# CROSSING CHANGES, DELTA MOVES
# AND SHARP MOVES ON WELDED KNOTS

SHIN SATOH

*Dedicated to Professor Yasutaka Nakanishi on the occasion of his* 60*th birthday*

ABSTRACT. We prove that the crossing changes, Delta moves and sharp moves are unknotting operations on welded knots.

**1. Introduction.** Virtual knots [9] and welded knots [3] are two extensions of classical knots in the Euclidian 3-space. In classical knot theory, invariants and local moves play important roles from algebraic and geometric viewpoints. Several algebraic invariants of classical knots are extended to those of virtual or welded knots. For example, the Jones polynomial is an invariant of a virtual knot but not that of a welded knot, and the knot group and knot quandle are invariants of both a virtual knot and a welded knot. As for local moves on virtual knots, there are many results in relation to finite type invariants. In particular, a replacement of a classical crossing with a virtual crossing is used in [5].

In this paper, we consider three types of classical local moves, called the *crossing change*, the *Delta move* and the *sharp move*, as shown in Figure 1. These local moves are known as unknotting operations for classical knots [10, 11]. On the other hand, the crossing change on a virtual knot is not an unknotting operation, cf., [4, 6]. Since a Delta and a sharp move are presented by crossing changes, neither of the moves is an unknotting operation for virtual knots, cf., [17]. The aim of this paper is to prove that the three local moves are unknotting operations for welded knots in the following sense.
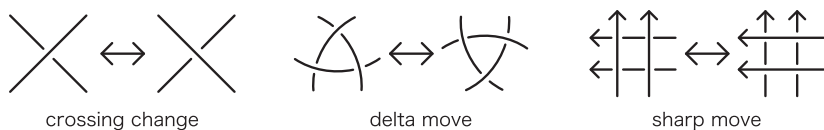
FIGURE 1.

**Theorem 1.1.** *For any diagram $D$ of a welded knot $K$, there is a diagram $D'$, such that*

   (i) *$D'$ is obtained from $D$ by the crossing changes at some classical crossings, and*
   (ii) *$D'$ presents the trivial welded knot.*

**Theorem 1.2.** *For any welded knot $K$, there is a finite sequence of welded knots $K = K_0, K_1, \ldots, K_n$, such that*

   (i) *$K_i$ is obtained from $K_{i-1}$ by a Delta move, $i = 1, 2, \ldots, n$, and*
   (ii) *$K_n$ is the trivial welded knot.*

**Theorem 1.3.** *For any welded knot $K$, there is a finite sequence of welded knots $K = K_0, K_1, \ldots, K_n$, such that*

   (i) *$K_i$ is obtained from $K_{i-1}$ by a sharp move, $i = 1, 2, \ldots, n$, and*
   (ii) *$K_n$ is the trivial welded knot.*

This paper is organized as follows. In Section 2, we prove that any descending diagram presents a trivial welded knot, which induces Theorem 1.1. In Sections 3 and 4, we prove that a replacement of a classical crossing with a welded crossing is accomplished by Delta moves and sharp moves, respectively, which induces Theorems 1.2 and 1.3.

**2. Crossing changes.** A *welded knot diagram* is a circle immersed in the plane $\mathbb{R}^2$ with transverse double points which are divided into two classes called *classical crossings* and *welded crossings*. A classical crossing has over-/under-information such that a small segment is removed from one of the paths intersecting at the crossing, and a welded crossing is indicated by putting a small circle on it. See Figure 2. A

welded knot diagram is called *trivial* if it has no classical and welded crossings, that is, it is an embedding of a circle in $\mathbb{R}^2$.



classical crossing       welded crossing

FIGURE 2.

We consider eight types of local moves on welded knot diagrams called *Reidemeister moves*, as shown in Figure 3. The first three moves C1–C3 are classical Reidemeister moves. The next three moves V1–V3 are obtained from C1–C3 by replacing all of the classical crossings with welded ones. The moves V4 and W are also obtained from C3 by replacing two or one classical crossing(s) with welded one(s), respectively. Here, the arc with two welded crossings passes the classical crossing for V4, and the arc with two classical crossings passes over (not under) the welded crossing for W.
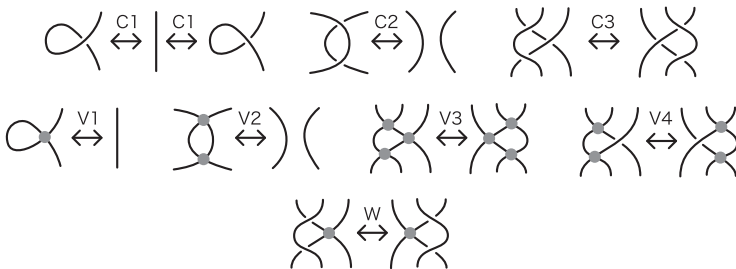


FIGURE 3.

We say that two welded knot diagrams $D$ and $D'$ are *equivalent* if there is a finite sequence of welded knot diagrams $D = D_0$, $D_1, \ldots, D_n = D'$ such that $D_i$ is obtained from $D_{i-1}$ by performing a Reidemeister move C1–C3, V1–V4 or W on $D_{i-1}$, $i = 1, 2, \ldots, n$. A *welded knot* is an equivalence class of welded knot diagrams under these Reidemeister moves. A welded knot is called *trivial* if it is presented by a trivial diagram.

A *Gauss diagram* is the union of a circle and $n$ signed and oriented chords for some $n \geq 0$ connecting $n$ pairs of points on the circle. Let $D$ be a welded knot diagram with $n$ classical crossings. The Gauss diagram $G = G(D)$ associated with $D$ is defined to be the union of a circle covering $D$ and $n$ chords connecting the preimages of classical crossings. Each chord has the sign derived from that of the classical crossing and is oriented from the over-crossing to the under-crossing.

We consider four types of moves on Gauss diagrams corresponding to moves C1, C2, C3 and W on welded knot diagrams. We use the same notation to indicate the moves on Gauss diagrams as those on welded knot diagrams. The left and middle of Figure 4 show C1 and W on a Gauss diagram, respectively, where $\varepsilon$ and $\varepsilon'$ are any signs. A move C1 removes or adds a chord whose endpoints are adjacent to each other. Such a chord is called *trivial*. Also, a move W changes the positions of adjacent initial endpoints of two chords regardless of the signs. We remark that a crossing change on a welded knot diagram corresponds to the change of sign and orientation of the chord simultaneously. See the right side of the figure.
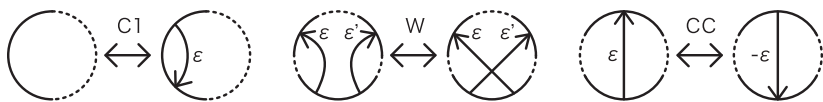


FIGURE 4.

It is known that two welded knot diagrams $D$ and $D'$ define the same Gauss diagram $G(D) = G(D')$ if and only if $D$ and $D'$ are related by a finite sequence of virtual Reidemeister moves V1–V4 [5].

**Lemma 2.1.** *Let $x$ be a classical crossing of a welded knot diagram $D$. We divide $D$ into two closed paths by cutting $D$ at $x$. Suppose that one of the paths thus obtained contains no under-crossing except $x$. Let $E$ be the welded knot diagram obtained from $D$ by replacing $x$ with a welded crossing. See the top of Figure 5. Then, $D$ is related to $E$ by a finite sequence of* C1, V1–V4 *and* W.

*Proof.* Let $\alpha$ be the closed path at $x$ on $D$ which contains no under-crossings. Since the path on the Gauss diagrams $G(D)$ corresponding

to $\alpha$ contains no terminal endpoints of chords, we shrink the chord corresponding to $x$ to be trivial by applying W repeatedly. Then, the chord is removed by C1 so that we obtain the Gauss diagram $G(E)$ of $E$. See the bottom of the figure. Therefore, $D$ and $E$ are related by C1 and W up to V1–V4. □
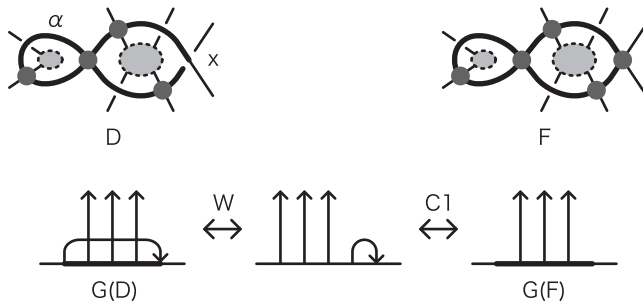


FIGURE 5.

We say that a welded knot diagram $D$ is *descending* if there is a base point and an orientation of $D$ such that, walking along $D$ from the base point with respect to the orientation, we meet the over-crossing for the first time and the under-crossing for the second time at every classical crossing.

**Proposition 2.2.** *Any descending diagram $D$ is related to the trivial diagram by a finite sequence of* C1, V1–V4 *and* W.

*Proof.* Let $x$ be the classical crossing of $D$ such that $x$ is the first under-crossing while walking along $D$ from the base point according to the orientation of $D$. Since $x$ satisfies the condition in Lemma 2.1, we can replace $x$ with a welded crossing by C1, V1–V4 and W. Since the diagram thus obtained is again descending, by repeating this process, $D$ is deformed into the diagram where all crossings are welded. Such a welded knot diagram is related to the trivial one by V1–V4. □

*Proof of Theorem* 1.1. By Proposition 2.2, it is sufficient to perform crossing changes on $D$ so that the obtained diagram is descending. □

Let $c(K)$ denote the minimal number of classical crossings for all diagrams of a welded knot $K$.

**Lemma 2.3.** *Any non-trivial welded knot $K$ satisfies* $c(K) \geq 3$.

*Proof.* If a Gauss diagram has at most two chords, then the chords can be removed by C1 and W. □

For a welded knot diagram $D$, we denote by $u(D)$ the minimal number of classical crossings of $D$ for which we perform the crossing changes to obtain a diagram presenting the trivial welded knot. The number $u(D)$ is well defined by Theorem 1.1. The *unknotting number* of a welded knot $K$ is the minimal number of $u(D)$ for all diagrams $D$ presenting $K$ and denoted by $u(K)$. The following is a generalization of the well-known result for a classical knot, cf., [**15**].

**Proposition 2.4.** *Any non-trivial welded knot $K$ satisfies* $u(K) \leq (c(K) - 1)/2$.

*Proof.* Let $D$ be a welded knot diagram of $K$ with $c(D) = c(K)$, and $x$ a classical crossing of $D$. From Lemma 2.3, we have $c(D) \geq 3$.

We choose a pair of points $p_1$ and $p_2$ on the over-path at $x$ from one side of $x$ to the other. Let $S_i$, $i = 1, 2$, be the set of classical crossings of $D$ such that crossing changes are performed at $S_i$ on $D$ to obtain the descending diagram with the base point $p_i$ and the orientation from $p_i$ to $p_j$, $j \neq i$. Since $S_1 \cap S_2 = \emptyset$ and $|S_1| + |S_2| = c(D) - 1$, it follows from Proposition 2.2 that

$$u(K) \leq u(D) \leq \frac{c(D) - 1}{2} = \frac{c(K) - 1}{2}. \qquad \square$$

## 3. Delta moves.

**Proposition 3.1.** *Let $x$ be a classical crossing of a welded knot diagram $D$ and $E$ the welded knot diagram obtained from $D$ by the crossing change at $x$. Then, $D$ and $E$ are related by a finite sequence of Reidemeister moves and Delta moves.*

*Proof.* Fix an orientation of $D$. Then, $D$ is regarded as a band sum of a positive or negative Hopf link diagram $H$ as shown in Figure 6. This replacement is realized by C2. We remark that each band is on the left hand side of the attaching arc on $E$.
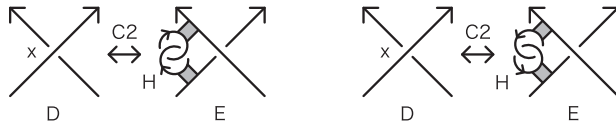


FIGURE 6.

We slide the attaching arcs of the bands along $E$ so that they are adjacent to each other on $E$. In the top row of Figure 7, an attaching arc passes a crossing of $E$ while making a pair of classical or welded crossings. In the bottom of the figure, an attaching arc passes an intersection of $E$ and a band while making a quadruple of classical or welded crossings. The deformations are accomplished by C2 and V2.
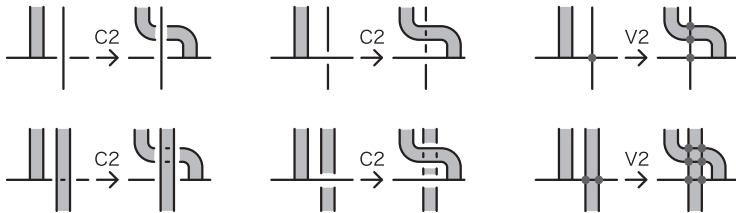


FIGURE 7.

The Hopf link diagram $H$ can slide anywhere between the bands. In Figure 8, $H$ passes an intersection of $E$ and a band by performing C2, C3, V2 and V4. The case that $H$ passes an intersection between bands is similarly proved by replacing the horizontal segment of $E$ in the figure with a band.

A *banded* Reidemeister move is a local deformation obtained from an original Reidemeister move as in Figure 3 by replacing some strings with bands. It is easy to see that any banded Reidemeister move except C1 is accomplished by some original Reidemeister moves. In order to proceed with the proof of Proposition 3.1, we prepare the following classical deformations.
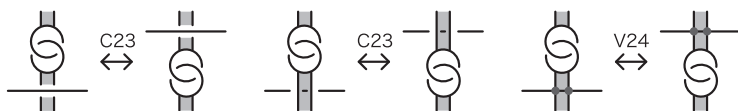
FIGURE 8.

**Lemma 3.2** ([**18**])**.** *For a band sum of E and H, we have the following.*

(i) *A banded Reidemeister move* C1 *is realized by classical Reidemeister moves and a Delta move up to a slide of H.*

(ii) *A crossing change between E and a band or between bands is realized by classical Reidemeister moves and a Delta move up to a slide of H.*

*Proof.* We slide $H$ near the portion where the modification will be applied. Then a banded C1 move and a crossing change between $E$ and a band are accomplished by classical Reidemeister moves and Delta moves as shown in Figure 9. A crossing change between bands is similarly proved by duplicating the horizontal segment of $E$ in the figure.                                                                  □
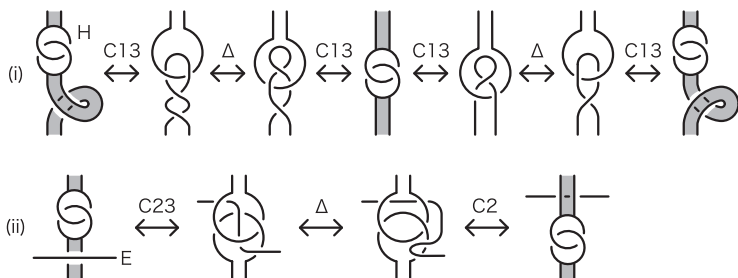


FIGURE 9.

Now, we continue the proof. From Lemma 3.2 (ii), we perform crossing changes so that the union of two bands are descending, that is, walking from one attaching arc on $E$ to the other, we meet:

(i) a pair of over-crossings at every classical intersection between $E$ and a band, and

(ii) a quadruple of over-crossings for the first time and a quadruple of under-crossings for the second time at every classical intersection between bands.
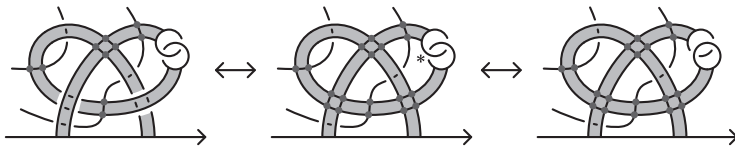
See the left side of Figure 10.



FIGURE 10.

We use the same technique as in the proof of Lemma 2.1 for the cores of the bands. By performing banded Reidemeister moves C1 and W by Lemma 3.2 (i), all classical intersections between bands are replaced with welded ones as shown in the middle of the figure. Since one of the classical crossings of $H$ satisfies the condition in Lemma 2.1, we perform a welding of the crossing and the inverse of another so that $H$ becomes unlinked. See the right hand side of the figure. Since the bands are removed by C2 and V2, $D$ is equivalent to $E$ up to Delta moves. This completes the proof of Proposition 3.1. □

The outline of the proof of Proposition 3.1 is similar to that for the classical case due to Murakami and Nakanishi [**11**]. The difference is that a band as shown in Figure 10 may have several welded crossings.

*Proof of Theorem* 1.2. From Proposition 2.2, it is sufficient to perform crossing changes on $D$ so that the obtained diagram is descending. Such crossing changes are accomplished by Reidemeister and Delta moves by Proposition 3.1. □

In classical knot theory, a single Delta move necessarily changes the knot type [**14**]. On the other hand, there is a pair of diagrams of the same welded knot which are related by a single Delta move. The two diagrams as shown in Figure 11 present the trivial welded knot.
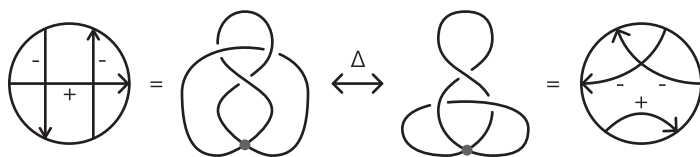
FIGURE 11.

By a similar argument to the proof of Proposition 3.1, any welded *link* diagram is transformed into one such that all self-crossings of the same component are welded by a finite sequence of Reidemeister moves and Delta moves. It is known [**11**] that two classical links are related by a finite sequence of classical Reidemeister and Delta moves if and only if their pairwise linking numbers coincide.

**Question 3.3.** *Can we detect whether two welded links are related by Reidemeister moves and Delta moves by some algebraic invariants?*

## 4. Sharp moves.

**Proposition 4.1.** *Let $x$ be a classical crossing of a welded knot diagram $D$ and $E$ the welded knot diagram obtained from $D$ by the crossing change at $x$. Then, $D$ and $E$ are related by a finite sequence of Reidemeister moves and sharp moves.*

*Proof.* The proof follows that of Proposition 3.1. It is sufficient to prove the following which is analogous to Lemma 3.2.     □

**Lemma 4.2.** *For a band sum of $E$ and $H$, we have the following.*

(i) *A banded Reidemeister move* C1 *is realized by classical Reidemeister moves and sharp moves up to a slide of $H$.*

(ii) *A crossing change between $E$ and a band or between bands is realized by classical Reidemeister moves sharp move up to a slide of $H$.*

*Proof.* We consider four types of local moves: a *pass move* [**2, 7**], a $t_4$ *move*, a $\overline{t_4}$ *move* [**16**] and a $\Gamma$ *move* [**7, 8**], as shown in Figure 12. It is well known that:

- a pass move is realized by classical Reidemeister moves and sharp moves [11];
- a $t_4$ move is realized by classical Reidemeister moves and a sharp move, cf., [12];
- a $\overline{t_4}$ move is realized by classical Reidemeister moves and a pass move (and, hence, sharp moves); and
- a $\Gamma$ move is realized by classical Reidemeister moves and a pass move (and, hence, sharp moves) [7].
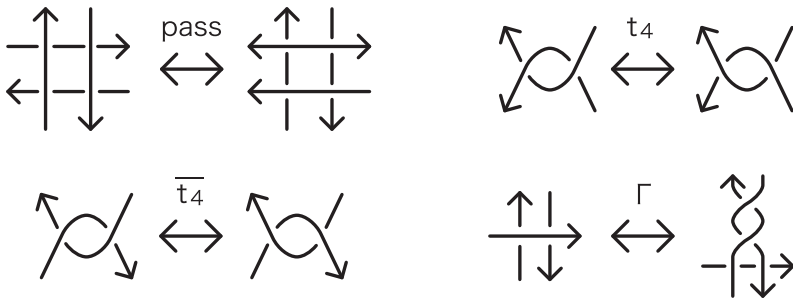


FIGURE 12.

Now, we slide $H$ near the portion where the modification will be applied. Then a banded C1 move and a crossing change between $E$ and a band are accomplished by classical Reidemeister moves and sharp moves, as shown in Figure 13. A crossing change between bands is exactly the same as a pass move. □

*Proof of Theorem* 1.3. From Proposition 2.2, it is sufficient to perform crossing changes on $D$ so that the diagram thus obtained is descending. Such crossing changes are accomplished by Reidemeister and sharp moves by Proposition 4.1. □

In classical knot theory, a single sharp move necessarily changes the knot type [10]. On the other hand, there is a pair of diagrams of the same welded knot which are related by a single sharp move. The two diagrams as shown in Figure 14 present the trivial welded knot.

By a similar argument to the proof of Proposition 4.1, any welded *link* diagram is transformed into one such that all the self-crossings of
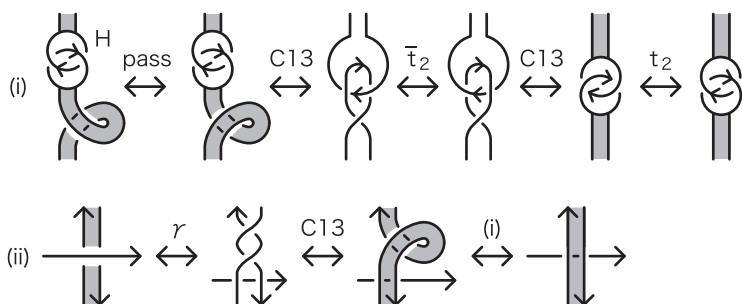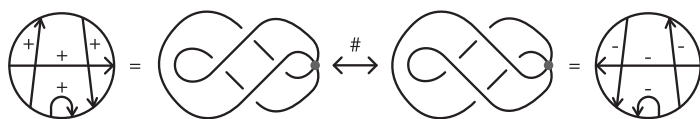
FIGURE 13.



FIGURE 14.

the same component are welded by a finite sequence of Reidemeister and sharp moves. The necessary and sufficient conditions are known for classical links to be related by classical Reidemeister moves and sharp moves in terms of their linking numbers [**11**].

**Question 4.3.** *Can we detect whether two welded links are related by Reidemeister moves and sharp moves by some algebraic invariants?*

**Remark 4.4.** After this work was finished, Yasuhara pointed out that, by combining the results due to Aida [**1**] and Nakanishi [**13**], a Delta move is given by a finite sequence of sharp moves. Therefore, Theorem 1.3 immediately follows Theorem 1.2.

**Remark 4.5.** In classical category, it is known that any classical knot is equivalent to the trivial or trefoil knot up to pass moves with respect to the Arf invariant [**7**]. On the other hand, in welded category, we can prove that any welded knot is equivalent to the trivial knot up to pass moves. The author and Akira Yasuhara are preparing a paper containing this result.

## REFERENCES

**1**. H. Aida, *The oriented $\Delta_{ij}$-moves on links*, Kobe J. Math. **9** (1992), 163–170.

**2**. D. Erle, *Die quadratische Form eines Knotens und ein Satz über Knoten-mannigfaltigkeiten*, J. reine angew. Math. **236** (1969), 174–218.

**3**. R. Fenn, R. Rimányi and C. Rourke, *The braid-permutation group*, Topology **36** (1997), 123–135.

**4**. R. Fenn and V. Turaev, *Weyl algebras and knots*, J. Geom. Phys. **57** (2007), 1313–1324.

**5**. M. Goussarov, M. Polyak and O. Viro, *Finite-type invariants of classical and virtual knots*, Topology **39** (2000), 1045–1068.

**6**. T. Kadokami, *Some numerical invariants of flat virtual links are always realized by reduced diagrams*, J. Knot Th. Ramif. **15** (2006), 289–297.

**7**. L.H. Kauffman, *Formal knot theory*, Math. Notes **30** (1983).

**8**. _____ , *On knots*, Ann. Math. Stud. **115** (1987).

**9**. _____ , *Virtual knot theory*, Europ. J. Combin. **20** (1999), 663–690.

**10**. H. Murakami, *Some metrics on classical knots*, Math. Ann. **270** (1985), 35–45.

**11**. H. Murakami and Y. Nakanishi, *On a certain move generating link-homology*, Math. Ann. **284** (1989), 75–89.

**12**. T. Nakamura and Y. Nakanishi, *Notes on sharp moves for knots*, J. Knot Th. Ramif. **21** (2012), 1250068, 20 pages.

**13**. Y. Nakanishi, *Replacements in the Conway third identity*, Tokyo J. Math. **14** (1991), 197–203.

**14**. M. Okada, *Delta-unknotting operation and the second coefficient of the Conway polynomial*, J. Math. Soc. Japan **42** (1990), 713–717.

**15**. M. Ozawa, *Ascending number of knots and links*, J. Knot Th. Ramif. **19** (2010), 15–25.

**16**. J.H. Przytycki, *$t_k$ moves on links*, Contemp. Math. **78** (1988).

**17**. S. Satoh and K. Taniguchi, *The writhes of a virtual knot*, Fund. Math. **225** (2014), 327–342.

**18**. K. Taniyama and A. Yasuhara, *Clasp-pass moves on knots, links and spatial graphs*, Topol. Appl. **122** (2002), 501–529.

Kobe University, Department of Mathematics, Rokkodai-cho 1-1, Nada-ku, Kobe 657-8501, Japan
**Email address**: **shin@math.kobe-u.ac.jp**