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RECURRENCE RELATION FOR COMPUTING A BIPARTITION FUNCTION

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ABSTRACT. Recently, Merca **[4]** found the recurrence relation for computing the partition function $p(n)$ which requires only the values of $p(k)$ for $k \leq n/2$. In this article, we find the recurrence relation to compute the bipartition function $p_{-2}(n)$ which requires only the values of $p_{-2}(k)$ for $k \leq n/2$. In addition, we also find recurrences for $p(n)$ and $q(n)$ (number of partitions of n into distinct parts), relations connecting $p(n)$ and $q_0(n)$ (number of partitions of n into distinct odd parts).

1. Introduction. A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n, $p_{-2}(n)$ denote the number of bipartitions of n, $q(n)$ denote the number of partitions of n into distinct parts and $q_o(n)$ denote the number of partitions of n into distinct odd parts. Throughout the paper, we set

$$p(0) = p_{-2}(0) = q(0) = q_o(0) = 1$$

and

$$p(x) = p_{-2}(x) = q(x) = q_o(x) = 0 \quad \text{if } x < 0.$$

The generating functions for $p(n)$, $p_{-2}(n)$, $q(n)$ and $q_o(n)$ are

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} p_{-2}(n)q^n = \frac{1}{(q;q)_\infty^2}, \tag{1.2}$$

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$$\sum_{n=0}^{\infty} q(n)q^n = \frac{1}{(q;q^2)_\infty} = (-q;q)_\infty, \tag{1.3}$$

and

$$\sum_{n=0}^{\infty} q_o(n)q^n = (-q;q^2)_\infty, \tag{1.4}$$

where $|q| < 1$ and $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$ is the q-shifted factorial.

Euler **[2]** invented generating function (1.1) which gives rise to a recurrence relation for $p(n)$,

$$\sum_{k=-\infty}^{\infty} (-1)^k p\left(n - \frac{k(3k-1)}{2}\right) = \delta_{0,n}, \tag{1.5}$$

where $\delta_{i,j}$ is the Kronecker delta. To compute partition function $p(n)$ using (1.5) requires the values of $p(k)$ with $k \le n-1$. Numerous mathematicians have given other recurrence relations for the partition function $p(n)$. In 2004, Ewell **[3]** found two recurrence relations for $p(n)$:

$$p(n) = \sum_{k=0}^{\infty} p\left(\frac{n - k(k+1)/2}{4}\right) + 2\sum_{k=1}^{\infty}(-1)^{k-1}p(n-2k^2) \tag{1.6}$$

and

$$\begin{aligned} p(n) &= \sum_{k=0}^{\infty} p\left(\frac{n-k(k+1)/2}{2}\right) \\ &\quad + \sum_{k=1}^{\infty}(-1)^{k-1}\{p(n-k(3k-1)) + p(n-k(3k+1))\}, \end{aligned} \tag{1.7}$$

which requires the values of $p(k)$ with $k \le n-2$ to compute $p(n)$. Over the years, it has been a challenge for mathematicians to find the recurrence relation for $p(n)$ that requires less number of values of $p(k)$ with $k < n$. In 2016, using Ramanujan's theta function, Merca **[4]** found the most efficient recurrence relation

$$p(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{j=-\infty}^{\infty} p(k)p\left(\left\lfloor \frac{n}{2} \right\rfloor - k - j(4j-2+(-1)^n)\right), \tag{1.8}$$

which requires only the values of $p(k)$ with $k \le n/2$ to compute $p(n)$.

Inspired by their relations, in this paper, we find the recurrence relation for bipartition function $p_{-2}(n)$ that requires only the values of $p_{-2}(k)$ with $k \leq n/2$. In addition, we also find recurrences for $p(n)$ and $q(n)$, the relation connecting $p(n)$ and $q_o(n)$.

Ramanujan's theta functions and Jacobi's identity play a key role in proving our main results. For $|q| < 1$, Ramanujan's theta functions [**1,** page 36, entry 22] are defined as

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \sum_{n=-\infty}^{\infty} q^{n(2n+1)} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \tag{1.9}$$

and

$$f(-q) = (q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}. \tag{1.10}$$

Lemma 1.1 (Jacobi's identity)**.** [**1,** page 39, entry 24]*. We have*

$$(q;q)_\infty^3 = \sum_{m=0}^{\infty} (-1)^m (2m+1) q^{m(m+1)/2}. \tag{1.11}$$

Our main result is stated in the next theorem.

Theorem 1.2. *For each integer $n \geq 0$,*

$$\begin{aligned} p_{-2}(n) = \sum_{i,j=-\infty}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} & p_{-2}(k) \\ & \times p_{-2}\left(\left\lfloor \frac{n}{2} \right\rfloor - k - j(4j-1) - i(4i-2+(-1)^n)\right) \\ & + \sum_{i,j=-\infty}^{\infty} \sum_{k=0}^{\lfloor n-1/2 \rfloor} p_{-2}(k) \\ & \times p_{-2}\left(\left\lfloor \frac{n-1}{2} \right\rfloor - k - j(4j-3) - i(4i-2-(-1)^n)\right). \end{aligned} \tag{1.12}$$

More explicitly, the above result may be written as

(1.13)

$$p_{-2}(2n) = \sum_{i,j=-\infty}^{\infty} \sum_{k=0}^{n} p_{-2}(k)p_{-2}(n-k-j(4j-1)-i(4i-1)) + \sum_{i,j=-\infty}^{\infty} \sum_{k=0}^{n-1} p_{-2}(k)p_{-2}(n-1-k-j(4j-3)-i(4i-3))$$

and

(1.14)

$$p_{-2}(2n+1) = 2\sum_{i,j=-\infty}^{\infty} \sum_{k=0}^{n} p_{-2}(k)p_{-2}(n-k-j(4j-3)-i(4i-1)).$$

Example 1.3. We see by Theorem 1.2 that the values of $p_{-2}(n)$ for $n \in \{0,1,2,3,4,5,6,7\}$ are:

$$\begin{aligned}
p_0 &= 1, \\
p_1 &= 2p_0^2 = 2, \\
p_2 &= p_0(p_0+2p_1) = 5, \\
p_3 &= 2p_0(p_0+2p_1) = 10, \\
p_4 &= 2p_0(p_0+p_1+p_2)+p_1^2 = 20, \\
p_5 &= 4p_0(p_1+p_2)+2p_1^2 = 36, \\
p_6 &= p_o(3p_0^2+4p_1+2p_2+2p_3)+p_1(p_1+2p_2) = 65, \\
p_7 &= 2p_0(p_0+2p_2+2p_3)+2p_1(p_1+2p_2) = 110,
\end{aligned}$$

where here, and throughout this example, we set $p_{-2}(n) = p_n$. With the above values in hand, we can compute the values of $p_{-2}(14)$ and $p_{-2}(15)$, i.e.,

$$\begin{aligned}
p_{-2}(14) = 2p_0(p_1+2p_2+3p_4+2p_5+p_6+p_7) \\
+ 2p_1(p_1+3p_3+2p_4+p_5+p_6) \\
+ p_2(3p_2+4p_3+2p_4+2p_5)+p_3(p_3+2p_4) = 2665
\end{aligned}$$

and

$$\begin{aligned}
p_{-2}(15) = 2p_0(p_0+2p_1+2p_2+2p_3+2p_4+2p_6+2p_7) \\
+ 2p_1(p_1+2p_2+2p_3+2p_5+2p_6) \\
+ 2p_2(p_2+2p_4+2p_5)+2p_3(p_3+2p_4) = 3956.
\end{aligned}$$

2. Proof of Theorem 1.2. We write

$$\frac{1}{(q;q)_\infty} = \frac{1}{(q;q)_\infty} = \frac{1}{(q^2;q^2)_\infty^2} \times \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{2.1}$$

Substituting (1.9) into (2.1), we obtain

$$\frac{1}{(q;q)_\infty} = \frac{1}{(q^2;q^2)_\infty^2} \sum_{k=-\infty}^{\infty} q^{k(2k+1)}. \tag{2.2}$$

Replacing q by $-q$ in equation (2.2), we find that

$$\frac{1}{(-q;-q)_\infty} = \frac{1}{(q^2;q^2)_\infty^2} \sum_{k=-\infty}^{\infty} (-1)^k q^{k(2k+1)}. \tag{2.3}$$

Therefore, we can write

$$\begin{aligned} \sum_{n=0}^{\infty} p(2n) q^{2n} &= \frac{1}{2}\left(\frac{1}{(q;q)_\infty} + \frac{1}{(-q;-q)_\infty} \right) \\ &= \frac{1}{2(q^2;q^2)_\infty^2} \sum_{k=-\infty}^{\infty} (1+(-1)^k) q^{k(2k+1)} \\ &= \frac{1}{(q^2;q^2)_\infty^2} \sum_{k=-\infty}^{\infty} q^{2k(4k+1)} \\ &= \sum_{n=0}^{\infty} p_{-2}(n) q^{2n} \sum_{k=-\infty}^{\infty} q^{2k(4k-1)}, \end{aligned} \tag{2.4}$$

which is equivalent to

$$\sum_{n=0}^{\infty} p(2n) q^{n} = \sum_{n=0}^{\infty} p_{-2}(n) q^{n} \sum_{k=-\infty}^{\infty} q^{k(4k-1)}. \tag{2.5}$$

Using the Cauchy product of two power series, we find that

$$\sum_{n=0}^{\infty} p(2n) q^{n} = \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} p_{-2}(n-k(4k-1)) q^{n}. \tag{2.6}$$

Equating coefficients of q^n, we obtain

$$p(2n) = \sum_{k=-\infty}^{\infty} p_{-2}(n-k(4k-1)). \tag{2.7}$$

In a similar fashion, considering

$$\sum_{n=0}^{\infty} p(2n+1)q^{2n+1} = \frac{1}{2}\left(\frac{1}{(q;q)_\infty} - \frac{1}{(-q;-q)_\infty}\right), \tag{2.8}$$

we derive the following expression of $p(2n+1)$ in terms of $p_{-2}(n)$:

$$p(2n+1) = \sum_{k=-\infty}^{\infty} p_{-2}(n-k(4k-3)). \tag{2.9}$$

Now, we consider

$$\frac{1}{(q;q)_\infty^2} = \frac{1}{(q^2;q^2)_\infty^2} \times \frac{1}{(q;q)_\infty} \times \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{2.10}$$

Using (1.1), (1.2), and (1.9) in (2.10), we find that

$$\sum_{n=0}^{\infty} p_{-2}(n)q^n = \sum_{k=0}^{\infty} p_{-2}(k)q^{2k} \sum_{n=0}^{\infty} p(n)q^n \sum_{j=-\infty}^{\infty} q^{j(2j+1)}.$$

Using the Cauchy product of power series, we have

$$\sum_{n=0}^{\infty} p_{-2}(n)q^n = \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\infty} p_{-2}(k)p(n-2k-j(2j+1))q^n. \tag{2.11}$$

Equating coefficients of q^n on both sides of (2.11), we obtain

$$\begin{aligned} p_{-2}(n) &= \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} p_{-2}(k)p(n-2k-j(2j+1)) \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} p_{-2}(k)p(n-2k-j(2j-1)) \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} p_{-2}(k)p(n-2k-2j(4j-1)) \\ &\quad + \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} p_{-2}(k)p(n-2k-2j(4j-3)-1). \end{aligned} \tag{2.12}$$

Replacing n by $2n$ and n by $2n+1$ in (2.12), we find that

$$p_{-2}(2n) = \sum_{j=-\infty}^{\infty}\sum_{k=0}^{n} p_{-2}(k)p(2n-2k-2j(4j-1)) + \sum_{j=-\infty}^{\infty}\sum_{k=0}^{n} p_{-2}(k)p(2n-2k-2j(4j-3)-1) \tag{2.13}$$

and

$$p_{-2}(2n+1) = \sum_{j=-\infty}^{\infty}\sum_{k=0}^{n} p_{-2}(k)p(2n+1-2k-2j(4j-1)) + \sum_{j=-\infty}^{\infty}\sum_{k=0}^{n} p_{-2}(k)p(2n-2k-2j(4j-3)). \tag{2.14}$$

Using (2.7) and (2.9) in (2.13) and (2.14), we arrive at (1.12).

3. New recurrences for $p(n)$ and $q(n)$.

Theorem 3.1. *For each nonnegative integer n, we have*

$$\sum_{k=0}^{\infty}(-1)^{k+n}(2k+1)p(n-k(k+1)) = \begin{cases} (-1)^{\ell+m} & \text{if } n = \ell(3\ell-1)/2 + 2m(3m-1), \\ 0 & \text{otherwise}, \end{cases} \tag{3.1}$$

where ℓ and m are integers.

Theorem 3.2. *For each integer $n \geq 0$, we have*

$$\sum_{k=0}^{\infty}(-1)^k q\left(n - \frac{k(3k-1)}{2}\right) = \begin{cases} (-1)^{\ell} & \text{if } n = \ell(3\ell-1), \\ 0 & \text{otherwise}, \end{cases} \tag{3.2}$$

where ℓ is an integer.

Proof of Theorem 3.1. We have

$$(-q;-q)_\infty = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty (q^4;q^4)_\infty},$$

that is,

$$\frac{(q^2;q^2)_\infty^3}{(-q;-q)_\infty} = (q;q)_\infty (q^4;q^4)_\infty.$$

Using (1.10) and (1.11) in the above equation, we obtain

$$\sum_{n,k=0}^{\infty} (-1)^{k+n}(2k+1)p(n)q^{n+k(k+1)} = \sum_{\ell,m=-\infty}^{\infty} (-1)^{\ell+m} q^{\ell(3\ell-1)/2+2m(3m-1)}. \tag{3.3}$$

Result (3.1) follows from (3.3) by extracting like powers of q. □

Proof of Theorem 3.2. We write

$$(q;q)_\infty = (q;q^2)_\infty (q^2;q^2)_\infty,$$

which is equivalent to

$$\frac{1}{(q;q^2)_\infty}(q;q)_\infty = (q^2;q^2)_\infty. \tag{3.4}$$

Substituting (1.3) and (1.10) into (3.4), we find that

$$\sum_{n=0}^{\infty} q(n)q^n \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2} = \sum_{\ell=-\infty}^{\infty} (-1)^\ell q^{\ell(3\ell-1)}, \tag{3.5}$$

from which the result (3.2) follows. □

4. Relation connecting $p(n)$ and $q_o(n)$.

Theorem 4.1. *For each $n \geq 0$,*

$$\sum_{k=-\infty}^{\infty} p\left(\left\lfloor \frac{n}{2} \right\rfloor - k(12k-3+(-1)^n 2)\right) - \sum_{k=-\infty}^{\infty} p\left(\left\lfloor \frac{n}{2} \right\rfloor - k(12k+14+(-1)^n 3) - 4 - (-1)^n 2\right) = q_o(n). \tag{4.1}$$

Proof. Equation (1.10) can be expressed as

$$(q;q^2)_\infty = \frac{1}{(q^2;q^2)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2}. \tag{4.2}$$

Replacing q by $-q$ in (4.2), we obtain

$$(-q;q^2)_\infty = \frac{1}{(q^2;q^2)_\infty} \sum_{k=-\infty}^{\infty} (-1)^{k(3k+1)/2} q^{k(3k-1)/2}. \tag{4.3}$$

However, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} q_o(2n) q^{2n} &= \frac{(q;q^2)_\infty + (-q;q^2)_\infty}{2} \\
&= \frac{1}{2(q^2;q^2)_\infty} \sum_{k=-\infty}^{\infty} ((-1)^k + (-1)^{k(3k+1)/2}) q^{k(3k-1)/2} \\
&= \frac{1}{(q^2;q^2)_\infty} \left(\sum_{k=-\infty}^{\infty} q^{2k(12k-1)} - \sum_{k=-\infty}^{\infty} q^{2k(12k+17)+12} \right),
\end{aligned}$$

which is equivalent to

$$\sum_{n=0}^{\infty} q_o(2n) q^{n} = \sum_{n=0}^{\infty} p(n) q^n \left(\sum_{k=-\infty}^{\infty} q^{k(12k-1)} - \sum_{k=-\infty}^{\infty} q^{k(12k+17)+6} \right).$$

Using the Cauchy product of two power series, we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} q_o(2n) q^{n} &= \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} p(n-k(12k-1)) q^n \\
&\quad - \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} p(n-k(12k+17)-6) q^n.
\end{aligned} \tag{4.4}$$

Equating coefficients of q^n on both sides of (4.4), we obtain

$$q_o(2n) = \sum_{k=-\infty}^{\infty} p(n-k(12k-1)) - \sum_{k=-\infty}^{\infty} p(n-k(12k+17)-6). \tag{4.5}$$

By taking

$$\sum_{n=0}^{\infty} q_o(2n+1)q^{2n+1} = \frac{(-q;q^2)_\infty - (q;q^2)_\infty}{2},$$

we also find in a similar fashion that

$$q_o(2n+1) = \sum_{k=-\infty}^{\infty} p(n-k(12k-5)) - \sum_{k=-\infty}^{\infty} p(n-k(12k+11)-2). \tag{4.6}$$

Combining (4.5) and (4.6), we arrive at (4.1). □

Example 4.2. If $n = 23$,

$$p(11) - p(8) - p(9) + p(4) = 9,$$

and $q_0(23)$ equals 9 since the nine partitions in question are:

$$\begin{gathered} 23,\ 19+3+1,\ 17+5+1, \\ 15+7+1,\ 15+5+3,\ 13+9+1, \\ 13+7+3,\ 11+9+3,\ 11+7+5. \end{gathered}$$

It would be interesting to find the recurrence relation for a t-tuple partition function denoted by $p_{-t}(n)$, which would lead to a generalization of (1.8) and (1.12).

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