theorem 2.2], Obersnel and Omari proved the existence of two sequences of solutions for the Dirichlet problem (for $p = 2$) under some constraints on the potential at infinity. One of their hypotheses implies a sign condition on the nonlinear term f. More precisely, the nonlinearity f is assumed to be definitively positive on the real half-line.

Inspired by the previous research, Molica Bisci and Pizzimenti **[25,** Theorems 5.1, 5.7] studied the existence of infinitely many weak solutions for the unperturbed Dirichlet problem (1.2) under the crucial assumption

$$-\limsup_{t\to L} \frac{\int_0^t f(z)\,dz}{t^p} < \kappa(p,N) \liminf_{t\to L} \frac{\int_0^t f(z)\,dz}{t^p},$$

where, either $L = 0^+$ or $L = +\infty$ and

$$\kappa(p,N) := \left(2^{N+p}N\int_{1/2}^{1} z^{N-1}(1-z)^p dz\right).$$

The existence of sequences of weak solutions for fractional nonlocal equations, with no symmetry hypothesis on the nonlinear term f, has only recently been investigated in the literature. In this sense, the results presented here may be seen as an extension of some recent nonlinear analysis theorems to the case of elliptic equations driven by nonlocal fractional operators.

More precisely, in our paper there are some computations, mostly straightforward, similar to those performed by Kristály and Moroşanu **[20]** and adapted here to the nonlocal fractional case. However, due to the presence of the fractional operator $(-\Delta)^s_p$, our abstract approach, as well as the setting of the main results, is different from the results found in **[20]**, where the authors studied competition phenomena for elliptic equations involving the Laplacian operator. A crucial point along the proof of the main results is that the truncation procedure developed in **[20,** Theorem 2.1] can be adapted to the fractional nonlocal setting. Of course, some technical difficulties naturally appear in this paper due to the nature of the fractional Gagliardo norm (see, for instance, Theorem 4.1).

Contrary to the classical literature dedicated to boundary value problems involving the Laplacian operator or some of its generalizations, up until the present, to our knowledge, only a few papers consider the existence of infinitely many weak solutions to nonlocal equations involving fractional nonlinear operators. For instance, Molica Bisci **[24]** studied the existence of a sequence of nontrivial weak solutions for exploiting the classical $\mathbb{Z}_2$-symmetric version of the Mountain pass theorem. In order to make the nonlinear methods work, careful analysis of the fractional spaces involved is necessary. As a particular case, we derive an existence theorem for the fractional Laplacian, finding nontrivial solutions of the equation

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

As far as we know, all of these results are new and represent a fractional version of classical theorems obtained working with Laplacian equations.

In **[40]**, Servadei studied the existence of infinitely many solutions for a nonlocal, nonlinear equation with homogeneous Dirichlet boundary data. In particular, the main result concerns the following model problem:

$$\begin{cases}(-\Delta)^s u - \lambda u = |u|^{q-2}u + h & \text{in } \Omega,\\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,\end{cases}$$

where $s \in (0,1)$ is a fixed parameter, $(-\Delta)^s$ is the fractional Laplacian operator, which (up to normalization factors) may be defined as

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy, \quad x \in \mathbb{R}^n,$$

while λ is a real parameter, the exponent $q \in (2, 2_s^*)$, with $2_s^* = 2N/(N-2s)$, $N > 2s$, the function h belongs to the space $L^2(\Omega)$ and, finally, the set Ω is an open, bounded subset of $\mathbb{R}^N$ with Lipschitz boundary.

Adapting the classical variational techniques used in order to study the standard Laplace equation with subcritical growth nonlinearities to the nonlocal framework, in the present paper, we prove that this problem admits infinitely many weak solutions $\{u_k\}_k$, with the property that the Sobolev norm goes to infinity as $k \to +\infty$, provided the exponent $q < 2_s^* - 2s/(N-2s)$. In this sense, the results presented here may be seen as an extension of some classical nonlinear analysis theorems to the case of fractional operators.

We consider different superlinear growth assumptions on the nonlinearity, starting from the well-known Ambrosetti-Rabinowitz condition. In this framework, we obtain three different results about the existence of infinitely many weak solutions for the problem under consideration, by using the Fountain theorem. All of these theorems extend the classical results for semilinear Laplacian equations to the nonlocal fractional setting.

This paper is organized as follows. In Section 2, we recall some preliminary notions and results. In Section 3, we discuss problem (1.1) under suitable asymptotic behavior of the potential either at zero or at infinity. Section 4, as well as Sections 5 and 6, will be devoted to the variational analysis of a suitable truncated problem $(\mathcal{P}_h^K)$ that will be crucial in order to study the existence of infinitely many solutions of problem (1.1) given in Theorems 3.1 and 3.2.

2. Preliminaries and functional setting. In this section, we recall some basic results related to fractional Sobolev spaces. In fact, the nonlocal analysis that we perform in this paper in order to use variational methods is quite general and may be suitable for other goals, too. Our proof will verify that the abstract approach developed in **[20]** is respected by the nonlocal framework. For this, we will develop a functional analytical setting that is inspired by (but not equivalent to) the fractional Sobolev spaces in order to correctly encode the Dirichlet boundary datum in the variational formulation. For more details regarding this topic, the reader is referred to **[15, 27]**.

Let $u : \mathbb{R}^N \to \mathbb{R}$ be a measurable function. We say that u belongs to the space $W^{s,p}(\mathbb{R}^N)$ if $u \in L^p(\mathbb{R}^N)$ and

$$[u]^p_{W^{s,p}(\mathbb{R}^N)} := \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy < +\infty.$$

Then, $W^{s,p}(\mathbb{R}^N)$ is a Banach space with respect to the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := [\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}]^{1/p}.$$

We will work in the following, closed linear subspace

$$X(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) : u(x) = 0 \text{ almost everwhere in } \mathbb{R}^N \setminus \Omega\},$$

which can be equivalently renormed by setting

$$\|u\| = [u]_{W^{s,p}(\mathbb{R}^N)}.$$

Now, the following, crucial results are recalled.

Theorem 2.1. $(X(\Omega), \|\cdot\|)$ *is a uniformly convex Banach space.*

Theorem 2.2. *Let $s \in (0,1)$ and $p \in (1,\infty)$ be such that $sp < N$. The embedding $X(\Omega) \subset L^r(\Omega)$ is continuous when $r \in [1, p_s^*]$, and compact for $r \in [1, p_s^*)$.*

We denote by $(X^*(\Omega), \|\cdot\|_*)$ the dual space of $(X(\Omega), \|\cdot\|)$. We define the nonlinear operator $A : X \to X^*$ by setting

$$\langle A(u), v\rangle := \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}(v(x)-v(y))\,dx\,dy$$

for $u, v \in X(\Omega)$. Here, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $X(\Omega)$ and its dual $X^*(\Omega)$.

Finally, we introduce a special function which will be useful in proving our main results. Fix $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, r) \subset \Omega$, where $B(x_0, r)$ is the open ball of radius R and center x_0. For any $t > 0$, we define the function z_t as follows:

$$z_t(x) := \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, r), \\ \frac{2t}{r}(r - |x - x_0|) & \text{if } x \in B(x_0, r) \setminus B(x_0, r/2), \\ t & \text{if } x \in B(x_0, r/2). \end{cases} \tag{2.1}$$

We set $z_t = 0$ in $\mathbb{R}^N \setminus \Omega$. Then, it is clear that $z_t \geq 0$ in $\mathbb{R}^N$ and $\|z_t\|_{L^\infty(\Omega)} = t$. Moreover, $z_t \in W_0^{1,p}(\Omega)$ and $\|z_t\|^p_{W_0^{1,p}(\Omega)} = C(r, p, N)t^p$ for some positive constant $C(r, p, N)$. By using [**15,** Proposition 2.2], we can infer that $z_t \in W^{s,p}(\Omega)$.

Since $z_t = 0$ outside the compact $\overline{B(x_0, r)}$, we can use [**15,** Lemma 5.1] to deduce that $z_t \in X(\Omega)$. In particular, the following holds:

$$\|z_t\|^p \leq C_0(r, s, p, N)\|z_t\|^p_{W_0^{1,p}(\Omega)} \leq C(r, s, p, N)t^p, \tag{2.2}$$

where $C(r, s, p, N)$ is a positive constant depending only upon p, r, s and N.

3. Main results. This section is devoted to the main results of the paper, where we prove the existence of infinitely many solutions for problem (1.1) in these two different contexts:

- f oscillating near the origin and $q \geq p - 1$,
- f oscillating at infinity and $0 < q \leq p - 1$;

while, in the remaining cases, that is, when

- f oscillates near the origin and $0 < q < p - 1$,
- f oscillates at infinity and $q > p - 1$;

we show the existence of at least a finite number of solutions. In all of these cases, we assume that $f : [0, +\infty) \to \mathbb{R}$ is a continuous function. Also, we denote by F the function

$$F(t) := \int_0^t f(\tau)\, d\tau \tag{3.1}$$

for any $t > 0$.

3.1. Oscillation near the origin. In this framework, we assume that the following conditions are satisfied:

$$\liminf_{t\to 0^+} \frac{f(t)}{t^{p-1}} =: -\ell_0 \in [-\infty, 0)\,; \tag{3.2}$$

$$-\infty < \liminf_{t\to 0^+} \frac{F(t)}{t^p} \leq \limsup_{t\to 0^+} \frac{F(t)}{t^p} = +\infty. \tag{3.3}$$

Our main result can be stated as follows.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $N > sp$, $\lambda \in \mathbb{R}$. Assume that $\beta \in L^\infty(\Omega)$ and $f \in C([0, +\infty); \mathbb{R})$ satisfy* (3.2) *and* (3.3)*. If either*

(a) *$q = p - 1$, $\ell_0 \in (0, +\infty)$ and $\lambda\beta(x) < \lambda_0$ almost everywhere $x \in \Omega$ for some $\lambda_0 \in (0, \ell_0)$, or*
(b) *$q = p - 1$, $\ell_0 = +\infty$ and $\lambda \in \mathbb{R}$ is arbitrary, or*
(c) *$q > p - 1$, and $\lambda \in \mathbb{R}$ is arbitrary,*

then there exists a sequence $\{u_j\}_j$ in $X(\Omega) \cap L^\infty(\Omega)$ of distinct weak solutions of problem (1.1) *such that*

$$\lim_{j\to+\infty} \|u_j\| = \lim_{j\to+\infty} \|u_j\|_{L^\infty(\Omega)} = 0. \tag{3.4}$$

Assumption (3.2) yields the existence of solutions for problem (1.1), while (3.3) allows us to deduce some information regarding the number of the solutions. In addition, we note that assertion (b) also covers the case when the power q is critical or supercritical, that is, the case when $q \geq p_s^*$, where

$$p_s^* := \frac{Np}{N - sp}, \quad N > sp, \tag{3.5}$$

is the Sobolev critical exponent.

3.2. Oscillation at infinity. In this framework, we assume that the following assumptions hold:

$$\liminf_{t\to+\infty} \frac{f(t)}{t^{p-1}} =: -\ell_\infty \in [-\infty, 0); \tag{3.6}$$

$$-\infty < \liminf_{t\to+\infty} \frac{F(t)}{t^p} \leq \limsup_{t\to+\infty} \frac{F(t)}{t^p} = +\infty. \tag{3.7}$$

In this setting, the counterpart of Theorem 3.1 may be stated as follows.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $N > sp$, $\lambda \in \mathbb{R}$. Assume that $\beta \in L^\infty(\Omega)$, and that $f \in C([0, +\infty); \mathbb{R})$ satisfies* (3.6), (3.7) *and $f(0) = 0$. If either*

(a) *$q = p - 1$, $\ell_\infty \in (0, +\infty)$ and $\lambda\beta(x) < \lambda_\infty$ almost everywhere $x \in \Omega$ for some $\lambda_\infty \in (0, \ell_\infty)$, or*
(b) *$q = p - 1$, $\ell_\infty = +\infty$, and $\lambda \in \mathbb{R}$ is arbitrary, or*
(c) *$0 < q < p - 1$, and $\lambda \in \mathbb{R}$ is arbitrary,*

then there exists a sequence $\{u_j\}_j$ in $X(\Omega) \cap L^\infty(\Omega)$ of distinct weak solutions of problem (1.1) *such that*

$$\lim_{j\to+\infty} \|u_j\|_{L^\infty(\Omega)} = +\infty. \tag{3.8}$$

A special case of the above result is as follows.

Corollary 3.3. *Let $q \leq p - 1$, and let all of the assumptions of Theorem* 3.2 *be satisfied. In addition, assume that*

$$\sup_{t\in[0,+\infty)} \frac{|f(t)|}{1 + t^{p_s^*-1}} < +\infty, \tag{3.9}$$

where p_s^* is as given in (3.5)*. Then,*

$$\lim_{j\to+\infty} \|u_j\| = +\infty,$$

where $\{u_j\}_j$ is the sequence of distinct weak solutions of problem (1.1), *given by Theorem* 3.2.

As in the case when there is oscillation near the origin, here, assumption (3.6) is used in order to prove the existence of solutions for problem (1.1), while (3.7) guarantees that these solutions are infinitely many, when $0 < q \leq p-1$, and at least one finite number, if $q > p-1$.

In all of the situations, that is, when there is an oscillation near zero or at infinity, and for any value of q, the idea is to prove the existence of solutions for problem (1.1) using the variational method. More precisely, we first consider an auxiliary problem and, under suitable assumptions on the data, we prove the existence of solutions for this equation studying the associated energy functional and proving that this functional admits a minimum, using the direct methods of the calculus of variations (see Theorem 4.1).

Next, we apply Theorem 4.1 to problem (1.1) in order to obtain Theorems 3.1 and 3.2.

4. An auxiliary nonlocal problem. In this section, we consider the problem

$$(\mathcal{P}_h^K) \qquad \begin{cases} (-\Delta)_p^s u + K(x)|u|^{p-2}u = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Here, we assume that $K : \Omega \to \mathbb{R}$ is such that

$$K \in L^\infty(\Omega) \text{ with } \operatorname*{ess\,inf}_{x \in \Omega} K(x) > 0, \tag{4.1}$$

while $h : \Omega \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function satisfying the following conditions:

$$h(x,0) = 0 \quad \text{for almost every } x \in \Omega; \tag{4.2}$$

(a) there exists an $M > 0$ such that $|h(x,t)| \leq M$ for almost every $x \in \Omega$ and for any $t \geq 0$;
(b) there exist δ and η, with $0 < \delta < \eta$ such that $h(x,t) \leq 0$ for almost every $x \in \Omega$ and for any $t \in [\delta, \eta]$.

In the sequel, we extend the function h on the whole $\Omega \times \mathbb{R}$ by taking $h(x,t) = 0$ for almost every $x \in \Omega$ and $t < 0$.

The aim of this section is to prove the existence of a non-negative weak solution for problem (refauxiliary), that is, a non-negative solution to the following problem:

$$
\begin{cases} \langle A(u), \varphi \rangle + \int_\Omega K(x)|u(x)|^{p-2}u(x)\varphi(x)\,dx \\ \quad = \int_\Omega h(x,u(x))\varphi(x)\,dx & \text{for any } \varphi \in X(\Omega), \\ u \in X(\Omega). \end{cases} \tag{4.3}
$$

In order to achieve our aim, we look for critical points of the energy functional $\mathcal{J}_{K,h} : X(\Omega) \to \mathbb{R}$, defined by setting

$$
\mathcal{J}_{K,h}(u) = \frac{1}{p}\|u\|^p + \frac{1}{p}\int_\Omega K(x)|u(x)|^p dx - \int_\Omega H(x,u(x))\,dx, \tag{4.4}
$$

where

$$
H(x,t) := \int_0^t h(x,\tau)\,d\tau \quad \text{for any } t \in \mathbb{R}. \tag{4.5}
$$

By using (4.1)–(4.2) (a), (b) and Theorem 2.2, we can deduce that $\mathcal{J}_{K,h}$ is well defined, and $\mathcal{J}_{K,h}$ is of class C^1 on $X(\Omega)$.

Now, we introduce the set W^η, defined as follows

$$
\mathcal{W}^\eta := \big\{ u \in X(\Omega) : \|u\|_{L^\infty(\Omega)} \leq \eta \big\},
$$

where η is the positive parameter given in (4.2) (b).

The main result of this section is the following:

Theorem 4.1. *We assume that $K : \Omega \to \mathbb{R}$ is a function verifying* (4.1) *and that $h : \Omega \times [0,+\infty) \to \mathbb{R}$ is a Carathéodory function satisfying* (4.2) (a), (b). *Then, we have*:

(i) *the functional $\mathcal{J}_{K,h}$ is bounded from below on $\mathcal{W}^\eta$ and its infimum is attained at some $u_\eta \in \mathcal{W}^\eta$;*
(ii) *$u_\eta \in [0,\delta]$, where δ is the positive parameter given in* (4.2);
(iii) *u_η is a non-negative weak solution of problem $(\mathcal{P}_h^K)$.*

Proof.

(i) Firstly, we note that $\mathcal{W}^\eta$ is convex. Moreover, $\mathcal{W}^\eta$ is closed in $X(\Omega)$. In fact, let $\{u_j\}_j$ be a sequence in $\mathcal{W}^\eta$ such that $u_j \to u$ in $X(\Omega)$ as $j \to +\infty$. We aim to prove that $u \in \mathcal{W}^\eta$. Clearly, $u \in X(\Omega)$. Since $\{u_j\}_j$ is bounded in $L^\infty(\Omega)$, and $L^\infty(\Omega)$ is the dual space of $L^1(\Omega)$, which is a separable Banach space by [**5,** Corollary III.26], it follows that $u_j \to u$ in the weak* topology of $L^\infty(\Omega)$ as $j \to +\infty$. Then, by using [**5,** Proposition III.12], we obtain, up to a subsequence,

$$\liminf_{j\to+\infty} \|u_j\|_{L^\infty(\Omega)} \geq \|u\|_{L^\infty(\Omega)}.$$

This and

$$\|u_j\|_{L^\infty(\Omega)} \leq \eta,$$

for any $j \in \mathbb{N}$, imply that

$$\|u\|_{L^\infty(\Omega)} \leq \eta,$$

that is, $u \in \mathcal{W}^\eta$. Since we have proved that $\mathcal{W}^\eta$ is convex and closed in $X(\Omega)$, we can deduce that $\mathcal{W}^\eta$ is weakly closed in $X(\Omega)$ by [**5,** Theorem III.7].

Now, we consider the functional $\mathcal{J}_{K,h}$. It is clear that $\mathcal{J}_{K,h}$ is sequentially weakly lower semicontinuous. Moreover, $\mathcal{J}_{K,h}$ is bounded from below on $\mathcal{W}^\eta$. In fact, by using (4.1) and (4.2) (a), we have, for any $u \in \mathcal{W}^\eta$,

$$\begin{aligned}
\mathcal{J}_{K,h}(u) &= \frac{1}{p}\|u\|^p + \frac{1}{p}\int_\Omega K(x)|u(x)|^p\,dx - \int_\Omega H(x,u(x))\,dx \\
&\geq \frac{1}{p}\|u\|^p - \int_\Omega H(x,u(x))\,dx \\
&\geq -\int_\Omega H(x,u(x))\,dx \\
&\geq -M\int_\Omega |u(x)|\,dx \\
&\geq -\eta M\mathcal{L}(\Omega),
\end{aligned}$$

where, hereon, $\mathcal{L}(\Omega)$ denotes the Lebesgue measure of Ω.

Set

$$m_\eta := \inf_{u\in\mathcal{W}^\eta} \mathcal{J}_{K,h}(u) > -\infty. \tag{4.6}$$

Then, for every $k \in \mathbb{N}$, there exists a $u_k \in \mathcal{W}^\eta$ such that

$$m_\eta \leq \mathcal{J}_{K,h}(u_k) \leq m_\eta + \frac{1}{k}. \tag{4.7}$$

By using $u_k \in \mathcal{W}^\eta$, (4.2) (a) and (4.7), we can see that

$$\begin{aligned}\frac{1}{p}\|u_k\|^p + \frac{1}{p}\int_\Omega K(x)|u_k(x)|^p\,dx &= \int_\Omega H(x,u_k(x))\,dx + \mathcal{J}_{K,h}(u_k)\\ &\leq \eta M\mathcal{L}(\Omega) + \mathcal{J}_{K,h}(u_k)\\ &\leq \eta M\mathcal{L}(\Omega) + m_\eta + \frac{1}{k}\\ &\leq \eta M\mathcal{L}(\Omega) + m_\eta + 1,\end{aligned}$$

for every $k \in \mathbb{N}$. In view of (4.1), we obtain

$$\|u_k\|^p \leq p(\eta M\mathcal{L}(\Omega) + \alpha_\eta + 1), \tag{4.8}$$

for every $k \in \mathbb{N}$, that is, $\{u_k\}_k$ is bounded in $X(\Omega)$. Then, up to a subsequence, we may assume that

$$u_k \longrightarrow u_\eta \quad \text{weakly in } X(\Omega) \tag{4.9}$$

as $k \to +\infty$ for some $u_\eta \in X(\Omega)$.

Now, our claim is to prove that u_η is the minimum of $\mathcal{J}_{K,h}$. Since $\mathcal{W}^\eta$ is weakly closed in $X(\Omega)$, we can see that $u_\eta \in \mathcal{W}^\eta$. Then, we have

$$\mathcal{J}_{K,h}(u_\eta) \geq m_\eta. \tag{4.10}$$

Now, by using the sequential weak lower semicontinuity of $\mathcal{J}_{K,h}$, (4.7) and (4.9), we get

$$m_\eta \geq \liminf_{k\to+\infty} \mathcal{J}_{K,h}(u_k) \geq \mathcal{J}_{K,h}(u_\eta).$$

This and (4.10) yield

$$\mathcal{J}_{K,h}(u_\eta) = m_\eta.$$

(ii) Let δ be as in (4.2) (b), and let us define

$$A := \{x \in \Omega : u_\eta(x) \notin [0,\delta]\}.$$

Our aim is to prove that $\mathcal{L}(A) = 0$. Assume, by contradiction, that $\mathcal{L}(A) > 0$.

We introduce the map $\gamma : \mathbb{R} \to \mathbb{R}$, defined by

$$\gamma(t) := \min\{t_+, \delta\},$$

where $t_+ = \max\{t, 0\}$. We also define $w := \gamma \circ u_\eta$, that is,

$$w(x) = \begin{cases} \delta & \text{if } u_\eta(x) > \delta, \\ u_\eta(x) & \text{if } 0 \le u_\eta(x) \le \delta, \\ 0 & \text{if } u_\eta(x) < 0, \end{cases}$$

for almost every $x \in \Omega$, and $w(x) = 0$ almost everywhere $x \in \mathbb{R}^N$.

Since γ is a Lipschitz function with Lipschitz constant equals to 1, we obtain

$$\iint_{\mathbb{R}^{2N}} \frac{|\gamma(u_\eta(x)) - \gamma(u_\eta(y))|^p}{|x-y|^{N+sp}}\, dx\, dy \le \iint_{\mathbb{R}^{2N}} \frac{|u_\eta(x) - u_\eta(y)|^p}{|x-y|^{N+sp}}\, dx\, dy,$$

which implies that $w \in X(\Omega)$. Moreover, $0 \le w(x) \le |u_\eta(x)|$ for almost every $x \in \mathbb{R}^N$, $0 \leqslant w(x) \leqslant \delta$ for almost every Ω, and, by using the fact that $\delta < \eta$, in view of (4.2) (b), we can infer that $w \in \mathcal{W}^\eta$.

The following sets are defined as

$$A_1 := \{x \in \Omega : u_\eta(x) < 0\}$$

and

$$A_2 := \{x \in \Omega : u_\eta(x) > \delta\}.$$

Thus, it is clear that $A = A_1 \cup A_2$. Moreover, we can see that $w(x) = u_\eta(x)$ for almost every $x \in \Omega \setminus A, w(x) = 0$ for almost every $x \in A_1$, and $w(x) = \delta$ for almost every $x \in A_2$.

Now, we aim to show that

$$\|w\|^p - \|u_\eta\|^p \le 0. \tag{4.11}$$

We note that

(4.12)

$$\begin{aligned} \|w\|^p - \|u_\eta\|^p = &\iint_{\Omega\times\Omega} \frac{|w(x) - w(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x) - u_\eta(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \\ &+ 2 \iint_{\Omega\times(\mathbb{R}^N\setminus\Omega)} \frac{|w(x)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x)|^p}{|x-y|^{N+sp}}\, dx\, dy \\ \le &\iint_{\Omega\times\Omega} \frac{|w(x) - w(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x) - u_\eta(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \end{aligned}$$

$$
\begin{aligned}
&= \iint\limits_{A\times A} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad + \iint\limits_{A\times(\Omega\setminus A)} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad + \iint\limits_{(\Omega\setminus A)\times A} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dxdy \\
&=: I_1 + I_2 + I_3.
\end{aligned}
$$

Consider the first term I_1, and observe that

$$
\begin{aligned}
I_1 &= \iint\limits_{A_1\times A_1} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad + \iint\limits_{A_2\times A_2} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad + \iint\limits_{A_1\times A_2} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad + \iint\limits_{A_2\times A_1} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy.
\end{aligned}
\tag{4.13}
$$

Since $w(x) = 0$ for almost every $x \in A_1$, and $w(x) = \delta$ for almost every $x \in A_1$, it may be seen that

$$
\begin{aligned}
&\iint\limits_{A_1\times A_1} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&\qquad = -\iint\limits_{A_1\times A_1} \frac{|u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy
\end{aligned}
\tag{4.14}
$$

and

$$
\begin{aligned}
&\iint\limits_{A_2\times A_2} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&\qquad = -\iint\limits_{A_2\times A_2} \frac{|u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy.
\end{aligned}
\tag{4.15}
$$

On the other hand, $w(x) = \delta$ and $u_\eta(x) < 0$ for almost every $x \in A_1$, and $w(y) = \delta$ and $u_\eta(y) > \delta$ for almost every $x \in A_2$; thus, we have

$$|u_\eta(x) - u_\eta(y)|^p = (u_\eta(y) - u_\eta(x))^p \geq \delta^p,$$

which gives

$$\begin{aligned}\tag{4.16} &\iint_{A_1\times A_2} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\ &\qquad = \iint_{A_1\times A_2} \frac{\delta^p - |u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \leq 0.\end{aligned}$$

A similar argument shows that

$$\iint_{A_2\times A_1} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \leq 0. \tag{4.17}$$

Taking into account (4.13)–(4.17), we can deduce that $I_1 \leq 0$.

Now, we estimate I_2. Then, we can see that

$$\begin{aligned}\tag{4.18} I_2 = &\iint_{A_1\times(\Omega\setminus A)} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\ &+ \iint_{A_2\times(\Omega\setminus A)} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy.\end{aligned}$$

Since $w(x) = 0$ and $u_\eta(x) < 0$ for almost every $x \in A_1$, and $w(y) = u_\eta(y) \in [0, \delta]$ for almost every $y \in \Omega \setminus A$, we can see that

$$|u_\eta(x) - u_\eta(y)|^p = (u_\eta(y) - u_\eta(x))^p \geq u_\eta(y)^p = |u_\eta(y)|^p,$$

which implies that

$$\begin{aligned}\tag{4.19} &\iint_{A_1\times(\Omega\setminus A)} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\ &\qquad = \iint_{A_1\times(\Omega\setminus A)} \frac{|u_\eta(x)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x)-u_\eta(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\ &\qquad \leq 0.\end{aligned}$$

On the other hand, $w(x) = \delta$ and $u_\eta(x) > \delta$ for almost every $x \in A_2$, and $w(y) = u_\eta(y) \in [0, \delta]$ for almost every $y \in \Omega \setminus A$; thus, we obtain

$$|\delta - u_\eta(y)|^p = (\delta - u_\eta(y))^p \leq (u_\eta(x) - u_\eta(y))^p = |u_\eta(x) - u_\eta(y)|^p,$$

which gives

$$\begin{aligned}
&\iint\limits_{A_2 \times (\Omega \setminus A)} \frac{|w(x) - w(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x) - u_\eta(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \\
&\quad = \iint\limits_{A_2 \times (\Omega \setminus A)} \frac{|\delta - u(y)|^p}{|x-y|^{N+sp}} - \frac{|u_\eta(x) - u_\eta(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \\
&\quad \leq 0.
\end{aligned} \tag{4.20}$$

Combining (4.18), (4.19) and (4.20), we deduce that $I_2 \leq 0$.

In a similar fashion, we can prove that $I_3 \leq 0$. Then, by using (4.12) and the fact that all terms I_1, I_2, I_3 are nonpositive, we can conclude that (4.11) holds.

Hence, by using (4.11), we can see that

$$\begin{aligned}
\mathcal{J}_{K,h}(w) - \mathcal{J}_{K,h}(u_\eta) &= \frac{1}{p}\|w\|^p - \frac{1}{p}\|u\|^p \\
&\quad + \frac{1}{p}\int_\Omega K(x)(|w(x)|^p - |u_\eta(x)|^p)\, dx \\
&\quad - \int_\Omega (H(x, w(x)) - H(x, u_\eta(x)))\, dx \\
&\leq \frac{1}{p}\int_A K(x)(|w(x)|^p - |u_\eta(x)|^p)\, dx \\
&\quad - \int_A (H(x, w(x)) - H(x, u_\eta(x)))\, dx.
\end{aligned} \tag{4.21}$$

Recalling that $\operatorname{ess\,inf}_{x\in\Omega} K(x) > 0$ by (4.1), $u_\eta(x) > \delta$ for almost every $x \in A_2$ and $w(x) = 0$ for almost every $x \in A_1$, we can deduce that

$$\begin{aligned}
\int_A K(x)(|w(x)|^p - |u_\eta(x)|^p)\, dx &= -\int_{A_1} K(x)|u_\eta(x)|^p\, dx \\
&\quad + \int_{A_2} K(x)(\delta^p - |u_\eta(x)|^p)\, dx \leqslant 0.
\end{aligned} \tag{4.22}$$

Since $h(x,t)=0$ for almost every $x\in\Omega$ and all $t\le 0$, we get

$$\int_{A_1}(H(x,w(x))-H(x,u_\eta(x)))\,dx=0. \tag{4.23}$$

On the other hand, by the mean value theorem, for almost every $x\in A_2$, we can find $\theta(x)\in[\delta,u_\eta(x)]\subseteq[\delta,\eta]$ such that

$$\begin{aligned}H(x,w(x))-H(x,u_\eta(x))&=H(x,\delta)-H(x,u_\eta(x))\\&=h(x,\theta(x))(\delta-u_\eta(x)).\end{aligned}$$

Then, (4.2) (b) and the definition of B_2 yield

$$\int_{A_2}(H(x,w(x))-H(x,u_\eta(x)))\,dx=\int_{A_2}h(x,\theta(x))(\delta-u_\eta(x))\,dx\ge 0. \tag{4.24}$$

By combining (4.23) and (4.4), we can see that

$$\int_{A}(H(x,w(x))-H(x,u_\eta(x)))\,dx\ge 0. \tag{4.25}$$

Taking into account (4.21), (4.22) and (4.25), we obtain

$$\mathcal{J}_{K,h}(w)-\mathcal{J}_{K,h}(u_\eta)\le 0. \tag{4.26}$$

However, $w\in\mathcal{W}^\eta$; thus, $\mathcal{J}_{K,h}(w)\geqslant\mathcal{J}_{K,h}(u_\eta)$, and, by using (4.26), we obtain that

$$\mathcal{J}_{K,h}(w)=\mathcal{J}_{K,h}(u_\eta). \tag{4.27}$$

Now, taking into account (4.27) and the fact that all of the integrals on the right hand-side of (4.21) are non-negative, we can deduce that

$$\int_{A_1}K(x)|u_\eta(x)|^p=\int_{A_2}K(x)(|u_\eta(x)|^p-\delta^p)\,dx=0.$$

From the definition of A_1 and A_2, and using (4.1), we get $\mathcal{L}(A_1)=\mathcal{L}(A_2)=0$, that is, $\mathcal{L}(A)=0$, which gives a contradiction.

(iii) Take $\varphi\in C_0^\infty(\Omega)$, and let

$$\varepsilon_0:=\frac{\eta-\delta}{\|\varphi\|_{L^\infty(\Omega)}+1}>0,$$

where δ and η are as given in (4.2) (b). We introduce the functional $\mathcal{I}:[-\varepsilon_0,\varepsilon_0]\to\mathbb{R}$, defined by setting

$$\mathcal{I}(\varepsilon)=\mathcal{J}_{K,h}(u_\eta+\varepsilon\varphi).$$

Using (ii), we can see that, for any $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$,

$$\begin{aligned}|u_\eta(x) + \varepsilon\varphi(x)| &\le |u_\eta(x)| + |\varepsilon||\varphi(x)| \\ &\le u_\eta(x) + \frac{\eta - \delta}{\|\varphi\|_{L^\infty(\Omega)} + 1}\|\varphi\|_{L^\infty(\Omega)} \le \delta + \eta - \delta = \eta,\end{aligned}$$

for almost every $x \in \Omega$, that is $u_\eta + \varepsilon\varphi \in \mathcal{W}^\eta$.

Hence, by using Theorem 4.1 (i), we can see that $I(\varepsilon) \ge I(0)$ for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, which implies that zero is an interior minimum point for I. Since $\mathcal{I}$ is differentiable at zero, we obtain $\mathcal{I}'(0) = 0$ and $\langle \mathcal{J}'_{K,h}(u_\eta), \varphi\rangle = 0$. Since $C_0^\infty(\Omega)$ is dense in $X(\Omega)$ (see **[16]**), we see that $\langle \mathcal{J}'_{K,h}(u_\eta), \varphi\rangle = 0$ for any $\varphi \in X(\Omega)$, and this gives that u_η is a weak solution of problem $(\mathcal{P}_h^K)$ (that is, a solution of (4.3)). Finally, u_η is non-negative in Ω in view of Theorem 4.1 (ii). □

Note that $u \equiv 0$ is a weak solution of problem $(\mathcal{P}_h^K)$, due to the fact that $h(x, 0) = 0$ almost everywhere $x \in \Omega$ by (4.2). This means that Theorem 4.1 does not guarantee that the solution u_η of problem $(\mathcal{P}_h^K)$ is not trivial. For this reason, we will choose the nonlinear term f in a suitable way, and, by using Theorem 4.1, we will be able to deduce the existence of non-trivial solutions for the original problem (1.1).

Finally, we denote by the truncation function $\tau_\eta : [0, +\infty) \to \mathbb{R}$, defined as

$$\tau_\eta(t) := \min\{\eta, t\} \tag{4.28}$$

for any $t \ge 0$, where η is the positive constant given in assumption (4.2) (b). Clearly, τ_η is a continuous function in $[0, +\infty)$.

5. Oscillatory behavior near the origin. In this section, we study problem (1.1) in the case where the nonlinear term f oscillates near the origin. In order to prove Theorem 3.1, we first give an auxiliary result obtained as a consequence of Theorem 5.1. Precisely, we prove the existence of infinitely many solutions for problem $(\mathcal{P}_h^K)$ under the following assumptions on the function h:

A. there exists a $\bar{t} > 0$ such that $\sup_{t\in[0,\bar{t}]} |h(\cdot, t)| \in L^\infty(\Omega)$;

B. there exist two sequences $\{\delta_j\}_j$ and $\{\eta_j\}_j$, with $0 < \eta_{j+1} < \delta_j < \eta_j$ and $\lim_{j\to+\infty} \eta_j = 0$ such that $h(x, t) \le 0$ for almost every $x \in \Omega$

and for every $t \in [\delta_j, \eta_j]$, $j \in \mathbb{N}$;

$$-\infty < \liminf_{t\to 0^+} \frac{H(x,t)}{t^p} \le \limsup_{t\to 0^+} \frac{H(x,t)}{t^p} = +\infty \tag{5.1}$$

uniformly for almost every $x \in \Omega$, where H is the function given in (4.5). More precisely, our result is as follows.

Theorem 5.1. *Let us assume that* $K : \Omega \to \mathbb{R}$ *satisfies* (4.1) *and* $h : \Omega \times [0, +\infty) \to \mathbb{R}$ *is a Carathéodory function verifying* (4.2) *and assumptions* A, B *and* (5.1). *Then, there exists a sequence* $\{u_j\}_j \subset X(\Omega)$ *of distinct nontrivial, non-negative weak solutions of problem* $(\mathcal{P}_h^K)$ *such that*

$$\lim_{j\to+\infty} \|u_j\|_{X(\Omega)} = \lim_{j\to+\infty} \|u_j\|_{L^\infty(\Omega)} = 0. \tag{5.2}$$

Proof. By using assumption B, we know that $\eta_j \to 0$ as $j \to +\infty$; therefore, without loss of generality, we may assume that

$$\delta_j < \eta_j < \overline{s} \tag{5.3}$$

for j sufficiently large, where $\overline{t} > 0$ is from A.

Now, for every $j \in \mathbb{N}$, we introduce the function $h_j : \Omega \times [0, +\infty) \to \mathbb{R}$, defined by

$$h_j(x,t) = h(x, \tau_{\eta_j}(t)) \tag{5.4}$$

and

$$H_j(x,t) := \int_0^t h_j(x,z)\, dz$$

for almost every $x \in \Omega$ and $t \ge 0$, where τ_{η_j} is the function defined in (4.28) with $\eta = \eta_j$. In order to simplify the notation, in what follows, we denote by

$$\mathcal{J}_j := \mathcal{J}_{K,\, h_j} \quad j \in \mathbb{N}, \tag{5.5}$$

where $\mathcal{J}_{K,\, h_j}$ is the functional given in (4.4), with $h = h_j$.

We show that h_j satisfies the assumptions of Theorem 4.1, for $j \in \mathbb{N}$ large enough. From the regularity of h, the continuity of τ_η and (4.2), the function h_j is Carathéodory and such that $h_j(x,0) = 0$ almost every $x \in \Omega$. By assumption A, (5.3) and (5.4), h_j satisfies (4.2) (a). Finally, condition (4.2) (b) holds in view of assumption B.

Thus, we are in a position to apply Theorem 4.1, and, for j sufficiently large, we can find $u_j \in \mathcal{W}^{\eta_j}$ such that

$$\min_{u\in\mathcal{W}^{\eta_j}} \mathcal{J}_j(u) = \mathcal{J}_j(u_j) \tag{5.6}$$

$$u_j(x) \in [0, \delta_j] \quad \text{for almost every } x \in \Omega, \tag{5.7}$$

and

C. u_j is a non-negative weak solution of $(\mathcal{P}^K_{h_j})$.

Taking into account the definition of τ_η, (5.4) and $u_j(x) \le \delta_j < \eta_j$ almost everywhere $x \in \Omega$, we can deduce that

$$h_j(x, u_j(x)) = h(x, \tau_{\eta_j}(u_j(x)) = h(x, u_j(x))$$

almost everywhere $x \in \Omega$. In particular, by combining this and assumption C, we can deduce that u_j is a non-negative weak solution for $(\mathcal{P}^K_{h_j})$ and for problem $(\mathcal{P}^K_h)$.

Now, our claim is to prove that there are infinitely many distinct elements in the sequence $\{u_j\}_j$. Firstly, we show that

$$\mathcal{J}_j(u_j) < 0 \quad \text{for } j \in \mathbb{N} \text{ large enough.} \tag{5.8}$$

By (5.1), we can find $\ell > 0$ and $\zeta \in (0, \eta_1)$ such that

$$\operatorname*{ess\,inf}_{x\in\Omega} H(x,t) \ge -\ell t^p \quad \text{for all } t \in (0,\zeta), \tag{5.9}$$

and there exists a sequence $\{t_j\}_j$ such that $0 < t_j \to 0$ as $j \to +\infty$ (here, we use the definition of H to take $t_j > 0$) such that

$$\lim_{j\to+\infty} \frac{\operatorname{ess\,inf}_{x\in\Omega} H(x,t_j)}{t_j^p} = +\infty, \tag{5.10}$$

which gives, for any $L > 0$,

$$\operatorname*{ess\,inf}_{x\in\Omega} H(x,t_j) > L t_j^p \tag{5.11}$$

for $j \in \mathbb{N}$ large enough.

Since $\delta_j \searrow 0$ as $j \to +\infty$, up to a subsequence, we may assume that

$$t_j \le \delta_j \quad \text{for all } j \in \mathbb{N}. \tag{5.12}$$

Now, take $j \in \mathbb{N}$ sufficiently large, and set

$$z_j := z_{t_j} \in X(\Omega),$$

where z_t is the function defined as in (2.1) with $t = t_j$. Then, $z_j \in X(\Omega)$ and $\|z_j\|_{L^\infty(\Omega)} = t_j \le \delta_j < \eta_j$ by B and (5.12). As a consequence, $z_j \in \mathcal{W}^{\eta_j}$ and $0 \le z_j(x) \le t_j \le \delta_j < \eta_j$ almost everywhere $x \in \Omega$. In particular, we obtain

$$\int_0^{z_j(x)} h_j(x,y)\,dy = \int_0^{z_j(x)} h(x,\tau_{\eta_j}(y))\,dy = \int_0^{z_j(x)} h(x,y)\,dy$$

for almost every $x \in \Omega$. Then, by using (2.2), (4.1), (5.9), (5.11), and using the fact that $z_j(x) < \eta_j < \eta_1$ (being $\{\eta_j\}_j$ decreasing by (5.2)), we obtain for j sufficiently large

(5.13)

$$\begin{aligned}
\mathcal{J}_j(z_j) &= \frac{1}{p}\|z_j\|^p + \frac{1}{p}\int_\Omega K(x)|z_j(x)|^p dx - \int_\Omega H_j(x,z_j(x))\,dx \\
&= \frac{1}{p}\|z_j\|^p + \frac{1}{p}\int_\Omega K(x)|z_j(x)|^p dx - \int_\Omega H(x,z_j(x))\,dx \\
&\le C(r,s,p,N)\frac{1}{p}t_j^p + \frac{1}{p}\int_\Omega K(x)|z_j(x)|^p dx \\
&\quad - \int_{B(x_0,r/2)} H(x,t_j)\,dx - \int_{B(x_0,r)\setminus B(x_0,r/2)} H(x,z_j(x))\,dx \\
&\le \left(C(r,s,p,N)\frac{1}{p} + \|K\|_{L^\infty(\Omega)}\frac{\mathcal{L}(\Omega)}{p} - L(r/2)^N\omega_N + \ell\mathcal{L}(\Omega)\right)t_j^p.
\end{aligned}$$

Taking $L > 0$ sufficiently large such that

$$L(r/2)^N\omega_N > C(r,s,p,N)\frac{1}{p} + \|K\|_{L^\infty(\Omega)}\frac{\mathcal{L}(\Omega)}{p} + \ell\mathcal{L}(\Omega),$$

we deduce that, for j large enough,

$$\mathcal{J}_j(z_j) < 0.$$

Therefore, using (5.6), we get

$$\mathcal{J}_j(u_j) = \min_{u\in\mathcal{W}^{\eta_j}} \mathcal{J}_j(u) \le \mathcal{J}_j(z_j) < 0, \tag{5.14}$$

for j sufficiently large. Then, (5.8) holds, and this allows us to infer that $u_j \not\equiv 0$ since $\mathcal{J}_j(0) = 0$.

Now, we aim to show that

$$\lim_{j\to+\infty} \mathcal{J}_j(u_j) = 0. \tag{5.15}$$

Firstly, we note that, from the definition of H_j, (4.1), A, B, (5.3), (5.4) and (5.7), we have, for any $j \in \mathbb{N}$ sufficiently large,

$$\begin{aligned}\mathcal{J}_j(u_j) &\geq -\int_\Omega H_j(x, u_j(x))\, dx = -\int_\Omega \int_0^{u_j(x)} h(x,t)\, dt \\ &\geq -\int_\Omega \sup_{s\in[0,\bar{t}]} |h(x,t)|\, u_j(x)\, dx \\ &\geq -\mathcal{L}(\Omega)\Big\| \sup_{t\in[0,\bar{t}]} |h(\cdot,t)|\Big\|_{L^\infty(\Omega)} \delta_j.\end{aligned} \tag{5.16}$$

By using B, we know that $\lim_{j\to+\infty} \delta_j = 0$; thus, the above inequality and (5.14) imply that (5.15) holds.

Hence, taking into account (5.8) and (5.15), we can see that $\{u_j\}_j$ contains infinitely many distinct elements, which means that problem $(\mathcal{P}_h^K)$ possesses infinitely many distinct weak solutions.

Finally, we prove (5.2). Concerning the first limit, we can observe that, from (5.7), it follows that $\|u_j\|_{L^\infty(\Omega)} \leq \delta_j$ for $j \in \mathbb{N}$ sufficiently large. Then, recalling that $\lim_{j\to+\infty} \delta_j = 0$ (see B), we can infer that $\|u_j\|_{L^\infty(\Omega)} \to 0$ as $j \to +\infty$.

Now, we prove that the latter limit holds. Combining (4.1), A, (5.7) and (5.8), we obtain

$$\begin{aligned}\frac{1}{p}\|u_j\|^p &\leq \frac{1}{p}\|u_j\|^p + \frac{1}{p}\int_\Omega K(x)|u_j(x)|^p dx \\ &< \int_\Omega H_j(x,u_j(x))\, dx = \int_\Omega H(x,u_j(x))\, dx \\ &\leq \mathcal{L}(\Omega)\Big\| \sup_{t\in[0,\bar{t}]} |h(\cdot,t)|\Big\|_{L^\infty(\Omega)} \delta_j,\end{aligned}$$

and, by using B, we have

$$\lim_{j\to+\infty} \|u_j\|^p = 0. \qquad \square$$

Now, we prove Theorem 3.1. In order to do so, we will apply Theorems 4.1 and 5.1, choosing the functions h and K in a suitable way.

5.1. Proof of Theorem 3.1. Firstly, we prove that problem (1.1) admits infinitely many distinct weak solutions, provided $q \geq p-1$. We distinguish the cases $q = p-1$ and $q > p-1$; in both situations, we will apply Theorem 5.1.

We begin by proving that (a) holds. In this setting, we assume that $q = p-1$, $\ell_0 \in (0,+\infty)$ and $\lambda \in \mathbb{R}$ is such that $\lambda\beta(x) < \lambda_0$ almost everywhere $x \in \Omega$ for some $\lambda_0 \in (0,\ell_0)$. Take $\widetilde{\lambda}_0 \in (\lambda_0,\ell_0)$, and define

$$K(x) := \widetilde{\lambda}_0 - \lambda\beta(x) \quad \text{and} \quad h(x,t) := \widetilde{\lambda}_0 t^{p-1} + f(t), \tag{5.17}$$

almost everywhere $x \in \Omega$ and $t \geq 0$.

Now, we show that K and h verify the assumptions of Theorem 5.1. Since $\beta \in L^\infty(\Omega)$, it is clear that $K \in L^\infty(\Omega)$ and

$$\operatorname*{ess\,inf}_{x\in\Omega} K(x) \geq \widetilde{\lambda}_0 - \lambda_0 > 0,$$

that is, (4.1) is satisfied.

Regarding the function h, we can observe that the regularity of f implies that h is a continuous function in $\Omega \times [0,+\infty)$, and $h(x,0) = 0$ for any $x \in \Omega$. Hence, (4.2) holds. In addition, the continuity of $t \mapsto h(\cdot,t)$ along with the Weierstrass theorem yield assumption A. Since, for any $x \in \Omega$ and $t > 0$, the following holds

$$\frac{H(x,t)}{t^p} = \frac{\widetilde{\lambda}_0}{p} + \frac{F(t)}{t^p},$$

and we can deduce (5.1) in view of (3.3).

Then, we must show that h verifies assumption B. For this purpose, we note that, by (3.2), there exists a sequence $\{t_j\}_j$ such that $t_j \to 0$ and

$$\frac{f(t_j)}{t_j^{p-1}} \longrightarrow -\ell_0, \tag{5.18}$$

as $j \to +\infty$. Since we are assuming that $\widetilde{\lambda}_0 < \ell_0$, we can find $\bar{\varepsilon} > 0$ such that $\widetilde{\lambda}_0 + \bar{\varepsilon} < \ell_0$. By combining this and (5.18), we can see that, for j large enough, say $j \geq j^* \in \mathbb{N}$,

$$\frac{f(t_j)}{t_j^{p-1}} < -\widetilde{\lambda}_0. \tag{5.19}$$

Then, using the continuity of f, it is possible to find a neighborhood of t_j, say (δ_j, η_j), such that

$$h(x,t) = \widetilde{\lambda}_0 t^{p-1} + f(t) \le 0,$$

for any $x \in \Omega$ and all $t \in [\delta_j, \eta_j]$ and $j \ge j^*$. As a consequence, B is satisfied.

Therefore, we can apply Theorem 5.1 to problem $(\mathcal{P}_h^K)$ with h and K given in (5.17), and to obtain the existence of infinitely many distinct non trivial non-negative solutions $\{u_j\}_j$ for problem $(\mathcal{P}_h^K)$, satisfying condition (3.4). From the definitions of h and K, and recalling that $q = p-1$, we can see that u_j is a weak solution of problem (1.1). This concludes the proof of Theorem 3.1 in the case $q = p-1$.

Now, let us consider assertion (b). For this purpose, let $q = p-1$, $\ell_0 = +\infty$ and $\lambda \in \mathbb{R}$. We choose $\widetilde{\lambda}_0 \in (\lambda_0, \ell_0)$, and we define

$$K(x) := \widetilde{\lambda}_0 \quad \text{and} \quad h(x,t) := (\lambda\beta(x) + \widetilde{\lambda}_0)t^{p-1} + f(t), \tag{5.20}$$

almost everywhere $x \in \Omega$ and $t \ge 0$. Then, we can proceed as in the proof of assertion (a), merely replacing formula (5.19) with the following one

$$\frac{f(t_j)}{t_j^{p-1}} < -(|\lambda| \|\beta\|_{L^\infty(\Omega)} + \widetilde{\lambda}_0). \tag{5.21}$$

For j large enough, we take into account that

$$h(x,t) = (\lambda\beta(x) + \widetilde{\lambda}_0)t^{p-1} + f(t) \le (|\lambda| \|\beta\|_{L^\infty(\Omega)} + \widetilde{\lambda}_0)t^{p-1} + f(t).$$

Finally, we deal with assertion (c). Let $q > p-1$ and $\lambda \in \mathbb{R}$. Take $\widetilde{\lambda}_0 \in (0, \ell_0)$. We introduce the functions

$$K(x) := \widetilde{\lambda}_0 \quad \text{and} \quad h(x,t) := \lambda\beta(x)t^q + \widetilde{\lambda}_0 t^{p-1} + f(t) \tag{5.22}$$

for almost every $x \in \Omega$ and $t \ge 0$. Also, in this setting, our claim is to prove that the functions h and K, defined in (5.22), verify the conditions required by Theorem 5.1.

It is clear that (4.1) and (4.2) hold. By using $\beta \in L^\infty(\Omega)$, the continuity of $t \mapsto h(\cdot, t)$ and the Weierstrass theorem, we can see that assumption A is satisfied. Since, for almost every $x \in \Omega$ and $t > 0$, we have

$$\frac{H(x,t)}{t^p} = \lambda \frac{\beta(x)}{q+1} t^{q-p+1} + \frac{\widetilde{\lambda}_0}{p} + \frac{F(t)}{t^p},$$

we can use (3.3) and $q > p-1$, to deduce that (5.1) holds.

Concerning condition B, we can observe that, for almost every $x \in \Omega$ and any $t \geq 0$, we have

$$h(x,t) \leq |\lambda| \, \|\beta\|_{L^\infty(\Omega)} t^q + \widetilde{\lambda}_0 t^{p-1} + f(t). \tag{5.23}$$

This, (3.2) and $q > p-1$ imply

$$\liminf_{t\to 0^+} \frac{h(x,t)}{t^{p-1}} \leq \liminf_{s\to 0^+} \left(\lambda| \, \|\beta\|_{L^\infty(\Omega)} t^{q-p+1} + \widetilde{\lambda}_0 + \frac{f(t)}{t^{p-1}} \right) = \widetilde{\lambda}_0 - \ell_0 < 0 \tag{5.24}$$

uniformly almost everywhere $x \in \Omega$. Hence, we can find a sequence $\{t_j\}_j$ converging to 0 as $j \to +\infty$ such that $h(x,t_j) < 0$ for $j \in \mathbb{N}$ large enough and uniformly almost everywhere $x \in \Omega$. Thus, by using the continuity of $t \mapsto h(\cdot,t)$, there exist two sequences $\{\delta_j\}_j$ and $\{\eta_j\}_j$ such that $0 < \eta_{j+1} < \delta_j < t_j < \eta_j, \lim_{j\to+\infty} \eta_j = 0$ and $h(x,t) \leq 0$, for almost every $x \in \Omega$, all $t \in [\delta_j, \eta_j]$ and j large enough. This concludes the proof of (5.2).

Hence, we can argue as in the proof of assertion (a), and, by applying Theorem 5.1, we obtain that (c) is satisfied.

Example 5.2. Let us consider problem (1.1), when f is given by

$$f(t) = \begin{cases} \alpha t^{\alpha-1}(1-\sin t^{-\sigma}) + \sigma t^{\alpha-\sigma-1}\cos t^{-\sigma} - p\gamma t^{p-1} & \text{if } t > 0, \\ 0 & \text{if } t = 0, \end{cases}$$

where α, σ and γ are such that $1 < \sigma + 1 < \alpha < p$ and $\gamma > 0$. Note that f is continuous in $[0,+\infty)$, and F is the following function

$$F(t) = \int_0^t f(\tau)\,d\tau = t^\alpha(1-\sin t^{-\sigma}) - \gamma t^p, \quad t > 0.$$

Another prototype for f is given by

$$f(t) = \begin{cases} \alpha t^{\alpha-1}\cos^2 t^{-\sigma} - 2\sigma t^{\alpha-\sigma-1}\cos t^{-\sigma}\sin t^{-\sigma} - p\gamma t^{p-1} & \text{if } t > 0, \\ 0 & \text{if } t = 0, \end{cases}$$

where α, σ and γ are such that $1 < \alpha < p$, $\sigma > 0$, $\alpha - \sigma > 1$ and $\gamma > 0$. Due to these choices of the parameters, f is continuous in $[0, +\infty)$. In addition, F is the following function

$$F(t) = \int_0^t f(\tau)\, d\tau = t^\alpha \cos^2 t^{-\sigma} - \gamma t^p, \quad t > 0.$$

In both examples, we deduce by direct calculation that f and F satisfy assumptions (3.2) and (3.3).

Remark 5.3. In Theorem 3.1, if $0 < q < p - 1$, then, for every $k \in \mathbb{N}$, there exists a $\Lambda_k > 0$ such that problem (1.1) has at least k distinct weak solutions $u_1, \ldots, u_k \in X(\Omega)$ such that

$$\|u_j\| \leq 1/j \quad \text{and} \quad \|u_j\|_{L^\infty(\Omega)} \leq 1/j, \tag{5.25}$$

$j = 1, \ldots, k$, provided that $|\lambda| < \Lambda_k$.

6. Oscillatory behavior at infinity. This section is devoted to the study of problem (1.1) in the case where f oscillates at infinity. In order to prove Theorem 3.2, we use some techniques developed in the previous section. Now, we again consider problem $(\mathcal{P}_h^K)$, under the following assumptions on function h:

A. for any $t \geq 0$, $\sup_{\tau \in [0,t]} |h(\cdot, \tau)| \in L^\infty(\Omega)$;

B. there exist two sequences $\{\delta_j\}_j$ and $\{\eta_j\}_j$ with $0 < \delta_j < \eta_j < \delta_{j+1}$ and $\lim_{j \to +\infty} \delta_j = +\infty$ such that $h(x, t) \leq 0$ for almost every $x \in \Omega$ and for all $t \in [\delta_j, \eta_j]$, $j \in \mathbb{N}$:

$$-\infty < \liminf_{t \to +\infty} \frac{H(x,t)}{t^p} \leq \limsup_{t \to +\infty} \frac{H(x,t)}{t^p} = +\infty \tag{6.1}$$
$$\text{uniformly for almost every } x \in \Omega,$$

where H is defined as in (4.5).

In this context, our existence result for problem $(\mathcal{P}_h^K)$ is given by the following theorem:

Theorem 6.1. *Let us assume that* $K : \Omega \to \mathbb{R}$ *satisfies* (4.1), *and* $h : \Omega \times [0, +\infty) \to \mathbb{R}$ *is a Carathéodory function verifying* (4.2) *and* A, B *and* (6.1). *Then, there exists a sequence* $\{u_j\}_j \subset X(\Omega)$ *of distinct non-negative weak solutions of problem* $(\mathcal{P}_h^K)$ *such that*

$$\lim_{j \to +\infty} \|u_j\|_{L^\infty(\Omega)} = +\infty. \tag{6.2}$$

Proof. By using assumptions A and B, we can see that, for any $j \in \mathbb{N}$, h_j (defined as in (5.4)) verifies the assumptions of Theorem 4.1. Thus, for every $j \in \mathbb{N}$, there is an element $u_j \in \mathcal{W}^{\eta_j}$ such that

C. u_j is the minimum point of the functional $\mathcal{J}_j$ on $\mathcal{W}^{\eta_j}$,

D. $u_j(x) \in [0, \delta_j]$ for almost every $x \in \Omega$, and

E. u_j is a non-negative weak solution of $(\mathcal{P}^K_{h_j})$.

Here, $\mathcal{J}_j$ is the functional defined as in (5.5).

Then, we may argue as in the proof of Theorem 5.1. Recalling the definition of h_j, and using B and D, we can obtain that

$$h_j(x, u_j(x)) = h(x, \tau_{\eta_j}(u_j(x))) = h(x, u_j(x)).$$

Hence, by using E, we can infer that u_j is a non-negative weak solution of problem $(\mathcal{P}^K_h)$.

Now, we prove that there exist infinitely many distinct elements in the sequence $\{u_j\}_j$. Firstly, we show that, up to a subsequence,

$$\lim_{j\to+\infty} \mathcal{J}_j(u_j) = -\infty. \tag{6.3}$$

Due to (6.1), we can find $\ell > 0$ and $\zeta > 0$ such that

$$\operatorname*{ess\,inf}_{x\in\Omega} H(x, t) \geq -\ell t^p \quad \text{for all } t > \zeta, \tag{6.4}$$

and there exists a sequence $\{t_j\}_j$ such that $\lim_{j\to+\infty} t_j = +\infty$ and

$$\limsup_{j\to+\infty} \frac{H(x, t_j)}{t_j^p} = +\infty,$$

namely, for any $L > 0$,

$$\operatorname*{ess\,inf}_{x\in\Omega} H(x, t_j) > L t_j^p \tag{6.5}$$

for $j \in \mathbb{N}$ sufficiently large.

In view of assumption B, we know that $\delta_j \nearrow +\infty$; thus, we can find a subsequence of $\{\delta_j\}_j$, still denoted $\{\delta_j\}_j$, such that, for all $j \in \mathbb{N}$,

$$t_j \leq \delta_j. \tag{6.6}$$

Let us fix $j \in \mathbb{N}$, and let $z_j := z_{t_j}$ be the function from (2.1) with $t = t_j$. Then, $z_j \in X(\Omega)$ and $\|z_j\|_{L^\infty(\Omega)} = t_j$. Moreover, by using B and (6.6), we obtain $0 \le z_j(x) \le \delta_j < \eta_j$ almost everywhere $x \in \Omega$.

Combining (2.2), (6.4) and (6.5), we obtain

(6.7)
$$\begin{aligned}
\mathcal{J}_j(z_j) &= \frac{1}{p}\|z_j\|^p + \frac{1}{p}\int_\Omega K(x)|z_j(x)|^p dx - \int_\Omega H_j(x, z_j(x))\, dx \\
&\le C(r,s,p,N)\frac{1}{p}t_j^p + \frac{1}{p}\int_\Omega K(x)|z_j(x)|^p dx - \int_{B(x_0,r/2)} H(x,t_j)\, dx \\
&\quad - \int_{(B(x_0,r)\setminus B(x_0,r/2))\cap\{z_j>\zeta\}} H(x,z_j(x))\, dx \\
&\quad - \int_{(B(x_0,r)\setminus B(x_0,r/2))\cap\{z_j\le\zeta\}} H(x,z_j(x))\, dx \\
&\le \left(C(r,s,p,N)\frac{1}{p} + \frac{\|K\|_{L^\infty(\Omega)}\mathcal{L}(\Omega)}{p} - L(r/2)^N\omega_N + \ell\mathcal{L}(\Omega)\right)t_j^p \\
&\quad + \Big\| \sup_{t\in[0,\,\zeta]} |h(\cdot,t)| \Big\|_{L^\infty(\Omega)}\mathcal{L}(\Omega)\zeta.
\end{aligned}$$

Then, taking $L > 0$ sufficiently large, such that

$$L(r/2)^N\omega_N > C(r,s,p,N)\frac{1}{p} + \frac{\|K\|_{L^\infty(\Omega)}\mathcal{L}(\Omega)}{p} + \ell\mathcal{L}(\Omega),$$

and exploiting the fact that $\lim_{j\to+\infty} t_j = +\infty$, we can see that (6.7) implies that

$$\lim_{j\to+\infty} \mathcal{J}_j(z_j) = -\infty. \tag{6.8}$$

Then, by using C and (6.8), we get

$$\mathcal{J}_j(u_j) = \min_{u\in\mathcal{W}^{\eta_j}} \mathcal{J}_j(u) \le \mathcal{J}_j(z_j) \longrightarrow -\infty,$$

that is, (6.3) is satisfied.

Now, we are ready to show that $\{u_j\}_j$ admits infinitely many distinct elements (and, in particular, $u_j \not\equiv 0$, being $\mathcal{J}_j(0) = 0$). Assume, by contradiction, that, in $\{u_j\}_j$, there is only a finite number of elements, say $\{u_1, \ldots, u_k\}$ for some $k \in \mathbb{N}$. Thus, the sequence $\{\mathcal{J}_j(u_j)\}_j$ reduces to at most the finite set $\{\mathcal{J}_1(u_1), \ldots, \mathcal{J}_k(u_k)\}$, and this contradicts

(6.3). As a consequence, problem $(\mathcal{P}_h^K)$ has infinitely many distinct weak solutions.

At this point, we show that (6.2) is true. We argue by contradiction, and we assume that, up to a subsequence, the following holds

$$\|u_j\|_{L^\infty(\Omega)} \leq L \tag{6.9}$$

for all $j \in \mathbb{N}$, and for some $L > 0$. Since $\eta_j \to +\infty$ as $j \to +\infty$, for j large enough, say $j \geq j^*$, with $j^* \in \mathbb{N}$, we have that $\eta_j \geq L$. Then, by using (6.9) and the fact that the sequence $\{\eta_j\}_j$ is increasing (see B), we get

$$u_j \in \mathcal{W}^{\eta_{j^*}} \quad \text{for any } j \geq j^*. \tag{6.10}$$

We also note that, from the monotonicity of $\{\eta_j\}_j$, it follows that, for $j < k$,

$$W^{\eta_j} \subseteq W^{\eta_k}. \tag{6.11}$$

In particular, (6.11) implies that for any $u \in \mathcal{W}^{\eta_j}$ we have

$$\begin{aligned} H_j(x, u(x)) &= \int_0^{u(x)} h(x, \tau_{\eta_j}(t))\, dt = \int_0^{u(x)} h(x, t)\, dt \\ &= \int_0^{u(x)} h(x, \tau_{\eta_k}(t))\, dt = H_k(x, u(x)), \end{aligned} \tag{6.12}$$

for almost every $x \in \Omega$.

Furthermore, we can show that $\{\mathcal{J}_j(u_j)\}_j$ is non-increasing. In fact, if $j < k$, we can see that (6.11) and (6.12) yield

$$\begin{aligned} \mathcal{J}_j(u_j) &= \min_{u \in \mathcal{W}^{\eta_j}} \mathcal{J}_j(u) = \min_{u \in \mathcal{W}^{\eta_j}} \mathcal{J}_k(u) \\ &\geq \min_{u \in \mathcal{W}^{\eta_k}} \mathcal{J}_k(u) = \mathcal{J}_k(u_k). \end{aligned} \tag{6.13}$$

Hence, by using (6.10)–(6.13), for any $j \geq j^*$, we get

$$\mathcal{J}_{j^*}(u_{j^*}) \geq \mathcal{J}_j(u_j) \geq \min_{u \in \mathcal{W}^{\eta_{j^*}}} \mathcal{J}_j(u) = \min_{u \in \mathcal{W}^{\eta_{j^*}}} \mathcal{J}_{j^*}(u) = \mathcal{J}_{j^*}(u_{j^*}),$$

which contradicts (6.3) Thus, we can conclude that $\|u_j\|_{L^\infty(\Omega)} \to +\infty$ as $j \to +\infty$. $\square$

Requiring the following, extra condition on the function h,

$$\sup_{t\in[0,+\infty)} \frac{|h(x,t)|}{1+t^{p_s^*-1}} < +\infty \tag{6.14}$$

uniformly almost everywhere $x \in \Omega$, where p_s^* is the critical Sobolev exponent given in (3.5), we have the next result:

Corollary 6.2. *Let all of the assumptions of Theorem* 6.1 *be satisfied. In addition, assume that* (6.14) *holds true. Then*

$$\lim_{j\to+\infty} \|u_j\| = +\infty,$$

where $\{u_j\}_j$ *is the sequence of distinct weak solutions of problem* $(\mathcal{P}_h^K)$ *given by Theorem* 6.1.

Proof. Assume, by contradiction, that, up to a subsequence, there exists an $L > 0$ such that, for any $j \in \mathbb{N}$,

$$\|u_j\| \le L. \tag{6.15}$$

Then, by using (6.14), (6.15), and by applying Theorem 2.2, we can see that, for any $j \in \mathbb{N}$,

$$\begin{aligned} |\mathcal{J}_j(u_j)| &\le \frac{1}{p}\|u_j\|^p + \|K\|_{L^\infty(\Omega)}\|u_j\|^p_{L^p(\Omega)} \\ &\quad + C_1 \int_\Omega \int_0^{u_j(x)} (1+|t|^{p_s^*-1})\,dt\,dx \\ &\le \frac{1}{p}\|u_j\|^p + \|K\|_{L^\infty(\Omega)}\|u_j\|^p_{L^p(\Omega)} \\ &\quad + C_2\|u_j\|_{L^1(\Omega)} + C_3\|u_j\|^{p_s*}_{L^{p_s^*}(\Omega)} \\ &\le \frac{1}{p}L^p + C_4\|K\|_{L^\infty(\Omega)}L^p + C_5L + C_6L^{p_s*}. \end{aligned} \tag{6.16}$$

Therefore, (6.16) implies that $\{\mathcal{J}_j(u_j)\}_j$ is a bounded sequence in $\mathbb{R}$, and this is impossible in view of (6.3). $\square$

As a consequence of Corollary 6.2, we have the following result:

Proof of Corollary 3.3. It is sufficient to apply Corollary 6.2 with h and K given in (6.17) when $q = p-1$, and in (6.19) in the case $0 < q < p-1$.

Now, we assume that $q = p-1$. Clearly, $q > 0$ since $p > 1$. By using (3.9), we have, for any $t \in [0,1]$,

$$\begin{aligned}\frac{|h(x,t)|}{1+t^{p_s^*-1}} &= \frac{\widetilde{\lambda}_\infty t^{p-1}}{1+t^{p_s^*-1}} + \frac{|f(t)|}{1+t^{p_s^*-1}} \\ &\le \widetilde{\lambda}_\infty t^{p-1} + \frac{|f(t)|}{1+t^{p_s^*-1}} \le \widetilde{\lambda}_\infty + \frac{|f(t)|}{1+t^{p_s^*-1}} \\ &\le \widetilde{\lambda}_\infty + \sup_{t\in[0,+\infty)} \frac{|f(t)|}{1+t^{p_s^*-1}} < +\infty.\end{aligned}$$

For $t > 1$, we get

$$\begin{aligned}\frac{|h(x,t)|}{1+t^{p_s^*-1}} = \frac{\widetilde{\lambda}_\infty t^{p-1}}{1+t^{p_s^*-1}} + \frac{|f(t)|}{1+t^{p_s^*-1}} &\le \widetilde{\lambda}_\infty t^{p-p_s^*} + \frac{|f(t)|}{1+t^{p_s^*-1}} \\ &\le \widetilde{\lambda}_\infty t^{p-p_s^*} + \sup_{t\in[0,+\infty)} \frac{|f(t)|}{1+t^{p_s^*-1}} < +\infty,\end{aligned}$$

since $p < p^*$ and $t^{p-p_s^*} \to 0$ as $t \to +\infty$. Therefore, (6.14) is satisfied.

If $0 < q < p-1$, we can proceed in a similar way, observing that $q < p-1 < p_s^* - 1$. $\square$

6.1. Proof of Theorem 3.2. Our strategy consists of applying Theorems 4.1 and 6.1 to problem $(\mathcal{P}_h^K)$ and choosing the functions h and K in a suitable way.

We begin with consideration of cases $q = p-1$ and $\ell_\infty \in (0,+\infty)$. Fix $\lambda \in \mathbb{R}$ such that $\lambda\beta(x) < \lambda_\infty$ almost everywhere $x \in \Omega$ for some $\lambda_\infty \in (0,\ell_\infty)$. We take $\widetilde{\lambda}_\infty \in (\lambda_\infty, \ell_\infty)$, and we define the following functions

$$K(x) := \widetilde{\lambda}_\infty - \lambda\beta(x) \quad \text{and} \quad h(x,t) := \widetilde{\lambda}_\infty t^{p-1} + f(t), \tag{6.17}$$

for almost every $x \in \Omega$ and $t \ge 0$. Arguing as in the proof of Theorem 3.1, we can see that h and K satisfy the assumptions of Theorem 6.1 (here, we also use the fact that $f(0) = 0$ by assumption), and then the assertion of Theorem 3.2 follows.

When $q = p-1$ and $\ell_\infty = +\infty$, we take $\lambda \in \mathbb{R}$, and we use Theorem 6.1 with

$$K(x) := \widetilde{\lambda}_\infty \quad \text{and} \quad h(x,t) := (\lambda\beta(x) + \widetilde{\lambda}_\infty)t^{p-1} + f(t), \tag{6.18}$$

for almost every $x \in \Omega$ and $t \geq 0$. The arguments are the same as those used in the previous case.

In the case $0 < q < p-1$, we choose h and K in Theorem 6.1 as follows:

$$K(x) := \widetilde{\lambda}_\infty \quad \text{and} \quad h(x,t) := \lambda\beta(x)t^q + \widetilde{\lambda}_\infty t^{p-1} + f(t), \tag{6.19}$$

for almost every $x \in \Omega$ and $t \geq 0$, where $\widetilde{\lambda}_\infty \in (0, \ell_\infty)$, and we argue as above.

When we consider the case $q > p-1$, we aim to apply Theorem 4.1 to problem $(\mathcal{P}_h^K)$, provided to appropriately choose functions h and K. Let $\widetilde{\lambda}_\infty \in (\lambda_\infty, \ell_\infty)$, where $\ell_\infty > 0$ is given in assumption (3.6), and set

$$K(x) := \widetilde{\lambda}_\infty \quad \text{and} \quad h(x,s,\lambda) := \lambda\beta(x)t^q + \widetilde{\lambda}_\infty t^{p-1} + f(t)$$

almost everywhere $x \in \Omega$, $t \geq 0$ and $\lambda \in \mathbb{R}$. Thus, we may proceed as in the proof of Theorem 3.1 to obtain the assertion.

Example 6.3. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $N > sp$, $\lambda \in \mathbb{R}$. Furthermore, let $\beta \in L^\infty(\Omega)$, and consider the following problem

$$\begin{cases} (-\Delta)_p^s u = \lambda\beta(x)u^q + f(u) & \text{in } \Omega, \\ u \geq 0, u \not\equiv 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{6.20}$$

where $f : [0, +\infty) \to \mathbb{R}$ is given by

$$f(t) := (p+1)(1-\sin t)t^p - t^{p+1}\cos t - pt^{p-1}.$$

Direct calculations show that f and the potential $F(t) = (1 - \sin t)t^{p+1} - t^p$ satisfy assumptions (3.6) and (3.7). Then, if $0 < q \leq p-1$, and $\lambda \in \mathbb{R}$ is arbitrary, then there exists a sequence $\{u_j\}_j$ in $X(\Omega) \cap L^\infty(\Omega)$ of distinct weak solutions of problem (6.20) such that

$$\lim_{j \to +\infty} \|u_j\|_{L^\infty(\Omega)} = +\infty.$$

Remark 6.4. In Theorem 3.2, if $q > p-1$, then, for every $k \in \mathbb{N}$, there exists a $\Lambda_k > 0$ such that problem (1.1) has at least k distinct weak solutions $u_1, \ldots, u_k \in X(\Omega)$ such that

$$\|u_j\|_{L^\infty(\Omega)} \geq j-1, \quad j=1,\ldots,k, \tag{6.21}$$

provided $|\lambda| < \Lambda_k$.

Remark 6.5. It is easy to see that Theorems 1.1 and 1.2 in the introduction immediately follow from Theorems 3.1 and 3.2. Finally, we point out that the results contained in this paper are a fractional counterpart of the main theorems proved in the recent paper **[26]** and are valid for elliptic equations involving the classical p-Laplacian operator. Also, see the quoted paper **[20]** where the competition phenomena analyzed here for nonlocal fractional equations was observed for the first time in literature, exploiting the existence of infinitely many weak solutions for elliptic problems driven by the classical Laplacian operator. Our methods are fully based on this abstract approach. See, for instance, **[18, 19, 28, 37]** for related topics.

Remark 6.6. If $p > N/s$, our hypotheses on the nonlinear term f can be relaxed. Indeed, for instance, if f is a non-negative continuous function, exploiting **[38,** Theorem 2.1], the existence of infinitely many weak solutions for the following problem

$$\begin{cases} (-\Delta)_p^s u = f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, s \end{cases}$$

is achieved, requiring that

$$\liminf_{\xi \to L} \frac{F(\xi)}{\xi^p} = 0 \quad \text{and} \quad \limsup_{\xi \to L} \frac{F(\xi)}{\xi^p} = +\infty,$$

where either $L = 0^+$ or $L = +\infty$. This case will be discussed in a forthcoming paper (see **[23, 25]** and the references therein for related topics).

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