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RATIONALLY CONNECTED VARIETIES OVER THE MAXIMALLY UNRAMIFIED EXTENSION OF p-ADIC FIELDS

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ABSTRACT. A result of Graber, Harris and Starr shows that a rationally connected variety defined over the function field of a curve over the complex numbers always has a rational point. Similarly, a separably rationally connected variety over a finite field or the function field of a curve over any algebraically closed field will have a rational point. Here, we show that rationally connected varieties over the maximally unramified extension of the p-adics usually, in a precise sense, have rational points. This result is in the spirit of Ax and Kochen's result, which states that the p-adics are usually C_2 fields. The method of proof utilizes a construction from mathematical logic called the ultraproduct.

1. Introduction. Let X be a proper, smooth variety over a field K and $\overline{K}$ an algebraic closure of K. A guiding principle in the study of K-rational points on X is given by Kollár [**18,** IV.6.3].

Principle 1.1. *If $\overline{X} = X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$ is rationally connected, then X should have many K-points, at least if K is nice, e.g., K is a finite field, a function field of a curve or a sufficiently large number field.*

The term "nice" has since been replaced by many with the term quasi-algebraically closed. A field K is said to be *quasi-algebraically closed* or C_1 if every homogeneous polynomial over K with degree less

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than the number of variables has a nontrivial solution in K. Some well-known examples are finite fields, function fields in one variable over an algebraically closed field, the field of Laurent series over an algebraically closed field and the maximal unramified extension of p-adic fields $\mathbb{Q}_p^{nr}$.

A homogeneous polynomial in n variables over K defines a hypersurface in projective $(n-1)$-space $\mathbb{P}^{n-1}$. The hypersurface associated to a form with degree less than the number of variables, when smooth, is a Fano variety; thus, it is rationally connected, see **[3, 19]**. Since smooth rationally connected hypersurfaces defined over quasi-algebraically closed fields always have a K-rational point, it is natural to ask:

Question 1.2. [**26,** 1.11]. *Let X be a proper, smooth separably rationally connected variety over a field K where K is a quasi-algebraically closed field. Is $X(K) = \emptyset$?*

Affirmative answers to this question have been given when:

(1) K is the function field of a curve defined over an algebraically closed field of characteristic 0 [**13**].
(2) K is the function field of a curve defined over an algebraically closed field of positive characteristic [**8**].
(3) K is a finite field [**11**].
(4) $K = k((t))$ is the field of Laurent series over an algebraically closed field [**6**].
(5) If X is a smooth, proper, rational *surface* over a quasi-algebraically closed field K, then $X(K) \neq \emptyset$ [**5, 21**].

Colliot-Thélène and Madore have shown that there exist fields K of cohomological dimension 1 and del Pezzo surfaces X_d of degrees $d = 2, 3, 4$ such that $X_d(K) = \emptyset$ [**7**]. In particular, these fields are examples of fields of cohomological dimension 1 which are not C_1. This seems to rule out a cohomological proof that separably rationally connected varieties over C_1 fields have points.

We recall some basic facts about rational connectivity. Suppose that X is a smooth, projective variety defined over an arbitrary field K. We say that X is *separably rationally connected* if there is a variety Y and a morphism $u : Y \times \mathbb{P}^1 \to X$ such that

$$u^{(2)} : (Y \times \mathbb{P}^1) \times (Y \times \mathbb{P}^1) \longrightarrow X \times X$$

is dominant and smooth at the generic point. If the field K is algebraically closed, we can simplify the definition and say that X is *separably rationally connected* if there is a rational curve, called a *very free curve*,

$$f : \mathbb{P}^1 \longrightarrow X$$

such that f^*T_X is ample [**18,** IV.3.7]. Over an arbitrary field K, we say that X is *rationally connected* if there is a family of proper algebraic curves

$$g : U \longrightarrow Y,$$

whose geometric fibers are irreducible rational curves with cycle morphism

$$u : U \longrightarrow X$$

such that $u^{(2)}$ is dominant. If the field K is uncountable and algebraically closed, then we say that X is rationally connected if, for very general closed points $x_1, x_2 \in X$, there is a morphism

$$f : \mathbb{P}^1 \longrightarrow X$$

such that $x_1, x_2 \in f(\mathbb{P}^1)$, [**18,** IV.3.6]. Over any field of characteristic 0, the notions of rationally connected and separably rationally connected are equivalent [**18,** IV.3.3].

The proof that rational connectivity and separable rational connectivity are equivalent over characteristic 0 relies in some way upon generic smoothness, which fails in characteristic p. In fact, in positive characteristic, there are smooth, projective varieties for which a rational curve may be found through any two closed points; however, the variety does not contain any very free curves [**24**]. This may be thought of as rationally connected varieties containing many rigid rational curves, while separably rationally connected varieties have rational curves that freely deform. Another example of a rationally connected but not separably rationally connected variety over a field of positive characteristic is given by Kollár [**18,** V.5.19].

Due to the subtleties between rationally and separably rationally connected over characteristic p, we had to be careful stating Question 1.2. However, in this paper, we consider varieties defined over fields of characteristic 0 such that the terms are interchangeable. The question considered in this article is whether or not a smooth, projective, rationally connected variety over the maximal unramified extension of the p-adics $\mathbb{Q}_p^{nr}$ has a rational point. Lang's theorem asserts that this is true for Fano hypersurfaces [**20**]. Here, we prove a partial result.

Theorem 1.3. *Fix a numerical polynomial P. There is a finite set of exceptional primes $e(P)$, depending only upon P such that, if X is a smooth, projective, rationally connected variety defined over $\mathbb{Q}_p^{nr}$ with Hilbert polynomial P, then $X(\mathbb{Q}_p^{nr}) \neq \emptyset$ when $p \notin e(P)$.*

A polynomial $P(z) \in \mathbb{Q}(z)$ is called a *numerical polynomial* if $P(n)$ is an integer for all sufficiently large integers n.

This theorem is similar to Ax and Kochen's theorem [**2**] that the p-adic number fields are almost C_2. Artin conjectured that the p-adic fields $\mathbb{Q}_p$ are C_2. In general, a C_i field K is one for which any form in $K[x_1, \dots, x_n]_d$ with $n > d^i$ has a nontrivial 0. In [**25**], Terjanian found a counterexample to Artin's conjecture, see for instance, [**23**]. However, using methods of logic, Ax and Kochen were able to show that $\mathbb{Q}_p$ is almost C_2 in the following sense.

Theorem 1.4. [**2**]. *Fix an integer $d > 0$. Then, there exist a finite number of primes $p_0, \dots, p_m$, such that, for all forms $f \in \mathbb{Q}_p[x_1, \dots, x_n]_d$ with $n > d^2$ and $p \neq p_0, \dots, p_m$, f represents 0 over $\mathbb{Q}_p$.*

Their method of proof uses mathematical logic to make precise the analogy that $\mathbb{Q}_p$ is like $\mathbb{F}_p((t))$. Then, using the fact that the field $\mathbb{F}_p((t))$ is C_2 [**14**] is sufficient for Ax and Kochen to conclude the above theorem.

We similarly make an analogy between the asymptotic properties of $\mathbb{Q}_p^{nr}$ when p goes to infinity and the properties of $\mathbb{C}((t))$, the field of Laurent expansions over the complex numbers. Then, we use the fact that every rationally connected variety over $\mathbb{C}((t))$ contains a $\mathbb{C}((t))$-point [**6**].

It should be noted that, in a recent paper, Denef proved Theorem 1.4 using only algebraic geometry [**9**].

2. Model theory and algebraic geometry. The main tool from model theory we will use is the ultraproduct. A more thorough introduction to ultrafilters and ultraproducts is given in [**17**].

Definition 2.1. Let S be a set, and let Σ be a collection of non-empty subsets of S. Then Σ is called a *non-principal filter* if the following hold:

(1) $S_1, S_2 \in \Sigma$ implies $S_1 \cap S_2 \in \Sigma$;
(2) $S_1 \in \Sigma$ and $S_2 \supset S_1$ imply $S_2 \in \Sigma$;
(3) for each $s \in S$, there is a set $S_1 \in \Sigma$ such that $s \notin S_1$.

Σ is called a *non-principal ultrafilter* if it is maximal among the class of all non-principal filters on S, or equivalently,

(i) $S_1 \notin \Sigma$ implies $S - S_1 \in \Sigma$.

Conditions (1), (2) and (i) define an *ultrafilter*.

A simple, but important, property of ultrafilters to keep in mind is that, if S is the disjoint union of subsets $S_1, \ldots, S_n$, then precisely one of these subsets is in Σ. This observation follows from properties (1) and (i), namely, at least one of the S_i is in Σ by property (i). Moreover, two disjoint subsets cannot both be in Σ since then so would their intersection; however, Σ consists only of nonempty subsets of S.

Given any subset $S_0 \subset S$, it will be useful to know whether we can find a non-principal ultrafilter on S containing S_0. Certainly, if S_0 is a finite set, then properties (3) and (i) of Definition 2.1 will prevent us from finding a non-principal ultrafilter containing S_0. However, this is the only obstruction as the next lemma asserts.

Lemma 2.2. *Given any infinite subset $S_0 \subset S$, there exists a non-principal ultrafilter containing S_0.*

Proof. Let Σ consist of all of the subsets of S that contain all but a finite number of points in S_0. It is easy to verify that Σ is a non-principal filter on S containing S_0. The desired non-principal ultrafilter is any maximal filter containing Σ. $\square$

Our usage of ultrafilters will be for an auxiliary construction called the ultraproduct. In particular, given a collection of fields indexed by a set S, and an ultrafilter Σ on S, we construct a new field via the ultraproduct. We use a similar construction for modules.

Definition 2.3. Given an ultrafilter Σ on S and a collection of rings $\{R_i\}_{i\in S}$, we can form a new ring denoted

$$\prod_{i\in S} R_i/\Sigma,$$

defined by componentwise addition and multiplication under the equivalence condition that

$$a, b \in \prod_{i\in S} R_i$$

are equivalent if they agree on a set of indices in Σ. This new ring is called the *ultraproduct* of the R_is with respect to Σ. The same definition may be used for groups, modules, etc.

A nice aspect of ultraproducts is that an ultraproduct of fields is itself a field. Moreover, statements in the language of fields can be transferred between the ultraproduct and its components, which leads to the fundamental property of ultraproducts. Let $\{k_p\}_{p\in S}$ be a collection of fields. Then Łoš's theorem [**12,** 7.7.1] applied to the particular cases of an ultraproduct of fields can be stated as:

Theorem 2.4. *A first-order formula in the language of rings is true in the ultraproduct of fields*

$$\prod_{p\in S} k_p/\Sigma,$$

if and only if the set of indices p such that the formula is true in the field k_p is a member of Σ.

Intuitively, a first-order formula is one that quantifies only over elements of the field, not over subsets, sets of subsets, etc. In order to get a feeling for why Łoš's theorem is true, even for structures more general than fields, consider the next lemma.

Lemma 2.5. *Let N be a positive integer. For each $i \in S$, let M_i be a free module of rank less than N over a ring R_i. Then, an ultraproduct of the M_is is a free module of rank less than N over the corresponding ultraproduct of the R_is.*

Proof. First, assume that the rank of the M_is are all $m > 0$. Then, for any ultrafilter Σ on S,

$$M := \prod_{i \in S} M_i/\Sigma$$

will be a free module of rank m over

$$R := \prod_{i \in S} R_i/\Sigma.$$

In order to see this, let $e_{i1}, \ldots, e_{im}$ be an R_i basis for M_i. Then, note that M has basis $(e_{i1})_{i \in S}, \ldots, (e_{im})_{i \in S}$ over R.

Now, generally, consider the subsets $S_k \in S$ consisting of those $i \in S$ such that the rank of M_i is k. Then, S is the disjoint union of $S_1, \ldots, S_N$. By the remarks on the definition of ultrafilter, there is only one such subset contained in Σ, say $S_m \in \Sigma$. It follows that M has rank m over R. □

The success of Lemma 2.5 is based upon the boundedness of the statement (that the rank is less than N). The similar statement that the ultraproduct of finite rank free modules is of finite rank is actually false (say, if the rank of the free modules keeps increasing). Łoš's theorem does not apply to such a statement since it is not a first-order statement.

Next, we develop some basic algebraic geometry over a general ultraproduct of fields

$$F = \prod_{i \in S} F_i/\Sigma$$

of characteristic 0. Suppose that we are given a scheme X of finite type over F. There is a natural process for obtaining schemes X_i of finite type over F_i, and for almost every $i \in S$, X_i is nicely related to X. However, the X_i are not unique.

We first assume that X is an affine scheme corresponding to the F-algebra

$$F[x_0, \ldots, x_n]/I(X).$$

Suppose that $f_1, \ldots, f_k$ are generators for $I(X)$. We may write each generator as

$$f_j = \sum a_{j,I} x^I, \quad I \in \mathbb{N}^{n+1}.$$

Let

$$(a^i_{j,I})_{i\in S} \in \prod_{i\in S} F_i$$

be a representative for $a_{j,I}$. Setting

$$f^i_j = \sum a^i_{j,I} x^I,$$

we define X_i as the affine scheme associated to the ideal generated by the f^i_j. These schemes are not unique as they depend on the choice of representatives for the $a_{j,I}$.

We can perform a similar construction in reverse, namely, given schemes X_i defined over F_i for each $i \in S$, we can define their ultraproduct

$$X = \prod_{i\in S} X_i/\Sigma$$

by taking the f^i_j and lifting them to $f \in F[x_0, \dots, x_n]$. This new object is not necessarily pretty, for example, when the degrees of the f^i_j are not bounded. However, we will see that, under certain circumstances, the ultraproduct is a scheme of finite type, and other nice properties of the X_is will be inherited by X.

Let $X \subseteq \mathbb{P}^n$ be a projective variety defined over a field F with homogeneous ideal $J(X)$, and let $S(X) = F[x_0, \dots, x_n]/J(X)$ denote its homogeneous coordinate ring. For each integer ℓ, we define the Hilbert function φ_X of X by

$$\varphi_X(\ell) = \dim_F S(X)_\ell.$$

A theorem of Hilbert and Serre ([**15,** I.7.5]) states that there exists a unique numerical polynomial $P(z) \in \mathbb{Q}[z]$ such that $\varphi_X(\ell) = P(\ell)$ for all $l \gg 0$. By definition, the degree of the Hilbert polynomial is the dimension of the variety X. Chardin and Moreno-Socías characterize, in terms of their coefficients, which numerical polynomials are Hilbert polynomials of some projective scheme [**4**].

Lemma 2.6. *Given a collection of projective varieties $X_i \subset \mathbb{P}^n_{F_i}$, all with Hilbert polynomial P, the ultraproduct X is a projective variety in $\mathbb{P}^n_F$ with Hilbert polynomial P.*

Proof. Let $J_i \subset F_i[x_0, \ldots, x_n]$ be the homogeneous ideal of $X_i \subset \mathbb{P}^n_{F_i}$. For each degree $d > 0$, consider the F_i-vector space $J_{i,d}$ of the homogeneous polynomials of degree d in J_i.

Now, define

$$J_d := \prod_{i \in S} J_{i,d}/\Sigma.$$

A property of the Hilbert polynomial ensures that, for sufficiently large d, the rank of $J_{i,d}$ is the same for each $i \in S$, i.e., the Hilbert functions of the X_i are equal for sufficiently large d [**18,** I.1.5]. Then, the proof of Lemma 2.5 shows that, for $d \gg 0$, the rank of J_d equals the rank of $J_{i,d}$. This yields a homogeneous ideal

$$J := \bigoplus_{d>0} J_d \subset F[x_0, \ldots, x_n].$$

The corresponding projective variety X, denoted by

$$X := \prod_{i \in S} X_i/\Sigma,$$

has Hilbert polynomial P. $\square$

Other desirable results on properties of varieties preserved under the ultraproduct may be found in [**1, 22**].

3. Proof of the main theorem. Now that we have established some basic knowledge of ultraproducts, we can prove the main theorem of this paper, Theorem 1.3, in a manner similar to that of Ax and Kochen's proof of Theorem 1.4. First, we must recall a theorem of Ax, Kochen and Eršov [**2, 10**].

Theorem 3.1. [**2, 10**]. *Let K and K' be two Henselian-valued fields of residual characteristic* 0. *Assume that their residue fields k, k' and their value groups Γ, Γ' are elementary equivalent, that is, they have the same set of true sentences in the language of rings, respectively, ordered abelian groups. Then, K and K' are elementary equivalent, that is, they satisfy the same set of formulae in the language of valued fields.*

Proof of Theorem 1.3. Fix a numerical polynomial P. Let $\mathbb{Q}_p^{nr}$ denote the maximally unramified extension of the p-adics, and let S be the set of all primes. Suppose, by way of contradiction, that there is an infinite subset of primes $e(P) \subset S$ such that, for each $p \in e(P)$, there is a smooth, projective, rationally connected variety $X_p \in \mathbb{P}^n$ defined over $\mathbb{Q}_p^{nr}$ having Hilbert polynomial P and $X(\mathbb{Q}_p^{nr}) = \emptyset$. Now, by Lemma 2.2 there is a non-principal ultrafilter Σ containing $e(P)$, and we can define the non-principal ultraproduct

$$K = \prod_{p \in S} \mathbb{Q}_p^{nr} / \Sigma.$$

Both K and $\mathbb{C}((t))$ are Henselian-valued fields of characteristic 0. The residue field of $\mathbb{C}((t))$ is simply $\mathbb{C}$, and thus, algebraically closed of characteristic 0. The residue field of K is a non-principal ultraproduct of the algebraic closures of the finite fields $\mathbb{F}_p$ and is known to be algebraically closed of characteristic 0 [**2,** Lemma 4]. Thus, the residue fields of K and $\mathbb{C}((t))$ are elementary equivalent. Note that the lemma of Ax and Kochen requires the ultrafilter to be non-principal, which is the reason $e(P)$ must be infinite. K has value group a non-principal ultraproduct of the integers, otherwise known as an ultrapower of $\mathbb{Z}$, and hence, is elementary equivalent to $\mathbb{Z}$, the value group of $\mathbb{C}((t))$. Thus, by Theorem 3.1, K and $\mathbb{C}((t))$ are elementary equivalent.

For the numerical polynomial P fixed above, consider all varieties $X \subset \mathbb{P}_k^n$ with Hilbert polynomial P. It is possible to choose a uniform $M \gg 0$ such that the Hilbert function $\varphi_X(M)$ is equal to $P(M)$ for all varieties with Hilbert polynomial P [**16,** 12.47]. Let Gr denote the Grassmannian of codimension-$P(M)$ subspaces of the space of polynomials of degree M in $n+1$ variables:

$$\mathrm{Gr} := \mathrm{Grass}\left(\binom{n+M}{M} - P(M),\, k[x_0, \ldots, x_n]_M\right).$$

Let $J(X)_M \subset k[x_0, \ldots, x_n]_M$ denote the degree M polynomials vanishing on X. Then, $J(X)_M$ defines a point in the Grassmannian Gr, and the set of all projective varieties with Hilbert polynomial P is parameterized by a projective variety known as the Hilbert scheme, $\mathcal{H}ilb_P \subset \mathrm{Gr}$. Let g denote the smooth morphism that injects $\mathcal{H}ilb_P$ into Gr, $g : \mathcal{H}ilb_P \hookrightarrow \mathrm{Gr}$. Note that, since P is a numerical polynomial, everything here is defined over $\mathbb{Q}$. Over any field L of characteristic 0, every projective L-variety X with Hilbert polynomial P is realized as a fiber g_b of g above a point $b = J(X)_M \in \mathrm{Gr}$. Thus, X has an L-point if and only if g_b has an L-point.

If the fiber g_b is rationally connected for some $b \in \mathrm{Gr}$, then there is an open neighborhood $b \in U \subset \mathrm{Gr}$ such that the fiber g_u is rationally connected if $u \in U$ [**18,** IV.3.11]. Thus, over $\mathbb{Q}$, the set Z of points $b \in \mathrm{Gr}$ such that the fiber g_b is rationally connected is an open subset of Gr. Since Gr is a variety and Z is an open subset of Gr, over any field L of characteristic 0, $Z(L)$ is definable in the ring language. Furthermore, that definition is actually the same one given over $\mathbb{Q}$. Also, for any field L of characteristic 0 and any rationally connected variety X over L with Hilbert polynomial P, there is an L-point $z \in Z(L)$ such that $X = g_z$ and

$$X(L) \neq \emptyset \iff g_z(L) \neq \emptyset.$$

Now, let H_L be the set of points $b \in \mathrm{Gr}(L)$ such that the fiber over b is rationally connected and contains an L-point:

$$H_L = \{b \in \mathrm{Gr}(L) : b \in Z(L),\ g_b(L) \neq \emptyset\}.$$

Then, $H_L \subset Z(L)$ is a definable subset of $\mathrm{Gr}(L)$ characterized by the following formula over $\mathbb{Q}$ with an existential quantifier:

$b \in H_L$ if and only if the formula

$$b \in Z \text{ and there exists an } x \in \mathcal{H}ilb_P \text{ such that } g_P(x) = b$$

is true in the field L. This formula does not depend on the field L, but only upon the polynomial P.

The result of Colliot-Thélène [**6**] $\mathbb{C}((t))$ tells us that the formula

$$b \in Z \text{ and there exists an } x \in A_P \text{ such that } g_P(x) = b$$

is true over $\mathbb{C}((t))$. Then, by elementary equivalence, it holds over the ultraproduct K. However, Łoš's theorem tells us that this statement is false over K by the manner in which our non-principal ultrafilter Σ was constructed. Thus, we arrive at our contradiction, and we have shown that the set of primes $e(P) \subset S$ such that, for each $p \in e(P)$, there is a smooth, projective, rationally connected variety $X_p \in \mathbb{P}^n$ defined over $\mathbb{Q}_p^{nr}$ having Hilbert polynomial P and $X(\mathbb{Q}_p^{nr}) = \emptyset$, is finite. □

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