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α-POSITIVE/α-NEGATIVE DEFINITE FUNCTIONS ON GROUPS

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ABSTRACT. In this paper, we introduce the notions of an α-positive/α-negative definite function on a (discrete) group. We first construct the Naimark-GNS type representation associated to an α-positive definite function and prove the Schoenberg type theorem for a matricially bounded α-negative definite function. Using a J-representation on a Krein space $(\mathcal{K}, J)$ associated to a nonnegative normalized α-negative definite function, we also construct a J-cocycle associated to a J-representation. Using a J-cocycle, we show that there exist two sequences of α-positive definite functions and proper (α, J)-actions on a Krein space $(\mathcal{K}, J)$ corresponding to a proper matricially bounded α-negative definite function.

1. Introduction. In operator algebras, positive definite functions on a group correspond to a completely positive linear map on a C^*-algebra. This has many applications in mathematics, particularly, in harmonic analysis **[3]**. Moreover, a positive definite function on a group naturally occurs in the unitary representation theory of a group on a Hilbert space due to GNS construction. In this paper, we are concerned with an α-positive/α-negative definite function on a discrete group, where positive definiteness is replaced by a more weakened positivity which we call "α-positive definiteness." Structures and properties of local quantum field theories without positivity were studied in **[13, 14]** in connection with their Euclidean formulations.

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A Krein space is a Hilbert space equipped with an indefinite inner product, which arises naturally in massless or gauge field theory where the positivity condition must be abandoned. Due to such physical facts, GNS construction of a Krein space is of increasing interest in the general (axiomatic) quantum field theory **[2, 4, 5]**. It is also known that the geometry of a Krein space is much richer than that of an ordinary space with an indefinite metric. Motivated by a metric operator introduced by Jakobczyk and Strocchi **[13]** and a P-functional in **[2]**, Heo, Hong and Ji **[9]** introduced the notion of an α-completely positive linear map between two C^*-algebras, and Heo and Ji **[10]** proved a Radon-Nikodým type theorem for α-completely positive linear maps. In **[7]**, the author introduced a notion of an α-completely positive linear map of a topological group with an involution into a C^*-algebra and proved a (covariant) Naimark-KSGNS representation theorem for a (covariant) α-completely positive linear map. Recently, Joita **[15]** also studied an operator valued α-completely positive map on a group.

The purpose of this paper is to introduce the notion of an α-positive/α-negative definite function on a group and to study properties of these functions. Such an α-positive definite function may be regarded as a generalization of a positive definite function as well as a counterpart of an α-completely positive linear map on a C^*-algebra **[9, 10, 11]**. We now give a brief overview of the organization of the paper.

In Section 2, we introduce the notion of an α-positive definite function on a locally compact group with an involution. This function may be regarded as a generalization of a positive definite function as well as a function with more weakened positivity. We study some properties of an α-positive definite function on a discrete group and prove a Naimark-GNS type theorem for an α-positive definite function, which implies that an α-positive definite function naturally gives rise to a J-unitary representation on a Krein space.

In Section 3, we also introduce the notion of an α-negative definite function, which may be regarded as a generalization of a negative definite function as well as a notion corresponding to an α-positive definite function. We investigate various properties of an α-negative definite function and prove a Schoenberg type theorem for a normalized and matricially bounded α-negative definite function.

In Section 4, we prove the main theorem which states that a J-cocycle exists for a J-representation on a Krein space associated with a normalized α-negative definite function. Given a proper and matricially bounded α-negative definite function, we construct a sequence of α-positive definite functions and a proper (α, J)-action on a Krein space.

2. α-positive definite functions on groups. We now introduce the notion of an α-positive definite function on a locally compact group which may be regarded as the counterpart of an α-completely positive linear map between (locally) C^*-algebras [**9, 10, 11**].

Definition 2.1. Let G be a locally compact group with an involution α, that is, $\alpha^2 = \mathrm{id}_G$, $\alpha(g)^{-1} = \alpha(g^{-1})$ and $\alpha(e) = e$, where e is a unit element of G. A function

$$\phi : G \longrightarrow \mathbb{C}$$

is called *α-positive definite* if

(i) $\phi(\alpha(g_1)\alpha(g_2)) = \phi(\alpha(g_1 g_2)) = \phi(g_1 g_2)$ for all $g_1, g_2 \in G$;

(ii) for any $n \in \mathbb{N}$ and $g_1, \dots, g_n \in G$, the $n \times n$ matrix $[\phi(\alpha(g_i^{-1})g_j)]$ is positive semi-definite, i.e.,

$$\sum_{i,j=1}^{n} \overline{\lambda}_i \lambda_j \phi\big(\alpha(g_i^{-1})g_j\big) \geq 0 \quad \text{for all } \lambda_1, \dots, \lambda_n \in \mathbb{C}; \tag{2.1}$$

(iii) for all $g, g_1, \dots, g_n \in G$, there exists a constant $M(g) > 0$ such that

$$[\phi(\alpha(gg_i)^{-1}\, gg_j)] \leq M(g)[\phi(\alpha(g_i)^{-1} g_j)].$$

Let G be a locally compact group with a left invariant Haar measure μ and an involution α. If Φ is continuous on G, then Φ is α-positive definite if and only if

$$\int_G \int_G \Phi(\alpha(g)^{-1}h)\overline{\varphi(g)}\varphi(h)\, d\mu(g)\, d\mu(h) \geq 0$$

for all compactly supported continuous functions φ on G.

Throughout this paper, Γ and e denote a discrete group with an involution α and a unit element of Γ, respectively, unless specified otherwise.

If $\phi : \Gamma \to \mathbb{C}$ is α-positive definite, then the following properties hold true:

(a) $\phi(e) \geq 0$,

(b) $\overline{\phi(g)} = \phi(\alpha(g^{-1}))$ for all $g \in \Gamma$,

(c) $|\phi(g)|^2 \leq \phi(e)\phi(\alpha(g^{-1})g)$ for all $g \in \Gamma$,

(d) $\overline{\phi(\alpha(g^{-1})h)} = \phi(\alpha(h^{-1})g)$ for all $g, h \in \Gamma$.

Proposition 2.2. *If $\phi : \Gamma \to \mathbb{C}$ is α-positive definite, then*

(i) $|\phi(g)-\phi(h)|^2 \leq \phi(e)[\phi(\alpha(g^{-1})g)+\phi(\alpha(h^{-1})h)-2\mathrm{Re}(\phi(\alpha(g^{-1})h))]$ *for all $g, h \in \Gamma$.*

If, in addition, $\phi(e) = 1$, then the following inequalities are valid for all $g, h \in \Gamma$:

(ii) $|\phi(g)|^2 \leq \phi(\alpha(g^{-1})g)$;

(iii) $|\phi(g) - \phi(h)|^2 \leq [\phi(\alpha(g^{-1})g) + \phi(\alpha(h^{-1})h) - 2\mathrm{Re}(\phi(\alpha(g^{-1})h))]$;

(iv) $|\phi(\alpha(g^{-1})h) - \overline{\phi(g)}\phi(h)|^2 \leq (\phi(\alpha(g^{-1})g) - |\phi(g)|^2)(\phi(\alpha(h^{-1})h) - |\phi(h)|^2)$.

Proof. In order to show the inequality in (i), take three elements $g_1 = e$, $g_2 = g$ and $g_3 = h$ in equation (2.1). Then, we have that the following matrix is positive semi-definite:

$$\begin{pmatrix} \phi(e) & \phi(g) & \phi(h) \\ \phi(\alpha(g^{-1})) & \phi(\alpha(g^{-1})g) & \phi(\alpha(g^{-1})h) \\ \phi(\alpha(h^{-1})) & \phi(\alpha(h^{-1})g) & \phi(\alpha(h^{-1})h) \end{pmatrix} \geq 0. \tag{2.2}$$

For any $x \in \mathbb{R}$, we take $\lambda_1 = 1$, $\lambda_2 = x|\phi(g) - \phi(h)|/(\phi(g) - \phi(h))$ and $\lambda_3 = -\lambda_2$. Then, we see that the inequality

$$x^2[\phi(\alpha(g^{-1})g)+\phi(\alpha(h^{-1})h)-2\mathrm{Re}\phi(\alpha(g^{-1})h)]+2x|\phi(g)-\phi(h)|+\phi(e) \geq 0$$

is valid for all $x \in \mathbb{R}$ so that the discriminant of the polynomial in x on the left is non-positive, i.e.,

$$|\phi(g) - \phi(h)|^2 - \phi(e)[\phi(\alpha(g^{-1})g) + \phi(\alpha(h^{-1})h) - 2\mathrm{Re}\ \phi(\alpha(g^{-1})h)] \leq 0,$$

which gives the inequality in (i).

The inequality in (ii) immediately follows from (c). Since $\phi(e) = 1$, it follows from the positive semi-definiteness of matrix (2.2) that

(2.3)
$$\begin{aligned}&\phi(\alpha(g^{-1})g)\phi(\alpha(h^{-1})h) + \phi(g)\phi(\alpha(g^{-1})h)\overline{\phi(h)} + \overline{\phi(g)\phi(\alpha(g^{-1})h)}\phi(h)\\ &\quad \geq |\phi(h)|^2\phi(\alpha(g^{-1})g) + |\phi(g)|^2\phi(\alpha(h^{-1})h) + |\phi(\alpha(g^{-1})h)|^2.\end{aligned}$$

Thus, we have that

$$\begin{aligned}|\phi(\alpha(g^{-1})h) - \overline{\phi(g)}\phi(h)|^2 &\leq \phi(\alpha(g^{-1})g)\phi(\alpha(h^{-1})h) - |\phi(g)|^2\phi(\alpha(h^{-1})h)\\ &\quad - |\phi(h)|^2\phi(\alpha(g^{-1})g) + |\phi(g)|^2|\phi(h)|^2\\ &= (\phi(\alpha(g^{-1})g) - |\phi(g)|^2)(\phi(\alpha(h^{-1})h) - |\phi(h)|^2),\end{aligned}$$

where the inequality follows from inequality (2.3). Hence, the inequality in (iv) is valid for all $g, h \in \Gamma$. □

Remark 2.3. In Proposition 2.2, if, in addition, $\phi(\alpha(g^{-1})g) = 1$ for all $g \in \Gamma$, then we have that, for every $g, h \in \Gamma$,

(ii)′ $|\phi(g)|^2 \leq 1$,

(iii)′ $|\phi(g) - \phi(h)|^2 \leq 2[1 - \mathrm{Re}\ \phi(\alpha(g^{-1})h)]$,

(iv)′ $|\phi(\alpha(g^{-1})h) - \overline{\phi(g)}\phi(h)|^2 \leq (1 - |\phi(g)|^2)(1 - |\phi(h)|^2)$.

Let J be a (fundamental) symmetry on a Hilbert space $\mathcal{H}$ with an inner product $\langle \cdot, \cdot \rangle$, i.e., $J = J^* = J^{-1}$. We define an indefinite inner product $[\cdot, \cdot]_J$ on $\mathcal{H}$ by

$$[\xi, \eta]_J = \langle J\xi, \eta \rangle, \quad \xi, \eta \in \mathcal{H}.$$

In this case, the pair $(\mathcal{H}, J)$ is called a *Krein space*. For any $T \in \mathcal{B}(\mathcal{H})$, there exists an operator $T^J \in \mathcal{B}(\mathcal{H})$ such that

$$[T\xi, \eta]_J = [\xi, T^J\eta]_J, \quad \xi, \eta \in \mathcal{H}.$$

The operator T^J is called the *J-adjoint* of T, and we see that $T^J = JT^*J$.

Let $(\mathcal{H}, J)$ be a Krein space. A homomorphism $\pi : \Gamma \to \mathcal{B}(\mathcal{H})$ is called a *representation* on $\mathcal{H}$. A *unitary representation* $\pi : \Gamma \longrightarrow \mathcal{B}(\mathcal{H})$ is a representation on $\mathcal{H}$ such that $\pi(g^{-1}) = \pi(g)^*$ for all $g \in \Gamma$. A representation $\pi : \Gamma \to \mathcal{B}(\mathcal{H})$ on $\mathcal{H}$ is called a *J-unitary representation* on $(\mathcal{H}, J)$ if π is a representation on $\mathcal{H}$ and $\pi(g^{-1}) = \pi(g)^J$ for all $g \in \Gamma$.

The next theorem is a GNS type construction, and its proof is standard; however, we will give a sketch of the proof for the reader's convenience.

Theorem 2.4. *If $\phi : \Gamma \to \mathbb{C}$ is α-positive definite, then there exist a Krein space $(\mathcal{H}_\phi, J_\phi)$, a J_ϕ-unitary representation π_ϕ and a vector $\xi_\phi \in \mathcal{H}_\phi$ such that*

(i) *$\phi(g) = \langle \pi_\phi(g)\xi_\phi, \xi_\phi\rangle$ for all $g \in \Gamma$;*

(ii) *the linear span of the set $\{\pi_\phi(g)\xi_\phi : g \in \Gamma\}$ is dense in $\mathcal{H}_\phi$.*

Proof. Let $\mathbb{C}[\Gamma]$ be a group algebra, and let $\phi : \Gamma \to \mathbb{C}$ be α-positive definite. We define a sesquilinear form on $\mathbb{C}[\Gamma]$ by

$$\langle f_1, f_2\rangle = \sum_{g,g' \in G} \overline{f_2(g)} f_1(g')\phi(\alpha(g^{-1})g').$$

The kernel $N_\phi = \{f \in \mathbb{C}[\Gamma] : \langle f, f\rangle = 0\}$ is a vector space, and the equation $\langle f_1 + N_\phi, f_2 + N_\phi\rangle = \langle f_1, f_2\rangle$ defines an inner product on the quotient space $\mathbb{C}[\Gamma]/N_\phi$. We denote by $\mathcal{H}_\phi$ the Hilbert space obtained by the completion of $\mathbb{C}[\Gamma]/N_\phi$ with respect to the induced norm.

We see that the involution α on Γ induces an involutive map J from $\mathbb{C}[\Gamma]$ into itself defined by $J(f)(g) := f(\alpha(g))$ for any $g \in G$. We define an indefinite inner product $[\cdot, \cdot]$ on $\mathbb{C}[\Gamma]/N_\phi$ by

$$\left[f_1 + N_\phi, f_1 + N_\phi\right] = \sum_{g,g' \in G} \overline{f_2(g)} f_1(g')\phi(g^{-1}g').$$

Since $J(N_\phi) \subseteq N_\phi$, J induces a map on $\mathbb{C}[\Gamma]/N_\phi$ such that $[f_1 + N_\phi, f_2 + N_\phi] = \langle J(f_1 + N_\phi), f_2 + N_\phi\rangle$. In fact, J induces a (fundamental) symmetry on $\mathcal{H}_\phi$, which is denoted by J_ϕ. For each $g \in \Gamma$, we define a linear operator

$$\pi(g) : \mathbb{C}[\Gamma] \longrightarrow \mathbb{C}[\Gamma]$$

by

$$(\pi(g)f)(g') = f(g^{-1}g'), \quad g' \in \Gamma.$$

Since $\langle \pi(g)f, \pi(g)f\rangle \leq M(g)\langle f, f\rangle$ for $f \in \mathbb{C}[\Gamma]$ where $M(g)$ is the constant in condition (iii) of Definition 2.1, we see that $\pi(g)(N_\phi) \subseteq N_\phi$,

$g \in \Gamma$, such that each $\pi(g)$ can be extended to the entire space $\mathcal{H}_\phi$, denoted by $\pi_\phi(g)$. Moreover, π_ϕ becomes a J_ϕ-unitary representation of Γ on $(\mathcal{H}_\phi, J_\phi)$ since

$$\langle \pi(g)f_1, f_2\rangle = \langle f_1, J_\phi \pi(g^{-1})J_\phi(f_2)\rangle, \quad f_1, f_2 \in \mathbb{C}[\Gamma].$$

Similar to GNS construction, we can get a cyclic vector $\xi_\phi = \delta_e + N_\phi \in \mathcal{H}_\phi$ satisfying properties (i) and (ii). $\square$

Remark 2.5. In Theorem 2.4, it is not difficult to see that $J_\phi \pi_\phi(g) J_\phi = \pi_\phi(\alpha(g))$ for any $g \in G$. We also see that the cyclic vector ξ_ϕ in Theorem 2.4 is invariant under J_ϕ.

We should note that J_ϕ is different from the modular conjugation $\mathcal{J}$ in Tomita-Takesaki theory. Indeed, the modular conjugation $\mathcal{J}$ on $l^2(\Gamma)$ is given by $(\mathcal{J}f)(s) = \overline{f(s^{-1})}$. $\square$

The following proposition says that an α-positive definite function of Γ naturally occurs from a J-unitary representation on a Krein space, having additional properties.

Proposition 2.6. *Let $(\mathcal{H}, J)$ be a Krein space, and let ξ be a unit vector invariant under J. If $\pi : \Gamma \to \mathcal{B}(\mathcal{H})$ is a J-unitary representation such that $\pi(\alpha(g)) = J\pi(g)J$ for all $g \in \Gamma$, then $\phi : \Gamma \to \mathbb{C}$ given by $\phi(g) = \langle \pi(g)\xi, \xi\rangle$ is α-positive definite.*

Proof. Since $J\xi = \xi$ and $\pi(\alpha(g)) = J\pi(g)J$ $(g \in \Gamma)$, we have that, for any $g_1, g_2 \in \Gamma$,

$$\begin{aligned}
\phi(g_1 g_2) &= \langle \pi(g_1 g_2) J\xi, J\xi\rangle = \langle \pi(\alpha(g_1 g_2))\xi, \xi\rangle \\
&= \phi\big(\alpha(g_1 g_2)\big) = \langle \pi(g_1)\pi(g_2)J\xi, J\xi\rangle \\
&= \langle J\pi(g_1)J \cdot J\pi(g_2)J\xi, \xi\rangle \\
&= \phi\big(\alpha(g_1)\alpha(g_2)\big).
\end{aligned}$$

Since π is a J-unitary representation, we also see that $\pi(\alpha(g^{-1})) = \pi(g)^*$ for every $g \in \Gamma$.

Let $g_1, \dots, g_n \in \Gamma$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. Then, we have that

$$\sum_{i,j=1}^{n} \overline{\lambda}_i \lambda_j \phi(\alpha(g_i^{-1})g_j) = \sum_{i,j=1}^{n} \overline{\lambda}_i \lambda_j \langle \pi(\alpha(g_i^{-1})g_j))\xi, \xi\rangle$$
$$= \sum_{i,j=1}^{n} \overline{\lambda}_i \lambda_j \langle \pi(g_j)\xi, \pi(g_i)\xi\rangle \geq 0.$$

Moreover, we obtain from the equality $\pi(\alpha(g^{-1})) = \pi(g)^*$ that

$$\sum_{i,j=1}^{n} \overline{\lambda}_i \lambda_j \phi(\alpha(gg_i)^{-1}gg_j) = \sum_{i,j=1}^{n} \overline{\lambda}_i \lambda_j \langle \pi(g)\pi(g_j)\xi, \pi(g)\pi(g_i))\xi\rangle$$
$$\leq \|\pi(g)\|^2 \sum_{i,j=1}^{n} \overline{\lambda}_i \lambda_j \phi(\alpha(g_i)^{-1}g_j),$$

so that

$$[\phi(\alpha(gg_i)^{-1}\ gg_j)] \leq \|\pi(g)\|^2 [\phi(\alpha(g_i)^{-1}g_j)].$$

This completes the proof. □

Example 2.7. Let G be a locally compact group with an involution α. Then, we see from Proposition 2.6 that any J-unitary representation of G on a Krein space $(\mathcal{H}, J)$ induces an α-positive definite function. In particular, if α is an identity map on G, then any α-positive definite function on G is positive definite.

Now we give a rather easy and concrete example. Let $G = \mathbb{C} \oplus \mathbb{C}$ be an additive group, and let $a \in \mathbb{C} \setminus \{\pm 1\}$ be fixed. We define an involution α on G by

$$\alpha(x \oplus y) = (ax + (1-a)y) \oplus ((1+a)x - ay).$$

Let $t_i \in \mathbb{C} \setminus \{0\}$ $(i = 1, 2)$ be such that $(1-a)t_1 = (1+a)t_2$. We define a function

$$\phi : \mathbb{C} \oplus \mathbb{C} \longrightarrow \mathbb{C}$$

by $\phi(x \oplus y) = t_1 x + t_2 y$. Then, ϕ is α-positive definite.

3. α-negative definite functions on groups. Negative definite functions on (semi-)groups were systematically discussed in **[3]** and have been used in several different contexts of operator algebras **[1, 6]**.

For detailed information on negative definite functions, we refer to **[3, 17]**. In this section, we introduce a notion of an α-negative definite function on a discrete group, regarded as a generalization of a negative definite function as well as one corresponding to an α-positive definite function. We also give some equivalent formulations of the notion of an α-negative definite function.

Definition 3.1. A function $\psi : \Gamma \to \mathbb{C}$ is called *α-negative definite* if

(i) $\psi(g_1 g_2) = \psi(\alpha(g_1 g_2)) = \psi(\alpha(g_1)\alpha(g_2))$ for all $g_1, g_2 \in \Gamma$;

(ii) for any $n \in \mathbb{N}$ and $g_1, \dots, g_n \in \Gamma$, the $n \times n$ matrix $[\overline{\psi(g_i)} + \psi(g_j) - \psi(\alpha(g_i^{-1})g_j)]$ is positive semi-definite, i.e.,

$$\sum_{i,j=1}^{n} \overline{\lambda}_i \lambda_j \big(\overline{\psi(g_i)} + \psi(g_j) - \psi(\alpha(g_i^{-1})g_j)\big) \geq 0 \quad \text{for all } \lambda_1, \dots, \lambda_n \in \mathbb{C}.$$

For an α-negative definite function $\psi : \Gamma \to \mathbb{C}$, we easily see the following properties:

(a) $\psi(e) \geq 0$;

(b) $\psi(\alpha(g^{-1})) = \overline{\psi(g)}$ for all $g \in \Gamma$;

(c) $\operatorname{Re} \psi(g) \geq \frac{1}{2}(\psi(e) + \psi(\alpha(g^{-1})g))$ for all $g \in \Gamma$;

(d) $\psi(\alpha(h^{-1})g) = \overline{\psi(\alpha(g^{-1})h)}$ for all $g, h \in \Gamma$.

Proposition 3.2. *If $\psi : \Gamma \to \mathbb{C}$ is α-negative definite, then*

$$\psi(\alpha(g^{-1})g) + \psi(\alpha(h^{-1})h) \leq \psi(\alpha(g^{-1})h) + \psi(\alpha(h^{-1})g) \tag{3.1}$$

for all $g, h \in \Gamma$.

Proof. Taking $g_1 = g$, $g_2 = h$ and $\lambda_1 = 1$, $\lambda_2 = -1$, we obtain from condition (ii) in Definition 3.1 that inequality (3.1) holds for all $g, h \in \Gamma$. □

Note that, if $\psi : \Gamma \to \mathbb{R}$ is an α-negative definite function satisfying $\psi(e) + \psi(\alpha(g^{-1})g) \geq 0$, then ψ is nonnegative throughout Γ. Indeed, for any $\lambda \in \mathbb{R}$ and $g \in \Gamma$, we have that

$$\begin{aligned} 0 \geq {} & \lambda \cdot \lambda\, \psi(e) + \lambda(-\lambda)\psi(\alpha(e)g) \\ & + (-\lambda)\lambda\, \psi(\alpha(g^{-1})e) + (-\lambda)(-\lambda)\psi(\alpha(g^{-1})g) \end{aligned}$$

$$
\begin{aligned}
&= \lambda^2\big[\psi(e) + \psi(\alpha(g^{-1})g)\big] - 2\lambda^2\psi(g) \\
&\geq -2\lambda^2\psi(g).
\end{aligned}
$$

Hence, we see that $\psi(g) \geq 0$.

Theorem 3.3. *A function $\psi : \Gamma \to \mathbb{C}$ is α-negative definite if and only if the following three properties hold true*:

(i) $\psi(g_1g_2) = \psi(\alpha(g_1g_2)) = \psi(\alpha(g_1)\alpha(g_2))$ *for all* $g_1, g_2 \in \Gamma$;

(ii) $\psi(e) \geq 0$ *and* $\overline{\psi(g)} = \psi(\alpha(g^{-1}))$ *for all* $g \in \Gamma$;

(iii) *for all* $n \in \mathbb{N}$, $g_1, \ldots, g_n \in \Gamma$ *and* $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$,

$$
\sum_{i=1}^{n} \lambda_i = 0 \implies \sum_{i,j=1}^{n} \overline{\lambda}_i\lambda_j\psi(\alpha(g_i^{-1})g_j) \leq 0.
$$

Proof. Assume that ψ is α-negative definite. Property (i) immediately follows from Definition 2.1 and property (ii) has already been observed. In order to prove that property (iii) holds, let $g_1, \ldots, g_n \in \Gamma$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ with $\sum_i \lambda_i = 0$. Then, we have that

$$
0 \leq \sum_{i,j=1}^{n} \overline{\lambda}_i\lambda_j(\overline{\psi(g_i)} + \psi(g_j) - \psi(\alpha(g_i^{-1})g_j)) = -\sum_{i,j=1}^{n} \overline{\lambda}_i\lambda_j\psi(\alpha(g_i^{-1})g_j).
$$

Conversely, suppose that the properties (i), (ii) and (iii) hold true. Let $g_1, \ldots, g_n \in \Gamma$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be given. We take $g_{n+1} = e$ and $\lambda_{n+1} = -\sum_{i=1}^{n} \lambda_i$. Then, we have that

$$
\begin{aligned}
0 &\geq \sum_{i,j=1}^{n+1} \overline{\lambda}_i\lambda_j\psi(\alpha(g_i^{-1})g_j) \\
&= |\lambda_{n+1}|^2\psi(e) + \sum_{j=1}^{n} \overline{\lambda}_{n+1}\lambda_j\psi(g_j) \\
&\quad + \sum_{i=1}^{n} \overline{\lambda}_i\lambda_{n+1}\psi(\alpha(g_i^{-1})) + \sum_{i,j=1}^{n} \overline{\lambda}_i\lambda_j\psi(\alpha(g_i^{-1})g_j),
\end{aligned}
$$

such that

$$
\begin{aligned}
0 &\leq |\lambda_{n+1}|^2\psi(e) \\
&\leq -\sum_{j=1}^{n} \overline{\lambda}_{n+1}\lambda_j\psi(g_j) - \sum_{i=1}^{n} \overline{\lambda}_i\lambda_{n+1}\psi(\alpha(g_i^{-1}))
\end{aligned}
$$

$$- \sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j \psi(\alpha(g_i^{-1}) g_j)$$
$$= \sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j \big(\psi(g_j) + \psi(\alpha(g_i^{-1})) - \psi(\alpha(g_i^{-1}) g_j)\big).$$

This completes the proof. □

Proposition 3.4.

(1) *If $\phi : \Gamma \to \mathbb{C}$ is α-positive definite, then the function ψ defined by $\psi(g) = \phi(e) - \phi(g)$ is α-negative definite.*

(2) *If $\psi : \Gamma \to \mathbb{C}$ is α-negative definite, the function Ψ given by $\Psi(g) = \psi(g) - \psi(e)$ is also α-negative definite.*

Proof.

(1) If ψ is defined by $\psi(g) = \phi(e) - \phi(g)$, then we easily see that $\psi(e) = 0$ and $\overline{\psi(g)} = \psi(\alpha(g^{-1}))$. Taking elements $g_1, \ldots, g_n \in \Gamma$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ with $\sum_{i=1}^{n} \lambda_i = 0$, we have that

$$\sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j \psi(\alpha(g_i^{-1}) g_j) = \sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j (\phi(e) - \phi(\alpha(g_i^{-1}) g_j))$$
$$= - \sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j \phi(\alpha(g_i^{-1}) g_j) \leq 0,$$

so that ψ is α-negative definite.

(2) Let Ψ be defined by $\Psi(g) = \psi(g) - \psi(e)$ for every $g \in \Gamma$. Then, we have that $\Psi(e) = \psi(e) - \psi(e) = 0$ and $\overline{\Psi(g)} = \overline{\psi(g)} - \overline{\psi(e)} = \psi(\alpha(g^{-1})) - \psi(e) = \Psi(\alpha(g^{-1}))$. Let $g_1, \ldots, g_n \in \Gamma$, and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ with $\sum_i \lambda_i = 0$. Then, we have that

$$\sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j \Psi(\alpha(g_i^{-1}) g_j) = \sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j (\psi(\alpha(g_i^{-1}) g_j) - \psi(e))$$
$$= \sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j \psi(\alpha(g_i^{-1}) g_j) \leq 0,$$

which implies that Ψ is α-negative definite. □

Schoenberg's fundamental result to positive/negative definite functions states that a complex-valued function ψ on a group is negative definite if and only if $\exp(-t\psi)$ is positive definite for each $t > 0$. However, in general, this fundamental result does not hold for an α-negative definite function due to the boundedness condition of an α-positive definite function. We need the boundedness condition of an α-negative definite function, which corresponds to (iii) in Definition 2.1.

Let $\psi : \Gamma \to \mathbb{C}$ be α-negative definite. We say that ψ is *matricially bounded* if, for any $g \in \Gamma$, there exists a constant $C(g) > 0$ such that, for all $g_1, \dots, g_n \in \Gamma$,

$$[\overline{\psi(gg_i)}+\psi(gg_j)-\psi(\alpha(gg_i^{-1})gg_j)]_{i,j} \leq C(g)[\overline{\psi(g_i)}+\psi(g_j)-\psi(\alpha(g_i^{-1})g_j)]_{i,j},$$

where $[\,\cdot\,]_{i,j}$ denotes an $n \times n$-matrix over $\mathbb{C}$.

The next theorem shows that, for α-positive definiteness of $\exp(-t\psi)$, it is sufficient that an α-negative definite function ψ be matricially bounded.

Theorem 3.5. *If $\psi : \Gamma \to \mathbb{C}$ is α-negative definite and matricially bounded, then the following properties hold true*:

(i) $\psi(\alpha(g_1g_2)) = \psi(\alpha(g_1)\alpha(g_2)) = \psi(g_1g_2)$ *for all* $g_1, g_2 \in \Gamma$;

(ii) $\psi(e) \geq 0$, $\overline{\psi(g)} = \psi(\alpha(g^{-1}))$ *for all* $g \in \Gamma$;

(iii) *the function* $g \in \Gamma \mapsto \exp(-t\psi(g))$ *is α-positive definite for all* $t > 0$.

Proof. Suppose that ψ is α-negative definite. Obviously, properties (i) and (ii) hold true. For any $g_1, \dots, g_n \in \Gamma$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, the $n \times n$ matrix $[\overline{\psi(g_i)} + \psi(g_j) - \psi(\alpha(g_i^{-1})g_j)]_{i,j}$ is positive semi-definite, so that the matrix

$$[\exp(\overline{\psi(g_i)} + \psi(g_j) - \psi(\alpha(g_i^{-1})g_j))]_{i,j}$$

is positive semi-definite. Thus, we have that

$$\begin{aligned}&\sum_{i,j=1}^{n} \overline{\lambda_i}\lambda_j \exp(-\psi(\alpha(g_i^{-1})g_j))\\ &\quad= \sum_{i,j=1}^{n} \overline{\lambda_i'}\lambda_j' \exp(\overline{\psi(g_i)} + \psi(g_j) - \psi(\alpha(g_i^{-1})g_j)) \geq 0\end{aligned}$$

where $\lambda_i' = \lambda_i \cdot \exp(-\psi(g_i))$ for $i = 1, \dots, n$. Moreover, for all $g, g_1, \dots, g_n \in \Gamma$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, we have that

$$\sum_{i,j=1}^{n} \overline{\lambda_i}\lambda_j \exp(-\psi(\alpha(gg_i^{-1})gg_j))$$
$$= \sum_{i,j=1}^{n} \overline{\lambda_i''}\lambda_j'' \exp(\overline{\psi(gg_i)} + \psi(gg_j) - \psi(\alpha(gg_i^{-1})gg_j))$$
$$\leq C''(g) \sum_{i,j=1}^{n} \overline{\lambda}_i\lambda_j \exp(-\psi(\alpha(g_i^{-1})g_j))$$

where

$$\lambda_i'' = \lambda_i \cdot \exp\bigl(-\psi(gg_i)\bigr),$$
$$C'(g) = \exp C(g) \quad \text{and} \quad C''(g) = \exp\{3C(g)\},$$

so that the function $\exp(-\psi)$ is α-positive definite. Furthermore, since $t\psi$ $(t > 0)$ is α-negative definite and matricially bounded, $\exp(-t\psi)$ is α-positive definite. □

4. J-cocyles associated to α-negative definite functions. We recall that a continuous real-valued negative definite function on $\mathbb{R}^n$ is uniquely given by its Lévy-Khintchine representation **[17]**. An α-negative definite function $\psi : \Gamma \to \mathbb{C}$ is called *normalized* if $\psi(\alpha(g^{-1})g) = 0$ for all $g \in \Gamma$. In this section, we prove the main theorem, which is a realization of a nonnegative normalized α-negative definite function using a J-cocycle on a Krein space.

Theorem 4.1. *If $\psi : \Gamma \to [0, \infty)$ is normalized and α-negative definite, then there exist a Krein space $(\mathcal{K}, J)$ and a map $c : \Gamma \to \mathcal{K}$ such that*

(i) $\psi(\alpha(g^{-1})h) = \|c(g) - c(h)\|^2$ *for all $g, h \in \Gamma$;*

(ii) $Jc(g) = c(\alpha(g)) = c(g)$ *for all $g \in \Gamma$.*

Proof. Let K be the set given by

$$K = \Bigl\{ f : \Gamma \longrightarrow \mathbb{C} \Bigm| \sum_{g \in \Gamma} f(g) = 0 \text{ and } \#\{g \in \Gamma | f(g) \neq 0\} < \infty \Bigr\}$$

where $\#S$ denotes the number of elements in a set S. Then, we see $K \neq \emptyset$ since the zero function is in K. For any $f_1, f_2 \in K$, we define a sesquilinear form

$$\langle f_1, f_2 \rangle = -\frac{1}{2} \sum_{g,h\in\Gamma} \overline{f_2(g)} f_1(h) \psi(\alpha(g^{-1})h). \tag{4.1}$$

Then, we see that

$$\begin{aligned}
\overline{\langle f_1, f_2 \rangle} &= -\frac{1}{2} \sum_{g,h\in\Gamma} f_2(g) \overline{f_1(h)}\ \overline{\psi(\alpha(g^{-1})h)} \\
&= -\frac{1}{2} \sum_{g,h\in\Gamma} \overline{f_1(h)} f_2(g) \psi(\alpha(h^{-1})g) \\
&= \langle f_2, f_1 \rangle.
\end{aligned}$$

Let the support of $f \in K$ be $\mathrm{supp}(f) = \{g \in \Gamma : f(g) \neq 0\}$. For any $f \in K$ with $\mathrm{supp}(f) = \{g_1, \ldots, g_n\}$, we have that

$$\langle f, f \rangle = -\frac{1}{2} \sum_{i,j=1}^{n} \overline{f(g_j)} f(g_i) \psi(\alpha(g_j^{-1})g_i) \geq 0.$$

We equip the quotient space K/N_ψ with the inner product $\langle f_1 + N_\psi, f_2 + N_\psi \rangle = \langle f_1, f_2 \rangle$ where $N_\psi = \{f \in K : \langle f, f \rangle = 0\}$ and obtain the Hilbert space $\mathcal{K}$ by completing the quotient space K/N_ψ with respect to the induced norm.

The map $J : K \to K$ is defined by $Jf(g) = f(\alpha(g))$, $g \in \Gamma$. For any $f \in K$, we have that

$$\begin{aligned}
\langle Jf, Jf \rangle &= -\frac{1}{2} \sum_{g,h\in\Gamma} \overline{f(\alpha(g))} f(\alpha(h)) \psi(\alpha(g^{-1})h) \\
&= -\frac{1}{2} \sum_{g,h\in\Gamma} \overline{f(g)} f(h) \psi(\alpha(g^{-1})h) = \langle f, f \rangle,
\end{aligned}$$

which implies that $J(N_\psi) \subseteq N_\psi$. Hence, J induces a map, still denoted by J, on K/N_ψ by $J(f + N_\psi) = Jf + N_\psi$. We define an indefinite inner product $[\cdot, \cdot]$ on K/N_ψ by

$$[f_1 + N_\psi, f_2 + N_\psi] = -\frac{1}{2} \sum_{g,h\in\Gamma} \overline{f_2(g)} f_1(h) \psi(g^{-1}h).$$

Then, we have that

$$\begin{aligned}[f_1 + N_\psi, f_2 + N_\psi] &= -\frac{1}{2} \sum_{g,h\in\Gamma} \overline{Jf_2(g)} f_1(h)\psi(\alpha(g^{-1})h) \\ &= \langle f_1 + N_\psi, J(f_2 + N_\psi)\rangle.\end{aligned}$$

Moreover, J can be extended to the entire space $\mathcal{K}$ where $\mathcal{K}$ is the completion of $\overline{K/N_\psi}$, and we have $[\xi, \eta] = \langle \xi, J\eta\rangle$ for all $\xi, \eta \in \mathcal{K}$, so that $(\mathcal{K}, J)$ becomes a Krein space.

We define a map $c : \Gamma \to \mathcal{K}$ by $c(g) = \delta_g - \delta_e + N_\psi$ where δ denotes the Dirac function. Since ψ is real valued, we have that, for any $g, h \in \Gamma$,

$$\psi(\alpha(g^{-1})h) = \overline{\psi(\alpha(g^{-1})h)} = \psi(\alpha(h^{-1})g). \tag{4.2}$$

We claim that $\psi(\alpha(g^{-1})h) = \|c(g) - c(h)\|^2$ for any $g, h \in \Gamma$. Indeed, we have that

$$\begin{aligned}&\|c(g) - c(h)\|^2 \\ &\quad = -\frac{1}{2}[\psi(\alpha(g^{-1})g) - \psi(\alpha(g^{-1})h) - \psi(\alpha(h^{-1})g) + \psi(\alpha(h^{-1})h)] \\ &\quad = \psi(\alpha(g^{-1})h).\end{aligned}$$

Moreover, we have that $\|c(\alpha(g)) - c(g)\|^2 = \psi(\alpha(\alpha(g)^{-1})g) = \psi(e) = 0$ so that $Jc(g) = c(\alpha(g)) = c(g)$ for all $g \in \Gamma$. $\square$

Let $(\mathcal{K}, J)$ be a Krein space, and let $\pi : \Gamma \to \mathcal{B}(\mathcal{K})$ be a J-unitary representation of Γ on $\mathcal{K}$. A function $c : \Gamma \to \mathcal{K}$ is called a *J-cocycle for π* if

$$c(gh) = \pi(g)(c(h)) + c(g) \quad \text{and} \quad Jc(g) = c(\alpha(g))$$

for all $g, h \in \Gamma$.

Proposition 4.2. *Let $(\mathcal{K}, J)$ be a Krein space, and let π be a J-unitary representation of Γ on $\mathcal{K}$. If $c : \Gamma \to \mathcal{K}$ is a J-cocycle for π and if $c(e)$ is an invariant vector under π, i.e., $\pi(g)c(e) = c(e)$ for all $g \in \Gamma$, then $\psi : \Gamma \to [0, \infty)$ defined by $\psi(g) = \|c(g) - c(e)\|^2$ satisfies*

$$\psi(\alpha(g^{-1})h) = \|c(g) - c(h)\|^2 \quad \textit{for all } g, h \in \Gamma,$$

and is normalized and α-negative definite.

Proof. For any $g, h \in \Gamma$, we have that

$$\begin{aligned}\psi(\alpha(g^{-1})h) &= \|\pi(\alpha(g^{-1}))c(h) + c(\alpha(g^{-1})) - c(e)\|^2 \\ &= \|c(h) + c(\alpha(g)\alpha(g^{-1})) - c(\alpha(g)) - c(e)\|^2 \\ &= \|c(g) - c(h)\|^2,\end{aligned}$$

which implies that $\psi(\alpha(g^{-1})g) = 0$ for all $g \in \Gamma$ so that ψ is normalized. We also have that

$$\psi(\alpha(g)\alpha(h)) = \|c(g^{-1}) - c(\alpha(h))\|^2 = \|Jc(g^{-1}) - Jc(\alpha(h))\|^2 = \psi(gh).$$

Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ with $\sum_{i=1}^n \lambda_i = 0$. For all $g_1, \dots, g_n \in \Gamma$, we obtain that

$$\begin{aligned}\sum_{i,j=1}^n \overline{\lambda}_i \lambda_j \psi(\alpha(g_i^{-1})g_j) &= \sum_{i,j=1}^n \overline{\lambda}_i \lambda_j (-\langle c(g_i), c(g_j)\rangle - \langle c(g_j), c(g_i)\rangle) \\ &= -\left\| \sum_{i=1}^n \lambda_i c(g_i) \right\|^2 - \left\| \sum_{i=1}^n \overline{\lambda}_i c(g_i) \right\|^2 \\ &\le 0.\end{aligned}$$

Moreover, it is easily seen that $\psi(e) = 0$ and $\psi(\alpha(g^{-1})) = \psi(g)$ for all $g \in \Gamma$. $\square$

A function $\psi : \Gamma \to \mathbb{R}^+$ is called *proper* if $\psi(g)$ goes to ∞ as $g \to \infty$. The following proposition gives a characterization of a proper α-negative definite function using a sequence of α-positive definite functions vanishing at infinity. Its proof is similar to the proof of [**1,** Theorem 10] in the case of negative definite functions; therefore, we provide a sketch of the proof for the convenience of the reader.

Proposition 4.3. *If there exists a sequence $\{\phi_n\}$ of α-positive definite functions in $c_0(\Gamma)$ such that $\phi_n(e) = 1$ and $\{\phi_n\}$ goes to 1 pointwisely, then there exists a proper α-negative definite function ψ on Γ. The converse is true if, in addition, $\psi : \Gamma \to \mathbb{R}^+$ is matricially bounded.*

Proof. Suppose that there exists a sequence $\{\phi_n\}$ of α-positive definite functions in $c_0(\Gamma)$ such that $\phi_n(e) = 1$ and $\phi_n(g) \to 1$ pointwise. We write

$$\Gamma = \bigcup_{n=1}^{\infty} K_n$$

where $K_1 \subset K_2 \subset \cdots$ are finite subsets containing e and set $K_0 = \{e\}$. Suppose that we have chosen increasing sequences $\{n_j\}_{j=1}^k$ and $\{i_j\}_{j=0}^k$ of integers such that

(i) $|1 - \phi_{n_j}(g)| < 2^{-j}$ for all $g \in K_{i_{j-1}}$;
(ii) $|\phi_{n_j}(g)| < 2^{-j}$ for all $g \notin K_{i_j}$.

We choose $\phi_{n_{k+1}}$ such that $|\phi_{n_{k+1}}(g) - 1| < 2^{-k-1}$ for all $g \in K_{i_k}$ since $\phi_n \to 1$ uniformly on K_{i_k}. As $\phi_{n_{k+1}}$ vanishes at infinity, we choose i_{k+1} such that

$$|\phi_{n_{k+1}}(g)| < 2^{-k-1} \quad \text{for } g \notin K_{i_{k+1}}.$$

It follows from Proposition 3.4 that the series $\sum_j (1 - \phi_{n_j})$ is α-negative definite. For any $g \in \Gamma \setminus K_{i_k}$, we have that $g \in \Gamma \setminus K_{i_j}$ for all $j \le k$, so that $|\phi_{n_j}(g)| < 1/2$ for all $j \le k$. Since

$$\sum_{j=1}^{\infty} (1 - \phi_{n_j}(g)) \ge \sum_{j=1}^{n} \frac{1}{2} = \frac{n}{2} \longrightarrow \infty,$$

the function $\psi = \sum_j (1 - \phi_{n_j})$ is proper and α-negative definite.

Conversely, assume that

$$\psi : \Gamma \longrightarrow \mathbb{R}^+$$

is proper, matricially bounded and α-negative definite. We may assume that $\psi(e) = 0$. Otherwise, $\psi' = \psi - \psi(e)$ is also proper, matricially bounded and α-negative definite. By Theorem 3.5, $\exp(-t\psi)$ is α-positive definite for each $t > 0$. For each $n \in \mathbb{N}$, we define a function ϕ_n on Γ by

$$\phi_n(g) = \exp\left(- \frac{1}{n} \psi(g) \right).$$

We see that each ϕ_n is α-positive definite, $\phi_n(e) = 1$ and

$$\lim_{n \to \infty} \phi_n(g) = \lim_{n \to \infty} \exp\left(- \frac{1}{n} \psi(g) \right) = 1.$$

Since $\psi(g) \to \infty$ as $g \to \infty$, each $\phi_n(g)$ goes to 0 as $g \to \infty$. $\square$

Let $(\mathcal{K}, J)$ be a Krein space. A map $\sigma : \Gamma \to \mathcal{B}(\mathcal{K})$ is an (α, J)-*affine action* of Γ on $\mathcal{K}$ if there exist a J-unitary representation π of Γ on $\mathcal{K}$ and a cocycle $c : \Gamma \to \mathcal{K}$ for π such that $\sigma(\alpha(g)) = J\sigma(g)J$ and $\sigma(g)\xi = \pi(g)\xi + c(g)$ for all $g \in \Gamma$ and $\xi \in \mathcal{K}$. The following proposition gives the existence of a proper affine action on a Krein space which has a J-cocycle as a translation part.

Proposition 4.4. *If $\psi : \Gamma \to \mathbb{R}^+$ is proper, normalized and α-negative definite, then there is a proper (α, J)-affine action σ of Γ on $\mathcal{K}$.*

Proof. Suppose that $\psi : \Gamma \to \mathbb{R}^+$ is proper, normalized and α-negative definite. By Theorem 4.1, there exist a Krein space $(\mathcal{K}, J)$ and a map $c : \Gamma \to \mathcal{K}$ such that

$$\psi(\alpha(g^{-1})h) = \|c(g) - c(h)\|^2 \quad \text{and} \quad Jc(g) = c(\alpha(g)) = c(g), \qquad \text{for all } g, h \in \Gamma.$$

For each $g \in \Gamma$, we define a linear map

$$\pi(g) : \mathcal{K} \longrightarrow \mathcal{K}$$

by

$$[\pi(g)\xi](h) = \xi(g^{-1}h), \quad h \in \Gamma.$$

It is obvious that $\pi(gh) = \pi(g)\pi(h)$. Let ξ, η be in $\mathcal{K}$, and let $\langle \cdot, \cdot \rangle$ be the inner product constructed in the proof of Theorem 4.1. We have that

$$\begin{aligned}\langle \pi(g)\xi, \eta \rangle &= -\frac{1}{2} \sum_{a,b\in\Gamma} \overline{\xi(a)}\ \eta(b)\psi(\alpha(a^{-1}g^{-1})b) \\ &= -\frac{1}{2} \sum_{a,b\in\Gamma} \overline{\xi(a)}\ [J\pi(g^{-1})J\eta](b)\psi(\alpha(a^{-1})b) \\ &= \langle \xi, J\pi(g^{-1})J\eta \rangle.\end{aligned}$$

Hence, $\pi(g^{-1}) = J\pi(g)^*J = \pi(g)^J$ for all $g \in \Gamma$, so that π is a J-unitary representation of Γ on $\mathcal{K}$. Similarly, we see that $\langle \pi(\alpha(g))\xi, \eta \rangle = \langle \xi, \pi(g^{-1})\eta \rangle$ so that $\pi(\alpha(g)) = J\pi(g)J$.

It may be seen from the construction that $c(g) = \delta_g - \delta_e + N_\psi$, where N_ψ is the kernel space. We claim that c is a cocycle for π, i.e.,

$c(gh) = \pi(g)(c(h)) + c(g)$. Indeed, for any $g' \in \Gamma$, we have that

$$\pi(g)(c(h))(g') + c(g)(g') = (\delta_{gh} - \delta_e)(g') + N_\psi = c(gh)(g').$$

Moreover, the map

$$\sigma : \Gamma \longrightarrow \mathcal{B}(\mathcal{K})$$

defined by $\sigma(g)\xi = \pi(g)\xi + c(g)$ is an (α, J)-affine action on $\mathcal{K}$. Indeed, for any $\xi \in \mathcal{K}$, we have that

$$\sigma(\alpha(g))\xi = J\pi(g)J\xi + Jc(g) = J\sigma(g)J\xi,$$

so that $\sigma(\alpha(g)) = J\sigma(g)J$ for all $g \in \Gamma$. Since

$$\|c(g)\|^2 = \psi(g) \longrightarrow \infty \quad \text{as } g \to \infty,$$

the action σ is proper. $\square$

Recall that the Haagerup approximation property of Γ **[6]** is equivalent to the existence of a proper affine isometric action of Γ on a Hilbert space. We may ask whether the existence of a proper (α, J)-affine isometric action of Γ is related to some approximation property.

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