ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 2, 2018

MULTIPLICITY OF SOLUTIONS FOR p-BIHARMONIC PROBLEMS WITH CRITICAL GROWTH

H. BUENO, L. PAES-LEME AND H. RODRIGUES

ABSTRACT. We prove the existence of infinitely many solutions for p-biharmonic problems in a bounded, smooth domain Ω with concave-convex nonlinearities dependent upon a parameter λ and a positive continuous function $f\colon \overline{\Omega} \to \mathbb{R}$. We simultaneously handle critical case problems with both Navier and Dirichlet boundary conditions by applying the Ljusternik-Schnirelmann method. The multiplicity of solutions is obtained when λ is small enough. In the case of Navier boundary conditions, all solutions are positive, and a regularity result is proved.

1. Introduction. In this work, we study the p-biharmonic equation with concave-convex nonlinearity and critical exponent

$$\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x)|u|^{q-2}u + |u|^{p^*-2}u \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded, smooth domain. We suppose that the exponents p and q are such that $1 < p < \infty$, $N > 2p$, $1 < q < p$, and that

$$p^* = \frac{Np}{N-2p}$$

denotes the Sobolev critical exponent for fourth-order problems. The parameters λ and $f : \overline{\Omega} \to \mathbb{R}$ are assumed to be continuous and positive.

Equation (1.1) is handled simultaneously with Dirichlet

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

2010 AMS *Mathematics subject classification.* Primary 35J35, 35J40, 35J91.
Keywords and phrases. Navier and Dirichlet boundary conditions, p-biharmonic operator, concave-convex nonlinearities, critical growth.
This research was supported by CNPq-Brazil and FAPEMIG.
Received by the editors on May 3, 2016, and in revised form on January 2, 2017.
DOI:10.1216/RMJ-2018-48-2-425

and Navier boundary conditions

$$u = \Delta u = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

The p-biharmonic operator Δ_p^2 has recently attracted the attention of many researchers (see [**5, 7, 12, 17**] and references therein). Looking for positive solutions $u, v > 0$ defined in a bounded, smooth domain Ω, it is sometimes associated with Hamiltonian systems (see [**4, 11**]):

$$\begin{cases} -\Delta u = v^p & \text{in } \Omega, \\ -\Delta v = u^q & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p, q \geq 1$ since, formally substituting the first equation

$$v = (-\Delta u)^{1/p}$$

into the second, we obtain

$$\begin{cases} -\Delta\Big(|-\Delta u|^{1/p-1}(-\Delta u)\Big) = -\Delta(-\Delta u)^{1/p} = u^q & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

The biharmonic operator, i.e., the case $p = 2$, can be viewed as a viscosity coefficient in Navier-Stokes equations, while the biharmonic equation $\Delta^2 u = 0$ appears in quantum mechanics as well as in the theory of linear elasticity modeling Stokes' flows.

Existence of solutions for p-biharmonic equations are mostly proved in cases of Steklov and Navier boundary conditions, see [**7, 17**]; existence and multiplicity of solutions for problems with Dirichlet boundary conditions in bounded, smooth domains are rare.

The main motivation for the present work comes from Bernis, García-Azorero and Peral [**2**], who in 1996 studied problems (1.2)–(1.3) in the case $p = 2$ and $f \equiv 1$. Thus, our study can be considered a generalization of these results for the p-biharmonic operator.

In the present paper, we apply Ljusternik-Schnirelmann methods to prove the existence of infinitely many solutions for problems (1.1)–(1.2) and (1.1)–(1.3). These methods were considered by many authors during the last decade, see [**10**] for Kirchhoff-type problems, [**15**] for non-autonomous elliptic semilinear equations, [**18**] for elliptic problems with nonlinear boundary data, [**1**] for systems in the whole $\mathbb{R}^N$ and [**14**] for systems in bounded domains. In **[6]**, a nondecreasing and unbounded sequence of eigenvalues of the p-biharmonic operator was obtained by considering the Ljusternik-Schnirelmann theory on C^1-manifolds.

Next, we state our main result.

Theorem 1.1. *There exists a constant* $\lambda_0 > 0$ *such that, for all* $\lambda \in (0, \lambda_0)$*, problems* (1.1)–(1.2) *and* (1.1)–(1.3) *admit infinitely many solutions. Furthermore, the solutions of the problem* (1.1)–(1.3) *are positive.*

In order to obtain our result, we consider the functional

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\Delta u|^p - \frac{\lambda}{q}\int_\Omega f(x)|u|^q - \frac{1}{p^*}\int_\Omega |u|^{p^*}. \tag{1.4}$$

In the case of problem (1.1)–(1.2), J_λ is defined in $W_0^{2,p}(\Omega)$, while J_λ is defined in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for the problem (1.1)–(1.3). Thus, let $\mathbf{E} = \mathbf{E}(\Omega)$ stand either for the space $W_0^{2,p}(\Omega)$ or the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, according to the problem with which we deal. The space $\mathbf{E}$ is endowed with the norm

$$\|u\| = \int_\Omega (|\Delta u|^p \mathrm{d}x)^{1/p}.$$

In order to handle both problems simultaneously, we apply a result by Gazzola, Grunau and Sweers **[9]**, which proves that the best constant for the immersion

$$W_0^{2,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$

equals the best constant for the immersion

$$W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega).$$

Critical points of J_λ are weak solutions of problems (1.1)–(1.2) and (1.1)–(1.3), see Section 4 for problem (1.1)–(1.3). Since the immersion of $\mathbf{E}$ into $L^{p^*}(\Omega)$ is not compact, we apply Lions' lemma (see Lemma 2.2), which implies that J_λ satisfies a local Palais-Smale (PS) condition below the level

$$\frac{2}{N} S^{N/(2p)} - D\lambda^\beta.$$

(We denote by S the best constant for the immersion of $\mathbf{E}$ into L^{p^*} and $\beta = p^*/(p^* - q)$; the constant D will be defined later on.)

The outline of this article is the following. In Section 2, we introduce the framework of both problems and prove a local PS-condition by applying a measure representation lemma obtained by Lions in the proof of the concentration-compactness principle (see Lemma 2.2). In Section 3, the application of Ljusternik-Schnirelmann methods allows us to establish the existence of infinitely many solutions for λ small enough. In Section 4, we prove a simple regularization result regarding equation (1.1) with Navier boundary conditions.

2. The local PS-condition via the concentration-compactness principle. We consider the "energy" functional

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\Delta u|^p - \frac{\lambda}{q}\int_\Omega f(x)|u|^q - \frac{1}{p^*}\int_\Omega |u|^{p^*}. \tag{2.1}$$

As mentioned above, J_λ is defined in $W_0^{2,p}(\Omega)$ in the case of problem (1.1)–(1.2) and in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ in the case of problem (1.1)–(1.3). Both spaces will be simply designated by $\mathbf{E} = \mathbf{E}(\Omega)$, according to the problem with which we deal. We denote

$$\|u\|_{\mathbf{E}} = \|u\| = \|\Delta u\|_p,$$

where $\|\cdot\|_p$ stands for the usual norm of $L^p(\Omega)$. In order to handle problems (1.1)–(1.2) and (1.1)–(1.3) simultaneously, we apply the following result by Gazzola, Grunau and Sweers **[9]**:

Theorem 2.1. *The best constant for the immersion* $W_0^{2,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ *is equal to the best constant for the immersion* $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.

From now on,

$$S = \inf\{\|u\|^p : u \in \mathbf{E} \text{ and } \|u\|_{p^*} = 1\}$$

indicates the best constant for the Sobolev's immersion of $\mathbf{E}$ into L^{p^*}. Thus, by definition,

$$\|u\|_{p^*} \le S^{-1/p}\|u\|.$$

For the reader's convenience, we state Lions' lemma (see **[13]**):

Lemma 2.2. *Let $\{u_n\}$ be a weakly convergent sequence with limit u such that*

(i) $|\Delta u_n|^p \to \mu$ *weakly-* in the sense of measures;*
(ii) $|u_n|^{p^*} \to \nu$ *weakly-* in the sense of measures,*

where μ and ν are non-negative and bounded measure. Then, for some finite set of indices I, we have

(a) $\nu = |u|^{p^*} + \sum_{k\in I} \nu_k \delta_{x_k}, \quad \nu_k > 0,$

(b) $\mu \geq |\Delta u|^p + \sum_{k\in I} \mu_k \delta_{x_k}, \quad \mu_k > 0,\ x_k \in \overline{\Omega},$

(c) $\nu_k^{p/p^*} \leq \mu_k S^{-1}.$

We apply this result to prove that the functional J_λ satisfies the Palais-Smale condition for levels below a certain constant.

We recall that $\beta = p^*/(p^* - q)$.

Theorem 2.3. *There exists a positive constant D such that any Palais-Smale sequence $\{u_n\} \subset \mathbf{E}$ for J_λ at the level c satisfying*

$$c < \frac{2}{N} S^{N/(2p)} - D\lambda^\beta,$$

has a subsequence that converges strongly in $\mathbf{E}$.

Proof. It is easy to conclude that $\{u_n\}$ is bounded in $\mathbf{E}$. Therefore, we may suppose that

$$u_n \rightharpoonup u \text{ weakly in } \mathbf{E},$$

and

$$\left.\begin{aligned} |\Delta u_n|^p &\rightharpoonup \mu \\ |u_n|^{p*} &\rightharpoonup \nu \end{aligned}\right\} \text{weakly-* in the sense of measures,}$$

for some bounded, non-negative measures μ and ν. Applying Lemma 2.2 (passing to a subsequence, if necessary) we also have

(2.2)
$$\begin{cases} u_n \to u \text{ in } L^r(\Omega) \text{ for } 1 < r < p^* \text{ and almost everywhere in } \overline{\Omega}, \\ |\Delta u_n|^p \rightharpoonup \mu \geq |\Delta u|^p + \sum_{k\in I} \mu_k \delta_{x_k}, \\ |u_n|^{p^*} \rightharpoonup \nu = |u|^{p^*} + \sum_{k\in I} \nu_k \delta_{x_k} \end{cases}$$

for some finite set I.

We claim that $I = \emptyset$. Supposing that $k \in I$, define $\psi \in C^\infty(\mathbb{R}^N)$ such that

$$\begin{cases} \psi := 1 \text{ in } B_\epsilon(x_k), \\ \psi := 0 \text{ out } B_{2\epsilon}(x_k), \\ |\nabla\psi| \leq \dfrac{2}{\epsilon}, \quad |\Delta\psi| \leq \dfrac{2}{\epsilon^2}. \end{cases} \tag{2.3}$$

Now, consider the bounded sequence in $\mathbf{E}$ given by $\{\phi u_n\}$, where $\phi(x) = \psi(x)\chi_\Omega(x)$. It follows that

$$\lim_{n\to\infty} \langle J'_\lambda(u_n), \phi u_n\rangle = 0.$$

Thus,

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega |\Delta u_n|^{p-2}\Delta u_n \Delta(\phi u_n)\,\mathrm{d}x \\ &\quad = \lambda \int_\Omega f(x)|u_n|^q \phi + \int_\Omega |u_n|^{p^*}\phi \\ &\quad = \lambda \int_\Omega f(x)|u|^q\phi\,\mathrm{d}x + \int_\Omega \phi\,\mathrm{d}\nu. \end{aligned} \tag{2.4}$$

However, the left-hand side of the last equation gives

$$\begin{aligned} &\int_\Omega |\Delta u_n|^{p-2}\Delta u_n(\phi\Delta u_n + 2\langle\nabla\phi, \nabla u_n\rangle + u_n\Delta\phi)\,\mathrm{d}x \\ &\quad = \int_\Omega |\Delta u_n|^p \phi\,\mathrm{d}x \\ &\qquad + \int_\Omega |\Delta u_n|^{p-2}\Delta u_n(2\langle\nabla\phi, \nabla u_n\rangle + u_n\Delta\phi)\,\mathrm{d}x, \end{aligned}$$

and taking the limit in n, we have

$$\begin{aligned}(2.5)\qquad \lim_{n\to\infty}\int_\Omega |\Delta u_n|^{p-2}\Delta u_n\Delta(\phi u_n)\,\mathrm{d}x &= \int_\Omega \phi\,\mathrm{d}\mu \\ &\quad + \lim_{n\to\infty}\int_\Omega |\Delta u_n|^{p-2}\Delta u_n(2\langle\nabla\phi,\nabla u_n\rangle + u_n\Delta\phi)\,\mathrm{d}x.\end{aligned}$$

We now prove that

$$\lim_{\epsilon\to 0}\left(\lim_{n\to\infty}\int_\Omega |\Delta u_n|^{p-2}\Delta u_n(2\langle\nabla\phi,\nabla u_n\rangle + u_n\Delta\phi)\,\mathrm{d}x\right) = 0.$$

In fact, by the Cauchy-Schwarz and Hölder inequalities, we have

$$\begin{aligned}0 &\le \lim_{n\to\infty}\left|\int_\Omega |\Delta u_n|^{p-2}\Delta u_n\langle\nabla\phi,\nabla u_n\rangle\,\mathrm{d}x\right| \\ &\le \lim_{n\to\infty}\left(\int_\Omega |\Delta u_n|^p\mathrm{d}x\right)^{(p-1)/p}\left(\int_\Omega |\nabla\phi|^p|\nabla u_n|^p\mathrm{d}x\right)^{1/p}.\end{aligned}$$

The weak convergence of $\{u_n\}$, Hölder's inequality and (2.3) thus imply

$$\begin{aligned}&\lim_{n\to\infty}\left(\int_\Omega |\Delta u_n|^p\mathrm{d}x\right)^{(p-1)/p}\left(\int_\Omega |\nabla\phi|^p|\nabla u_n|^p\mathrm{d}x\right)^{1/p} \\ &\quad\le C\left(\int_{B(x_k,2\epsilon)\cap\Omega} |\nabla\phi|^p|\nabla u|^p\mathrm{d}x\right)^{1/p} \\ &\quad\le C\left[\left(\int_{B(x_k,2\epsilon)\cap\Omega} |\nabla\phi|^N\mathrm{d}x\right)^{p/N}\right. \\ &\qquad\qquad \left.\times\left(\int_{B(x_k,2\epsilon)\cap\Omega} |\nabla u|^{Np/(N-p)}\mathrm{d}x\right)^{(N-p)/N}\right]^{1/p} \\ &\quad\le C\left(\int_{B(x_k,2\epsilon)\cap\Omega} |\nabla u|^{Np/(N-p)}\mathrm{d}x\right)^{(N-p)/Np} \longrightarrow 0 \text{ when } \epsilon\to 0.\end{aligned}$$

However, we also have that

$$\begin{aligned}
0 &\leq \lim_{n\to\infty}\left|\int_\Omega |\Delta u_n|^{p-2}(\Delta u_n)u_n\Delta\phi\,\mathrm{d}x\right| \\
&\leq \lim_{n\to\infty}\int_\Omega |\Delta u_n|^{p-1}|u_n\Delta\phi|\,\mathrm{d}x \\
&\leq \lim_{n\to\infty}\left(\int_\Omega |\Delta u_n|^p\mathrm{d}x\right)^{(p-1)/p}\left(\int_\Omega |\Delta\phi|^p|u_n|^p\mathrm{d}x\right)^{1/p} \\
&\leq C\left(\int_{B(x_k,2\epsilon)\cap\Omega}|\Delta\phi|^p|u|^p\mathrm{d}x\right)^{1/p} \\
&\leq C\left[\left(\int_{B(x_k,2\epsilon)\cap\Omega}|\Delta\phi|^{N/2}\mathrm{d}x\right)^{(2p)/N}\left(\int_{B(x_k,2\epsilon)\cap\Omega}|u|^{p^*}\mathrm{d}x\right)^{p/p^*}\right]^{1/p} \\
&\leq C\left(\int_{B(x_k,2\epsilon)\cap\Omega}|u|^{p^*}\mathrm{d}x\right)^{1/p^*} \longrightarrow 0 \text{ when } \epsilon\to 0,
\end{aligned}$$

as claimed.

Equations (2.4) and (2.5) imply

$$0=\lim_{\epsilon\to 0}\left\{\lambda\int_\Omega f(x)|u|^q\phi\,\mathrm{d}x+\int_\Omega\phi\,d\nu-\int_\Omega\phi\,d\mu\right\}=\nu_k-\mu_k.$$

Applying Lemma 2.2, we know that $\nu_k\geq S\nu_k^{p/p^*}$. So, $\nu_k\geq S^{N/(2p)}$. It follows from (2.4) and (2.5) that

$$\begin{aligned}
c &= \lim_{n\to\infty}J_\lambda(u_n)=\lim_{n\to\infty}\left\{J_\lambda(u_n)-\frac{1}{p}\langle J_\lambda'(u_n),u_n\rangle\right\} \\
&= \lim_{n\to\infty}\lambda\left(\frac{1}{p}-\frac{1}{q}\right)\int_\Omega f(x)|u_n|^q\mathrm{d}x+\left(\frac{1}{p}-\frac{1}{p^*}\right)\int_\Omega|u_n|^{p^*}\mathrm{d}x \\
&= \lambda\left(\frac{1}{p}-\frac{1}{q}\right)\int_\Omega f(x)|u|^q\mathrm{d}x+\frac{2}{N}\left(\int_\Omega|u|^{p^*}\mathrm{d}x+\sum_{k\in I}\nu_k\right) \\
&\geq \lambda\left(\frac{1}{p}-\frac{1}{q}\right)\int_\Omega f(x)|u|^q\mathrm{d}x+\frac{2}{N}\int_\Omega|u|^{p^*}\mathrm{d}x+\frac{2}{N}S^{N/(2p)}.
\end{aligned}\tag{2.6}$$

Since $1<q<p$, applying Hölder's inequality to (2.6) we obtain

$$c\geq\frac{2}{N}S^{N/(2p)}+\frac{2}{N}\int_\Omega|u|^{p^*}\mathrm{d}x-\lambda\left(\frac{1}{q}-\frac{1}{p}\right)\|f\|_\beta\left(\int_\Omega|u|^{p^*}\mathrm{d}x\right)^{q/p^*}.$$

We now consider the function $g(x) = \kappa_1 x^{p^*} - \lambda\kappa_2 x^q$ with

$$\kappa_1 = \frac{2}{N} \quad \text{and} \quad \kappa_2 = \left(\frac{1}{q} - \frac{1}{p}\right)\|f\|_\beta.$$

The function g attains its absolute minimum for $x > 0$ at

$$x_0 = \left(\frac{\lambda\kappa_2 q}{p^*\kappa_1}\right)^{1/(p^*-q)}.$$

Thus,

$$\begin{aligned} g(x) \geq g(x_0) &= \kappa_1\left(\frac{\lambda\kappa_2 q}{p^*\kappa_1}\right)^{p^*/(p^*-q)} - \lambda\kappa_2\left(\frac{\lambda\kappa_2 q}{p^*\kappa_1}\right)^{q/(p^*-q)} \\ &= \lambda^{p^*/(p^*-q)}\kappa_1\left(\frac{\kappa_2 q}{p^*\kappa_1}\right)^{p^*/(p^*-q)} - \lambda^{1+(q/(p^*-q))}\kappa_2\left(\frac{\kappa_2 q}{p^*\kappa_1}\right)^{q/(p^*-q)} \\ &= -D\lambda^{p^*/(p^*-q)}, \end{aligned}$$

where

$$D = \kappa_2\left(\frac{\kappa_2 q}{p^*\kappa_1}\right)^{q/(p^*-q)} - \kappa_1\left(\frac{\kappa_2 q}{p^*\kappa_1}\right)^{p^*/(p^*-q)}.$$

(It is easy to verify that $D > 0$.) Therefore, we conclude that

$$c \geq \frac{2}{N}S^{N/(2p)} - D\lambda^\beta,$$

thus reaching a contradiction with the hypothesis $c < (2/N)S^{N/(2p)} - D\lambda^\beta$. We conclude that $I = \emptyset$, and thus, (2.2) implies that

$$\int_\Omega |u_n|^{p^*}\mathrm{d}x \longrightarrow \int_\Omega |u|^{p^*}\mathrm{d}x \text{ when } n \to \infty.$$

Applying the Brézis-Lieb lemma, see **[3]**, we conclude that the convergence

$$u_n \longrightarrow u \quad \text{in } L^{p^*}(\Omega).$$

If we set

$$F_n := J'_\lambda(u_n) + \lambda|u_n|^{q-2}u_n + |u_n|^{p^*-2}u_n,$$

a straightforward computation shows that $\{F_{n_k}\}$ is a Cauchy sequence in $\mathbf{E}^*$. Since we have

$$\|u_n - u_m\| \leq \alpha \begin{cases} \|F_n - F_m\|_{\mathbf{E}^*}^{1/(p-1)} & \text{if } p \geq 2, \\ M^{2-p}\|F_n - F_m\|_{\mathbf{E}^*} & \text{if } 1 < p < 2, \end{cases}$$

where $\alpha = \alpha(p)$ and $M = \max\{\|u_n\|, \|u_m\|\}$, we deduce that $\{u_{n_k}\}$ is strongly convergent in $\mathbf{E}$. $\square$

3. Proof of Theorem 1.1. Assume that $1 < q < p$ and

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\Delta u|^p \mathrm{d}x - \frac{\lambda}{q}\int_\Omega f(x)|u|^q \mathrm{d}x - \frac{1}{p^*}\int_\Omega |u|^{p^*}\mathrm{d}x.$$

Then, by the Hölder and Sobolev inequalities we obtain:

$$\begin{aligned} J_\lambda(u) \geq \frac{1}{p}&\int_\Omega |\Delta u|^p \mathrm{d}x \\ &- \frac{\lambda}{q}\|f\|_\beta S^{-q/p}\left(\int_\Omega |\Delta u|^p \mathrm{d}x\right)^{q/p} \\ &- \frac{1}{p^*}S^{-p^*/p}\left(\int_\Omega |\Delta u|^p \mathrm{d}x\right)^{p^*/p}, \end{aligned}$$

where $\beta = p^*/(p^* - q)$. Consequently,

$$J_\lambda(u) \geq h(\|u\|),$$

where

$$h(x) = \frac{1}{p}x^p - \frac{\lambda}{q}\|f\|_\beta S^{-q/p}x^q - \frac{1}{p^*}S^{-p^*/p}x^{p^*}.$$

There exists a $\lambda_1 > 0$ such that, if $0 < \lambda < \lambda_1$, then h attains a local minimum and a local maximum. Let R_0 and R_1 be such that $r < R_0 < R < R_1$, where R is the value which h attains as its maximum and r is the value which h attains as its minimum, and $h(R_1) > h(r)$. (See Figure 1.)

We take the following truncation of the functional J_λ. Take

$$\tau : \mathbb{R}^+ \longrightarrow [0,1],$$

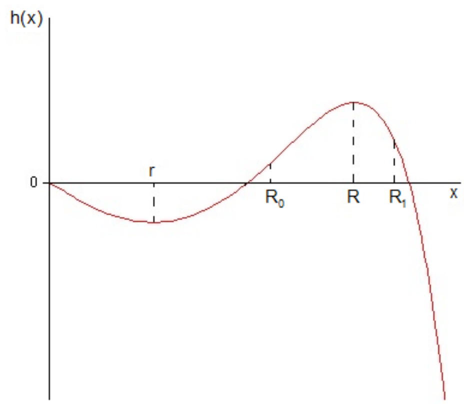

FIGURE 1. Graph $h(x) = (1/p)x^p - (\lambda/q)\|f\|_\beta S^{-q/p} x^q - (1/p^*) S^{-p^*/p} x^{p^*}$.

nonincreasing and C^∞, such that

$$\begin{cases} \tau(x) = 1 & \text{if } x \le R_0, \\ \tau(x) = 0 & \text{if } x \ge R_1. \end{cases}$$

Let $\varphi(u) = \tau(\|u\|)$. We consider the truncated functional

$$\widetilde{J}_\lambda(u) = \frac{1}{p}\int_\Omega |\Delta u|^p \mathrm{d}x - \frac{\lambda}{q}\int_\Omega f(x)|u|^q \mathrm{d}x - \frac{1}{p^*}\int_\Omega |u|^{p^*}\varphi(u)\,\mathrm{d}x.$$

Then we have, as before, $\widetilde{J}_\lambda(u) \ge \overline{h}(\|u\|)$, with

$$\overline{h}(x) = \frac{1}{p}x^p - \frac{\lambda}{q}\|f\|_\beta S^{-q/p} x^q - \frac{1}{p^*} S^{-p^*/p} x^{p^*}\tau(x).$$

Observe that $\overline{h} = h$, for $x \le R_0$, and

$$\overline{h}(x) = \frac{1}{p}x^p - \frac{\lambda}{q}\|f\|_\beta S^{-q/p} x^q \quad \text{for } x \ge R_1.$$

The main properties of $\widetilde{J}_\lambda$ are the following:

Lemma 3.1.

(i) $\widetilde{J}_\lambda \in C^1(\mathbf{E}, \mathbb{R})$.

(ii) *If* $\widetilde{J}_\lambda(u) \leq 0$, *then* $\|u\| < R_0$, *and* $J_\lambda(v) = \widetilde{J}_\lambda(v)$ *for all*

$$v \in B_{R_0} = \{u \in \mathbf{E} : \|u\| < R_0\}.$$

(iii) *There exists a* $\lambda_2 > 0$ *such that, if* $0 < \lambda < \lambda_2$, *then* $\widetilde{J}_\lambda$ *verifies the Palais-Smale condition for any level* $c < 0$.

Proof.

(i) and (ii) are immediate.

In order to prove (iii), let $\{u_n\} \subset \mathbf{E}$ be a Palais-Smale sequence for $\widetilde{J}_\lambda$:

$$\widetilde{J}_\lambda(u_n) \longrightarrow c \quad \text{and} \quad \widetilde{J}'_\lambda(u_n) \longrightarrow 0.$$

Since $c < 0$, we have that

$$\widetilde{J}_\lambda(u_n) \leq 0 \quad \text{for } n \text{ large enough.}$$

Consequently, by (ii), $\{u_n\} \subset B_{R_0}$. Let $\lambda_2 > 0$ be such that, for $0 < \lambda < \lambda_2$,

$$\frac{2}{N} S^{N/(2p)} - K\lambda^\beta \geq 0.$$

By definition,

$$J_\lambda = \widetilde{J}_\lambda \quad \text{in } B_{R_0};$$

hence, the sequence $\{u_n\}$ satisfies

$$J_\lambda(u_n) \longrightarrow c < 0 \leq \frac{2}{N} S^{N/(2p)} - D\lambda^\beta$$

and

$$J'_\lambda(u_n) \longrightarrow 0.$$

Therefore, by Theorem 2.3, the sequence $\{u_n\}$ admits a strongly convergent subsequence in $\mathbf{E}$. $\square$

Remark 3.2. Note that, if we find some negative critical value for $\widetilde{J}_\lambda$, then, by (ii), we have a negative critical value for J_λ.

Let Σ be the class of subsets of $\mathbf{E} \setminus \{0\}$ which are closed and symmetric with respect to the origin. For $A \in \Sigma$, we define the genus $\gamma(A)$ by

$$\gamma(A) = \min\{k \in \mathbb{N} : \text{there exists } \phi \in C(A, \mathbb{R}^k \setminus \{0\}),\ \phi(x) = -\phi(-x)\}$$

and, if such a minimum is not attained, we define $\gamma(A) = +\infty$.

The main properties of the genus are the following (see **[16]** for details):

Proposition 3.3. *Let $A, B \in \Sigma$. Then:*

(i) *if there exists an odd function $f \in C(A, B)$, then $\gamma(A) \leq \gamma(B)$.*
(ii) *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
(iii) *If there exists an odd homeomorphism between A and B, then $\gamma(A) = \gamma(B)$.*
(iv) *If $\mathbb{S}^{N-1}$ is the sphere in $\mathbb{R}^N$, then $\gamma(\mathbb{S}^{N-1}) = N$.*
(v) *$\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.*
(vi) *If $\gamma(B) < +\infty$, then $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$.*
(vii) *If A is compact, then $\gamma(A) < +\infty$, and there is a $\delta > 0$ such that $\gamma(A) = \gamma(N_\delta(A))$ where $N_\delta(A) = \{x \in \mathbf{E} : d(x, A) \leq \delta\}$.*
(viii) *If X is a subspace of $\mathbf{E}$ with codimension k, and $\gamma(A) > k$, then $A \cap X \neq \emptyset$.*

Now, we will construct an appropriate mini-max sequence of negative critical values for the functional $\widetilde{J}_\lambda$. The proof of the next result follows **[8]**.

Lemma 3.4. *Given $n \in \mathbb{N}$, there is an $\epsilon = \epsilon(n) > 0$, such that*

$$\gamma(\{u \in \mathbf{E} : \widetilde{J}_\lambda(u) \leq -\epsilon\}) \geq n.$$

Proof. Fix $n \in \mathbb{N}$, and let E_n be an n-dimensional subspace of $\mathbf{E}$. Take $u_n \in E_n$, with $\|u_n\| = 1$. For $0 < \rho < R_0$, we have:

$$\begin{aligned}\widetilde{J}_\lambda(\rho u_n) &= J_\lambda(\rho u_n)\\ &= \frac{1}{p}\rho^p - \frac{\lambda}{q}\rho^q \int_\Omega f(x)|u_n|^q \mathrm{d}x - \frac{1}{p^*}\rho^{p^*} \int_\Omega |u_n|^{p^*} \mathrm{d}x.\end{aligned}$$

Since all the norms in E_n are equivalent, we define

$$\alpha_n = \inf\left\{ \int_\Omega |u|^{p^*} \mathrm{d}x : u \in E_n, \|u\| = 1 \right\} > 0,$$

$$\beta_n = \inf\left\{\int_\Omega f(x)|u|^q \mathrm{d}x : u \in E_n, \|u\| = 1\right\} > 0.$$

Hence,

$$\widetilde{J}_\lambda(\rho u_n) \le \frac{1}{p}\rho^p - \frac{\lambda\beta_n}{q}\rho^q - \frac{\alpha_n}{p^*}\rho^{p^*},$$

and we can choose $\epsilon > 0$ (which depends upon n) and $0 < \eta < R_0$ such that

$$\widetilde{J}_\lambda(\eta u) \le -\epsilon \text{ if } u \in E_n \text{ and } \|u\| = 1.$$

Let $\mathbb{S}_\eta = \{u \in \mathbf{E} : \|u\| = \eta\}$ be such that

$$\mathbb{S}_\eta \cap E_n \subset \{u \in \mathbf{E} : \widetilde{J}_\lambda(u) \le -\epsilon\}.$$

Therefore, by Proposition 3.3, we have

$$\gamma(\{u \in \mathbf{E} : \widetilde{J}_\lambda(u) \le -\epsilon\}) \ge \gamma(\mathbb{S}_\eta \cap E_n) = n. \qquad \square$$

Let

$$\Sigma_k = \{C \subset \mathbf{E} \setminus \{0\} : C \text{ is closed}, C = -C, \gamma(C) \ge k\},$$

$$c_k = \inf_{C \in \Sigma_k} \sup_{u \in C} \widetilde{J}_\lambda(u)$$

and

$$K_c = \{u \in \mathbf{E} : \widetilde{J}'_\lambda(u) = 0, \widetilde{J}_\lambda(u) = c\}.$$

Lemma 3.5. *The c_ks are negative.*

Proof. In fact, for simplicity, set

$$\widetilde{J}_\lambda^{-\epsilon} = \{u \in \mathbf{E} : \widetilde{J}_\lambda(u) \le -\epsilon\}.$$

From Lemma 3.4, for all $k \in \mathbb{N}$, there exists an $\epsilon = \epsilon(k) > 0$ such that $\gamma(\widetilde{J}_\lambda^{-\epsilon}) \ge k$.

Since $\widetilde{J}_\lambda$ is continuous and even, $\widetilde{J}_\lambda^{-\epsilon} \in \Sigma_k$; then, $c_k \le -\epsilon(k) < 0$, for all k. However, $\widetilde{J}_\lambda$ is bounded from below; hence, $c_k > -\infty$ for all k. $\square$

The next result proves the existence of critical points.

Lemma 3.6. *Let* $\lambda_0 = \min\{\lambda_1, \lambda_2\}$, *and suppose that* $\lambda \in (0, \lambda_0)$. *If* $c = c_k = c_{k+1} = \cdots = c_{k+r}$, *then* $\gamma(K_c) \geq r + 1$.

Proof. We will use the classical deformation lemma (see **[16]**).

Assume that $c = c_k = c_{k+1} = \cdots = c_{k+r}$, and observe that $c < 0$; therefore, $\widetilde{J}_\lambda$ verifies the Palais-Smale condition in K_c. It is easy to see that K_c is compact.

Assume, for contradiction, that $\gamma(K_c) \leq r$. Thus, there exists a closed and symmetric set U, with $K_c \subset U$ such that $\gamma(U) = \gamma(K_c) \leq r$ (we can choose $U = N_\sigma(K_c)$ for some $\sigma > 0$).

By the deformation lemma, we have an odd homeomorphism

$$\eta : \mathbf{E} \longrightarrow \mathbf{E},$$

such that

$$\eta(\widetilde{J}_\lambda^{c+\delta} \setminus U) \subset \widetilde{J}_\lambda^{c-\delta} \quad \text{for some } 0 < \delta < -c.$$

By definition,

$$c = c_{k+r} = \inf_{C \in \Sigma_{k+r}} \sup_{u \in C} \widetilde{J}_\lambda(u).$$

Then, there exists an $A \in \Sigma_{k+r}$ such that $\sup_{u \in A} \widetilde{J}_\lambda(u) < c + \delta$, i.e., $A \subset \widetilde{J}_\lambda^{c+\delta}$ and

$$\eta(A \setminus U) \subset \eta(\widetilde{J}_\lambda^{c+\delta} \setminus U) \subset \widetilde{J}_\lambda^{c-\delta}. \tag{3.1}$$

However, by Proposition 3.3, we have

$$\gamma(\overline{A \setminus U}) \geq \gamma(A) - \gamma(U) \geq k$$

and

$$\gamma(\eta(\overline{A \setminus U})) = \gamma(\overline{A \setminus U}) \geq k.$$

Consequently, $\eta(\overline{A \setminus U}) \in \Sigma_k$. This contradicts (3.1) since $\eta(\overline{A \setminus U}) \in \Sigma_k$ implies

$$\sup_{u \in \eta(\overline{A \setminus U})} \widetilde{J}_\lambda(u) \geq c_k = c. \qquad \square$$

Proof of Theorem 1.1. It is a consequence of previous results. In fact, define $\lambda_0 = \min\{\lambda_1, \lambda_2\}$, and suppose that $\lambda \in (0, \lambda_0)$. By definition, we have

$$c_k \leq c_{k+1} \leq \cdots \leq c_{k+r} \leq \cdots < 0. \tag{3.2}$$

Now, we consider two cases.

Case (I). Suppose that all inequalities in (3.2) are strict. Since Lemma 3.6 proves that $\gamma(K_{c_k}) \geq 1$ for any $k \in \mathbb{N}$, the set K_{c_k} has at least one element. Thus, since the values of c_k are different from each other, we obtain a sequence of different critical points for $\widetilde{J}_\lambda$. Since Lemma 3.5 implies that the values of c_k are negative, Lemma 3.1 (ii) implies that critical points of $\widetilde{J}_\lambda$ are also critical points of J_λ.

Observe that, if $\mathbf{E} = W_0^{2,p}(\Omega)$ (respectively, $\mathbf{E} = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$), then critical points of J_λ are solutions of problem (1.1)–(1.2) (respectively, (1.1)–(1.3). See the next section for the second Navier boundary condition. Furthermore, by the maximum principle, the solutions of the problem (1.1)–(1.3) are positive.

Case (II). Suppose that there exist $k, r \in \mathbb{N}$, such that

$$c_k = c_{k+1} = \cdots = c_{k+r}.$$

In this case, Lemma 3.6 gives that $\gamma(K_{c_k}) \geq 2$. This means that the set K_{c_k} is connected, closed and symmetric with respect to the origin. Indeed, if K_{c_k} is disconnected, then $\gamma(K_{c_k}) = 1$ since we can define an odd function $f \in C(K_{c_k}, \mathbb{R} \setminus \{0\})$ as being 1 in a connected component and -1 in the other symmetric connected component. Therefore, we have an infinite number of distinct critical points of $\widetilde{J}_\lambda$. Analogously to Case (I), we obtain an infinite number of solutions for problems (1.1)–(1.2) and (1.1)–(1.3). □

4. On the Navier boundary condition. For all $\phi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, if $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfies

$$\int_\Omega |\Delta u|^{p-2} \Delta u \Delta \phi \, \mathrm{d}x = \lambda \int_\Omega f(x) |u|^{q-2} u \phi \, \mathrm{d}x + \int_\Omega |u|^{p^*-2} u \phi \, \mathrm{d}x,$$

we now show that $\Delta u = 0$ on $\partial\Omega$. For this, define

$$v = -|\Delta u|^{p-2} \Delta u \in L^{p/(p-1)}(\Omega)$$

and

$$g(u) = \lambda f(x)|u|^{q-2}u + |u|^{p^*-2}u \in L^{p^*/(p^*-1)}(\Omega) = L^r(\Omega),$$

where $r = p^*/(p^*-1) > 1$. Then, we have

$$\int_\Omega v(-\Delta\phi)\,\mathrm{d}x = \int_\Omega g(u)\phi\,\mathrm{d}x, \quad \text{for all } \phi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega). \tag{4.1}$$

Let $w \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ be the unique solution of the problem

$$\begin{cases} -\Delta w = g(u) & \text{on } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Therefore, we have, for all $\phi \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$,

$$\int_\Omega \nabla w \nabla \phi\,\mathrm{d}x = \int_\Omega w(-\Delta\phi)\,\mathrm{d}x = \int_\Omega g(u)\phi\,\mathrm{d}x. \tag{4.3}$$

Subtracting (4.1) from (4.3), we obtain

$$\int_\Omega (v-w)\Delta\phi\,\mathrm{d}x = 0 \quad \text{for all } \phi \in C_0^\infty(\Omega),$$

from which $v = w$ almost everywhere in Ω follows. Thus,

$$v = w \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega),$$

and we conclude that $v = 0$ on $\partial\Omega$.

REFERENCES

1. C.O. Alves, G.M. Figueiredo and M.F. Furtado, *Multiplicity of solutions for elliptic systems via local mountain pass method*, Comm. Pure Appl. Anal. **8** (2009), 1745–1758.

2. F. Bernis, J. García-Azorero and I. Peral, *Existence and multiplicity of nontrivial solutions in semilinear critical problems of fourth order*, Adv. Diff. Eqs. **1** (1996), 219–240.

3. H. Brézis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), 486–490.

4. E.M. dos Santos, *Multiplicity of solutions for a fourth-order quasilinear nonhomogeneous equations*, J. Math. Anal. Appl. **342** (2008), 277–297.

5. P. Drábek and M. Ôtani, *Global bifurcation result for the p-biharmonic operator*, Electr. J. Diff. Eqs. **2001**.

6. A. El Khalil, S. Kellati and A. Touzani, *On the spectrum of the p-biharmonic operator*, Electr. J. Diff. Eqs. **2002**, 161–170.

7. A. El Khalil, M.D. Morchid Alaoui and A. Touzani, *On the p-biharmonic operator with critical Sobolev exponent and nonlinear Steklov boundary condition*, Int. J. Anal. **2014**, Art. ID 498386.

8. J. García-Azorero and I. Peral, *Some results about the existence of a second positive solutions in a quasilinear critical problem*, Indiana Univ. Math. J. **43** (1994), 941–957.

9. F. Gazzola, H.-Ch. Grunau and G. Sweers, *Optimal Sobolev and Hardy-Rellich constants under Navier boundary conditions*, Ann. Mat. Pura Appl. **189** (2010), 475–486.

10. Y. He, G. Li and S. Peng, *Concentrating bound states for Kirchhoff type problems in $R3$ involving critical Sobolev exponents*, Adv. Nonlin. Stud. **14** (2014), 483–510.

11. H. He and J. Yang, *Asymptotic behavior of solutions for Henon systems with nearly critical exponent*, J. Math. Anal. Appl. **347** (2008), 459–471.

12. C. Ji and W. Wang, *On the p-biharmonic equation involving concave-convex nonlinearities and sign-changing weight function*, Electr. J. Qual. Th. Diff. Eqs. **2012**.

13. P.L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case*, Rev. Mat. Iber. **1** (1985), 145–201, 45–121.

14. D. Lü, *Multiple solutions for p-Laplacian systems with critical homogeneous nonlinearity*, Bound. Value Prob. **2012** (2012).

15. N. Qiao and Z.Q. Wang, *Multiplicity results for positive solutions to non-autonomous elliptic problems*, Electr. J. Diff. Eqs. (1999).

16. P.H. Rabinowitz, *Minimax methods in critical points theory with applications to differential equations*, CBMS Reg. Conf. Ser. Math. **65** (1986), American Mathematical Society, Providence, RI.

17. Y. Shen and J. Zhang, *Multiplicity of positive solutions for a Navier boundary-value problem involving the p-biharmonic with critical exponent*, Electr. J. Diff. Eqs. **2011**.

18. T.F. Wu, *Existence and multiplicity of positive solutions for a class of nonlinear boundary value problems*, J. Diff. Eqs. **252** (2012), 3403–3435.

Universidade Federal de Minas Gerais, Departamento de Matemática, Belo Horizonte, Minas Gerais, 30.123-970, Brazil
Email address: **hamilton@mat.ufmg.br**

Universidade Federal de Ouro Preto, Departamento de Matemática, Ouro Preto, 35.400-000, Brazil
Email address: **leandroleme.demat@gmail.com**

Universidade Federal de Minas Gerais, Departamento de Matemática, Belo Horizonte, Minas Gerais, 30.123-970, Brazil
Email address: **xhelder@mat.ufmg.br**