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CYCLICITY OF THE LEFT REGULAR REPRESENTATION OF A LOCALLY COMPACT GROUP

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ABSTRACT. We combine harmonic analysis and operator algebraic techniques to give a concise argument that the left regular representation of a locally compact group is cyclic if and only if the group is first countable, a result first proved by Greenleaf and Moskowitz.

1. Let G be a locally compact group, and let λ and ρ denote the (unitarily equivalent) *left and right regular representations* of G on $L^2(G)$, respectively. The *group von Neumann algebra* $VN(G)$ is the von Neumann algebra generated in $B(L^2(G))$ by $\lambda(G)$. It is well known that the commutant of $VN(G)$ is the von Neumann algebra generated by $\rho(G)$. In **[2]**, an operator algebraic argument viewing $VN(G)$ as arising from a left Hilbert algebra, in combination with a reduction argument using the structure theory of locally compact groups, is used to show that λ is cyclic when the group G is first countable. The converse, left open in **[2]**, was later established by the same authors in **[3]**. An alternative proof of the characterization, exploiting the structure theory of locally compact groups, is given in **[5]**.

The purpose of this note is to give a new and more economical proof of this equivalence, entirely avoiding the structure theory of locally compact groups. Our argument shows, moreover, that these conditions are equivalent to σ-finiteness of $VN(G)$, the latter condition naturally arising from our techniques. In the commutative case, it is well known that σ-finiteness of $L^\infty(G)$ characterizes σ-compactness of G, and it is our hope that further development of the techniques we employ

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will yield natural characterizations of σ-finiteness of a general locally compact quantum group.

Recall that the *support* of a normal state ω on a von Neumann algebra M is the minimal projection S_ω in M for which $\langle \omega, S_\omega \rangle = 1$ and that ω is *faithful* if $S_\omega = I$, the identity in M; equivalently, if ω takes strictly positive values on strictly positive operators. We record some elementary facts about these concepts. For a vector ξ in a Hilbert space, let ω_ξ denote the vector functional implemented by ξ. The notation $\langle X \rangle$ denotes the norm closed linear span of X. For a von Neumann algebra M on a Hilbert space $\mathcal{H}$, M' denotes the commutant of M.

Lemma 1.1. *Let $\mathcal{H}$ be a Hilbert space, let M be a von Neumann algebra in $B(\mathcal{H})$ and let $\xi, \eta \in \mathcal{H}$ be unit vectors. The following hold.*

(1) *The projection S_{ω_ξ} has range $\langle M'\xi \rangle$;*
(2) *$\langle \omega_\xi, S_{\omega_\eta} \rangle = 0$ if and only if ξ is orthogonal to $\langle M'\eta \rangle$;*
(3) *a projection P in M satisfies $P\xi = \xi$ if and only if $S_{\omega_\xi} \le P$;*
(4) *a normal state ω on M is faithful if and only if $\langle \omega, U \rangle = \langle \omega, I \rangle$ implies $U = I$, for any unitary U in M.*

Motivated by the following simple observation, we choose to characterize cyclicity of the right regular representation.

Lemma 1.2. *Let G be a locally compact group. A vector $\xi \in L^2(G)$ is cyclic for ρ if and only if ω_ξ is faithful on $VN(G)$.*

Proof. For $\xi \in L^2(G)$, we have $\langle \rho(G)\xi \rangle = \langle VN(G)'\xi \rangle$ since $\operatorname{span} \rho(G)$ is strong operator topology dense in $VN(G)'$. Consequently, the vector ξ is cyclic for ρ if and only if $\langle VN(G)'\xi \rangle = L^2(G)$. Since S_{ω_ξ} has range $\langle VN(G)'\xi \rangle$, the latter assertion is exactly that $S_{\omega_\xi} = I$. $\square$

Let $A(G)$ denote the *Fourier algebra* of a locally compact group G, which is the predual of $VN(G)$, and, for $T \in VN(G)$ and $u \in A(G)$, define $T \check{\cdot} u \in A(G)$ by

$$\langle T \check{\cdot} u, S \rangle = \langle u, \check{T} S \rangle, \quad S \in VN(G),$$

where $\check{T}$ is the image of T under the adjoint of the check map $u \mapsto \check{u}$ on $A(G)$ (here, $\check{u}(s) = u(s^{-1})$). See [**1,** page 213]. Proposition 3.17 of [**1**] shows that, for $u \in A(G) \cap L^2(G)$, we have $T \check{\cdot} u = Tu$, where the right hand side is the evaluation of the operator T at the vector u in $L^2(G)$. This fact is necessary in the following lemma, which is key to establishing the main result.

Lemma 1.3. *Let G be a locally compact group. Every nonzero projection in $VN(G)$ has a nonzero continuous function in its range.*

Proof. Let $P \in VN(G)$ be a nonzero projection, and choose a unit vector ξ in its range. Since positive functions span $\mathcal{C}_c(G)$, which is in turn dense in $L^2(G)$, we may find a positive $f \in \mathcal{C}_c(G)$ of norm one in $L^2(G)$ that is not orthogonal to $\langle \rho(G)\xi \rangle$ so that $\langle \omega_f, S_{\omega_\xi} \rangle \neq 0$ by Lemma 1.1. The function ω_f in $A(G)$ is positive definite and pointwise positive so that $\check{\omega_f} = \omega_f$ and is in $A(G) \cap L^2(G)$ since f has compact support, whence

$$\begin{aligned} S_{\omega_\xi}(\omega_f)(e) &= (S_{\omega_\xi} \check{\cdot} \omega_f)(e) = \langle S_{\omega_\xi} \check{\cdot} \omega_f, \lambda(e) \rangle = \langle \omega_f, \check{S_{\omega_\xi}} \rangle \\ &= \langle \check{\omega_f}, S_{\omega_\xi} \rangle = \langle \omega_f, S_{\omega_\xi} \rangle \neq 0. \end{aligned}$$

Thus, $S_{\omega_\xi}(\omega_f) = S_{\omega_\eta} \check{\cdot} \omega_f$ is nonzero and in $A(G)$, hence continuous, and is in the range of P since $S_{\omega_\xi} \leq P$, by Lemma 1.1. $\square$

Theorem 1.4. *Let G be a locally compact group. The following are equivalent*:

(1) *G is first countable*;
(2) *$VN(G)$ is σ-finite*;
(3) *the left* (*equivalently, right*) *regular representation is cyclic.*

Proof. Suppose that (1) holds. Let $(U_n)_{n=1}^\infty$ be a countable neighborhood base at the identity in G, and define $\omega_n = |U_n|^{-1}\omega_{\chi_{U_n}}$. We show that the normal state $\omega = \sum_{n=1}^\infty 2^{-n}\omega_n$ is faithful. Let T be a positive operator in $VN(G)$ with $\langle \omega, T \rangle = 0$, and let P be the range projection of T so that $\langle \omega, P \rangle = 0$ (see, e.g., [**4,** Remark 7.2.5]). Given any vector η in the range of T, we have $S_{\omega_\eta} \leq P$, and thus, $0 \leq \langle \omega_n, S_{\omega_\eta} \rangle \leq \langle \omega_n, P \rangle \leq \langle \omega, P \rangle = 0$, implying that η is orthogonal to $\langle \rho(G)\chi_{U_n} \rangle$ for each $n \geq 1$. If η is continuous, then

$$\eta(s) = \lim_n |U_n s|^{-1} \int_{U_n s} \eta = \lim_n |U_n s|^{-1} \langle \eta \mid \rho(s^{-1})\chi_{U_n} \rangle \Delta(s)^{1/2} = 0$$

for every $s \in G$. Thus, $P = 0$ by Lemma 1.3; hence, $T = 0$ and ω is faithful, so (2) holds.

Normal states on $VN(G)$, being positive definite functions in $A(G)$, are vector states so that statements (2) and (3) are equivalent by Lemma 1.2.

We provide the argument of **[6]** establishing that (2) implies (1). Suppose that (2) holds, and let ω be a faithful normal state on $VN(G)$. Fix a compact neighborhood K of the identity in G, and let V be any open neighborhood of the identity contained in K. We show that the sets $U_n = \{s \in K : |\omega(s) - 1| < 1/n\}$ form a neighborhood base at the identity, for which it suffices to establish that U_n is contained in V for some $n \geq 1$. For any $s \in G$ with $\omega(s) = 1$, Lemma 1.1 entails that $s = e$ since $\omega(s) = \langle \omega, \lambda(s) \rangle$. Compactness of $K \setminus V$ then implies that $\epsilon = \inf\{|\omega(s) - 1| : s \in K \setminus V\}$ is strictly positive. Choosing $N \geq 1$ with $1/N < \epsilon$, if $s \in U_N$, then $s \in K$ and $|\omega(s) - 1| < \epsilon$ together imply that $s \in V$. Thus, $U_N \subset V$, as required. $\square$

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