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A NOTE ON SKEW PRODUCT PRESERVING MAPS ON FACTOR VON NEUMANN ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ be a factor von Neumann algebra, with unit I, which contains a nontrivial projection P_1, and let $\psi : \mathcal{A} \to \mathcal{A}$ be a surjective map that satisfies one of the two conditions: $\psi(A)\psi(P) + \lambda\psi(P)\psi(A) = AP + \lambda PA$ and $\psi(A)\psi(P) + \lambda\psi(P)\psi(A)^* = AP + \lambda PA^*$ for all $A \in \mathcal{A}$ and $P \in \{P_1, I - P_1\}$ and $\lambda \in \{-1, 1\}$. Then, we determine the concrete form of ψ.

1. Introduction. Let $\mathcal{R}$ be a *-ring. The Jordan product, Lie product, *-Jordan product and *-Lie product of $A, B \in \mathcal{R}$ are defined as $A \circ B = AB + BA$, $[A, B] = AB - BA$, $A \bullet B = AB + BA^*$ and $[A, B]_* = AB - BA^*$, respectively. These products play an important role in different fields of research. The additive map

$$\psi : \mathcal{R} \longrightarrow \mathcal{R},$$

defined by $\psi(A) = AB - BA^*$ for all $A, B \in \mathcal{R}$, is a Jordan *- derivation, that is, it satisfies $\psi(A^2) = \psi(A)A^* + A\psi(A)$. The notion of Jordan *-derivations arose naturally in Šemrls' work **[7, 8]**, where he investigated the problem of representing quadratic functionals with sesquilinear functionals. Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ all of the bounded linear operators on $\mathcal{H}$. Motivated by the theory of rings (and algebras) equipped with a Lie product or a Jordan product, Molnar **[5]** studied the Lie product and gave a characterization of ideals of $\mathcal{B}(\mathcal{H})$ in terms of the Lie product. It is shown **[5]** that, if $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ is an ideal, then

$$\begin{aligned}\mathcal{N} &= \operatorname{span}\{AB - BA^* : A \in \mathcal{N}, B \in \mathcal{B}(\mathcal{H})\}\\ &= \operatorname{span}\{AB - BA^* : A \in \mathcal{B}(\mathcal{H}), B \in \mathcal{N}\}.\end{aligned}$$

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In particular, every operator in $\mathcal{B}(\mathcal{H})$ is a finite sum of $AB - BA^*$ type operators. Later, Beršar and Fsoňer [1] generalized the above results [5] to rings using different methods of involution. Let $\mathcal{A}$ be a factor von Neumann algebra and

$$\phi : \mathcal{A} \longrightarrow \mathcal{A}$$

the *-Jordan derivation on A. Then, in [11], we showed that ϕ is an additive *-derivation.

Recall that a map

$$\psi : \mathcal{R} \longrightarrow \mathcal{R}$$

is skew commutativity preserving if, for any $A, B \in \mathcal{R}$, $[A, B]_* = 0$ implies $[\psi(A), \psi(B)]_* = 0$. The problem of characterizing linear (or additive) bijective maps preserving skew commutativity has been studied intensively in various algebras (see [2, 3] and the references therein). More specifically, we say that a map

$$\psi : \mathcal{R} \longrightarrow \mathcal{R}$$

is strong skew commutativity preserving if $[\psi(A), \psi(B)]_* = [A, B]_*$ for all $A, B \in \mathcal{R}$. These maps are also called strong skew Lie product preserving maps in [4]. In [4], Cui and Park proved that, if $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a factor von Neumann algebra, then every strong skew commutativity preserving map ψ on $\mathcal{A}$ has the form

$$\psi(A) = \phi(A) + h(A)I \quad \text{for every } A \in \mathcal{A},$$

where $\phi : \mathcal{A} \to \mathcal{A}$ is a linear bijective map satisfying $[\phi(A), \phi(B)]_* = [A, B]_*$ for $A, B \in \mathcal{A}$ and h is a real functional on $\mathcal{A}$ with $h(0) = 0$. In particular, if $\mathcal{A}$ is a type I factor, then $\psi(A) = cA + h(A)I$ for every $A \in \mathcal{A}$, where $c \in \{-1, 1\}$. In addition, Qi and Hou [6] proved that, if $\mathcal{M}$ is a von Neumann algebra with no central summands of type I_1, then a surjective map

$$\Phi : \mathcal{M} \longrightarrow \mathcal{M}$$

satisfies

$$\Phi(A)\Phi(B) - \Phi(B)\Phi(A)^* = AB - BA^*$$

for all $A, B \in \mathcal{M}$ if and only if there exists a self-adjoint element Z in the center of $\mathcal{M}$ with $Z^2 = I$ such that $\Phi(A) = ZA$ for all $A \in \mathcal{M}$.

In [**9**], we investigated the *-additivity of

$$\psi : \mathcal{A} \longrightarrow \mathcal{B},$$

where $\mathcal{A}$ and $\mathcal{B}$ are two prime C^*-algebras and $\mathcal{A}$ contains a nontrivial projection P_1. We showed that, if ψ is a unital and bijective map and satisfies

$$\psi(AP + \lambda PA^*) = \psi(A)\psi(P) + \lambda\psi(P)\psi(A)^*$$

for all $A \in \mathcal{A}$, $P \in \{P_1, I - P_1\}$ and $\lambda \in \{-1, 1\}$, then ψ is a *-additive map, where $\mathcal{A}$ and $\mathcal{B}$ are two C^*-algebras such that $\mathcal{B}$ is prime. In [**10**], we investigated the additivity of map

$$\Phi : \mathcal{A} \longrightarrow \mathcal{B},$$

which is bijective, unital and satisfies

$$\Phi(AP + \eta PA^*) = \Phi(A)\Phi(P) + \eta\Phi(P)\Phi(A)^*$$

for all $A \in \mathcal{A}$ and $P \in \{P_1, I_{\mathcal{A}} - P_1\}$, where P_1 is a nontrivial projection in $\mathcal{A}$ and η is a non-zero complex number such that $|\eta| \neq 1$.

In this paper, we distinguish the concrete form of two types of strong skew-preserving maps on von Neumann algebras. Let $\mathcal{A}$ be a factor von Neumann algebra (with identity I) that contains a nontrivial projection P_1, and let $\psi : \mathcal{A} \to \mathcal{A}$ be a map. First, if ψ is surjective and satisfies the condition

$$\psi(A)\psi(P) + \lambda\psi(P)\psi(A) = AP + \lambda PA$$

for all $A \in \mathcal{A}$, $P \in \{P_1, I - P_1\}$ and $\lambda \in \{-1, 1\}$, then we will show that $\psi(T) = \alpha T$ for $\alpha \in \{-1, 1\}$ and for all $T \in \mathcal{A}$. Also, if $\mathcal{A}$ is a von Neumann algebra and $\psi : \mathcal{A} \to \mathcal{A}$ is not necessarily a surjective map satisfying the condition

$$\psi(A)\psi(P) + \lambda\psi(P)\psi(A)^* = AP + \lambda PA^*$$

for all $A \in \mathcal{A}$, $P \in \{P_1, I - P_1\}$ and $\lambda \in \{-1, 1\}$, then we will show that there exists a $Z \in \mathcal{A}$ with $Z^2 = I$ such that $\psi(A) = AZ$ for all $A \in \mathcal{A}$. Note that a subalgebra $\mathcal{A}$ from $\mathcal{B}(\mathcal{H})$ is called *von Neumann algebra* when it is closed in the weak topology of operators. A von Neumann algebra $\mathcal{A}$ is called a *factor* when its center is trivial, i.e., $\mathcal{Z}(\mathcal{A}) = \mathbb{C}I$. It is clear that, if $\mathcal{A}$ is a factor von Neumann algebra, then $\mathcal{A}$ is prime, that is, if $A\mathcal{A}B = \{0\}$, for $A, B \in \mathcal{A}$, then $A = 0$ or $B = 0$.

2. Statement of the main theorem. The statement of our main theorems follow.

Theorem 2.1. *Let $\mathcal{A}$ be a factor von Neumann algebra, with identity I, that contains a nontrivial projection P_1, and let $\psi : \mathcal{A} \to \mathcal{A}$ be a surjective map which satisfies*

$$\psi(A)\psi(P) + \lambda\psi(P)\psi(A) = AP + \lambda PA$$

for all $A \in \mathcal{A}$, $P \in \{P_1, I - P_1\}$ and $\lambda \in \{-1, 1\}$. Then, $\psi(T) = \alpha T$ for all $T \in \mathcal{A}$, where $\alpha \in \{-1, 1\}$.

Theorem 2.2. *Let $\mathcal{A}$ be a von Neumann algebra, with identity I, that contains a nontrivial projection P_1, and let*

$$\psi : \mathcal{A} \longrightarrow \mathcal{A}$$

be a map which satisfies

$$\psi(A)\psi(P) + \lambda\psi(P)\psi(A)^* = AP + \lambda PA^*$$

for all $A \in \mathcal{A}$, $P \in \{P_1, I - P_1\}$ and $\lambda \in \{-1, 1\}$. Then, there exists a $Z \in \mathcal{A}$ with $Z^2 = I$ such that $\psi(A) = AZ$ for all $A \in \mathcal{A}$.

For the above-determined P_1, let $P_2 = I - P_1$. By taking $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ for $i, j = 1, 2$, we can write

$$\mathcal{A} = \sum_{i,j=1,2} \mathcal{A}_{ij}.$$

We also note that each $\mathcal{A}_{ij}$ is nonempty, and their pairwise intersections are the set of zero.

Note, in addition, that, by the assumptions

$$A \circ B = AB + BA \quad \text{and} \quad [A, B] = AB - BA,$$

for $A, B \in \mathcal{A}$, we can show the condition of ψ in Theorem 2.1 as follows:

$$\psi(A) \circ \psi(P) = A \circ P \tag{2.1}$$

and

$$[\psi(A), \psi(P)] = [A, P] \tag{2.2}$$

for all $A \in \mathcal{A}$ and $P \in \{P_1, P_2\}$. Also, by the assumptions

$$A \bullet B = AB + BA^* \quad \text{and} \quad [A, B]_* = AB - BA^*,$$

for $A, B \in \mathcal{A}$, we show the condition of ψ in Theorem 2.2 as follows:

$$\psi(A) \bullet \psi(P) = A \bullet P \tag{2.3}$$

and

$$[\psi(A), \psi(P)]_* = [A, P]_* \tag{2.4}$$

for all $A \in \mathcal{A}$ and $P \in \{P_1, P_2\}$.

We prove Theorem 2.1 in two steps.

Step 1. There exist $\alpha_i, \beta_i \in \mathbb{C}$ with $\alpha_i \neq 0$ such that $\psi(P_i) = \alpha_i P_i + \beta_i I$ for $i = 1, 2$.

Proof. With simple computation, we can obtain

$$[P_1, [P_1, [A, P_1]]] = [A, P_1]$$

for all $A \in \mathcal{A}$. Thus, from equation (2.2), we have

$$[P_1, [P_1, [\psi(A), \psi(P_1)]]] = [\psi(A), \psi(P_1)].$$

Therefore,

$$[P_1, [P_1, [T, \psi(P_1)]]] = [T, \psi(P_1)]$$

for all $T \in \mathcal{A}$, as ψ is surjective.

Let $K = [T, \psi(P_1)]$. By simple calculation, from the above equation, we can obtain

$$P_1 K - 2P_1 K P_1 + K P_1 = K. \tag{2.5}$$

Multiplying by P_1 from both sides of equation (2.5), it follows that $P_1 K P_1 = 0$. This yields

$$P_1(T\psi(P_1) - \psi(P_1)T)P_1 = 0 \tag{2.6}$$

for all $T \in \mathcal{A}$.

Let $T = X_{11} \in \mathcal{A}_{11}$ in equation (2.6). We can write

$$X_{11}\psi(P_1)P_1 - P_1\psi(P_1)X_{11} = 0,$$

and thus,

$$X_{11}P_1\psi(P_1)P_1 = P_1\psi(P_1)P_1X_{11}$$

for all $X_{11} \in \mathcal{A}_{11}$. Hence, there exists a $\lambda_1 \in \mathbb{C}$ such that

$$P_1\psi(P_1)P_1 = \lambda_1 P_1 \tag{2.7}$$

since $\mathcal{A}$ is a factor. Replacing T by $X_{12} \in \mathcal{A}_{12}$ in equation (2.6), we have

$$P_1X_{12}\psi(P_1)P_1 = 0,$$

and thus,

$$P_1XP_2\psi(P_1)P_1 = 0$$

for all $X \in \mathcal{A}$. The primeness of $\mathcal{A}$ shows that

$$P_2\psi(P_1)P_1 = 0. \tag{2.8}$$

Similarly, by taking $T = X_{21}$ in equation (2.6), we can obtain

$$P_1\psi(P_1)P_2 = 0. \tag{2.9}$$

Also, from $P_1K - 2P_1KP_1 + KP_1 = K$, we can obtain $P_2KP_2 = 0$. Therefore,

$$P_2(T\psi(P_1) - \psi(P_1)T)P_2 = 0.$$

Let $T = X_{22} \in \mathcal{A}_{22}$ in the above equation. Similar to equation (2.7), we can write

$$P_2\psi(P_1)P_2 = \lambda_2 P_2 \tag{2.10}$$

for some $\lambda_2 \in \mathbb{C}$.

On the other hand, from

$$\psi(P_1) = P_1\psi(P_1)P_1 + P_1\psi(P_1)P_2 + P_2\psi(P_1)P_1 + P_2\psi(P_1)P_2,$$

and from equations (2.7)–(2.10), it follows that

$$\psi(P_1) = \lambda_1P_1 + \lambda_2P_2,$$

which yields $\alpha_1 = \lambda_1 - \lambda_2$ and $\beta_1 = \lambda_2$. The result $\psi(P_1) = \alpha_1P_1 + \beta_1I$ is derived.

Now, we show that $\alpha_1 \neq 0$. On the contrary, suppose that $\alpha_1 = 0$. Then, for all $B \in \mathcal{A}$, we have

$$[\psi(B), \psi(P_1)] = [\psi(B), \beta_1I] = 0.$$

Therefore,

$$[B, P_1] = o \Longrightarrow BP_1 = P_1 B.$$

Multiplying this latter equation on the left and right sides, respectively, by P_2, we obtain

$$B_{21} = B_{12} = 0$$

for all $B \in \mathcal{A}$, which is impossible. Thus, $\alpha_1 \neq 0$. Similarly, in this way, $\psi(P_2) = \alpha_2 P_2 + \beta_2 I$ and $\alpha_2 \neq 0$ can be obtained. □

Step 2. $\psi(T) = \alpha T$ for all $T \in \mathcal{A}$, where $\alpha^2 = 1$.

Proof. From Step 1, for all $T \in \mathcal{A}$, we have

$$\begin{aligned} TP_1 - P_1 T &= \psi(T)\psi(P_1) - \psi(P_1)\psi(T) \\ &= \psi(T)(\alpha_1 P_1 + \beta_1 I) - (\alpha_1 P_1 + \beta_1 I)\psi(T). \end{aligned}$$

Thus,

$$TP_1 - P_1 T = \alpha_1 \psi(T) P_1 - \alpha_1 P_1 \psi(T).$$

Multiplying this equation on the left and right sides, respectively, by P_2, we have

$$\begin{aligned} P_2 T P_1 &= \alpha_1 P_2 \psi(T) P_1 \\ P_1 T P_2 &= \alpha_1 P_1 \psi(T) P_2. \end{aligned}$$

Therefore,

$$\psi(T)_{21} = P_2 \psi(T) P_1 = \alpha T_{21} \tag{2.11}$$

and

$$\psi(T)_{12} = P_1 \psi(T) P_2 = \alpha T_{12}, \tag{2.12}$$

where $\alpha = 1/\alpha_1$.

On the other hand,

$$\begin{aligned} TP_1 + P_1 T &= \psi(T)\psi(P_1) + \psi(P_1)\psi(T) \\ &= \psi(T)(\alpha_1 P_1 + \beta_1 I) + (\alpha_1 P_1 + \beta_1 I)\psi(T) \\ &= \alpha_1 \psi(T) P_1 + \alpha_1 P_1 \psi(T) + 2\beta_1 \psi(T). \end{aligned}$$

Therefore, from this equation and equations (2.11) and (2.12), we have

$$\begin{aligned}2T_{11}+T_{21}+T_{12} &= \alpha_1\psi(T)_{11}+\alpha_1\psi(T)_{21}+\alpha_1\psi(T)_{11}\\ &\quad+\alpha_1\psi(T)_{12}+2\beta_1\psi(T)\\ &= 2\alpha_1\psi(T)_{11}+T_{21}+T_{12}+2\beta_1\psi(T).\end{aligned}$$

Hence,

$$\begin{aligned}T_{11} &= \alpha_1\psi(T)_{11}+\beta_1\psi(T)\\ &= \alpha_1\psi(T)_{11}+\beta_1(\psi(T)_{11}+\psi(T)_{12}\\ &\quad+\psi(T)_{21}+\psi(T)_{22}).\end{aligned}$$

If $\beta_1 \neq 0$, then, from the fact that the set of zero contains the pairwise intersections of $\mathcal{A}_{ij}$, we can obtain

$$\psi(T)_{12}=\psi(T)_{21}=\psi(T)_{22}=0$$

for all $T \in \mathcal{A}$. This is a contraction from the surjectivity of ψ. Thus, $\beta_1 = 0$, and we have

$$P_1\psi(T)P_1=\psi(T)_{11}=\alpha T_{11}. \tag{2.13}$$

Similarly, in this way, we can obtain

$$P_2\psi(T)P_2=\delta T_{22} \tag{2.14}$$

and also

$$P_1\psi(T)P_2=\delta T_{12},$$

where $\delta = 1/\alpha_2$. Hence, from the above equation and equation (2.12), we have $\alpha = \delta$ and so $\alpha_1 = \alpha_2$. Since

$$\psi(T)=P_1\psi(T)P_1+P_1\psi(T)P_2+P_2\psi(T)P_1+P_2\psi(T)P_2,$$

it follows from equations (2.11)–(2.14) that

$$\psi(T)=\alpha T$$

for all $T \in \mathcal{A}$. Thus, $\psi(P_1) = \alpha P_1$, and we also have $\psi(P_1) = \alpha_1 P_1 = P_1/\alpha$. Finally, this yields $1/\alpha = \alpha$, and thus, $\alpha^2 = 1$, which completes the proof of Theorem 2.1. □

Now, we will prove Theorem (2.2) by the following several steps.

Step 1. Under the assumptions of Theorem 2.2, ψ is additive on $\mathcal{A}$.

Proof. Letting $A = P = P_1$ in equations (2.3) and (2.4), we have

$$\psi(P_1) \bullet \psi(P_1) = P_1 \bullet P_1$$

and

$$[\psi(P_1), \psi(P_1)]_* = [P_1, P_1]_*.$$

Thus,

$$\begin{aligned}\psi(P_1)^2 + \psi(P_1)\psi(P_1)^* &= 2P_1\\ \psi(P_1)^2 - \psi(P_1)\psi(P_1)^* &= 0.\end{aligned}$$

Adding these equations, we have

$$\psi(P_1)^2 = P_1. \tag{2.15}$$

On the other hand, for all $A, B \in \mathcal{A}$, we have

$$\begin{aligned}&(\psi(A+B) - \psi(A) - \psi(B)) \bullet \psi(P_1)\\ &\quad = \psi(A+B) \bullet \psi(P_1) - \psi(A) \bullet \psi(P_1) - \psi(B) \bullet \psi(P_1)\\ &\quad = (A+B) \bullet P_1 - A \bullet P_1 - B \bullet P_1\\ &\quad = 0\end{aligned}$$

and

$$\begin{aligned}&[\psi(A+B) - \psi(A) - \psi(B), \psi(P_1)]_*\\ &\quad = [\psi(A+B), \psi(P_1)]_* - [\psi(A), \psi(P_1)]_* - [\psi(B), \psi(P_1)]_*\\ &\quad = [A+B, P_1]_* - [A, P_1]_* - [B, P_1]_*\\ &\quad = 0.\end{aligned}$$

Therefore,

$$(\psi(A+B)-\psi(A)-\psi(B))\psi(P_1)+\psi(P_1)(\psi(A+B)-\psi(A)-\psi(B))^*=0$$

and

$$(\psi(A+B)-\psi(A)-\psi(B))\psi(P_1)-\psi(P_1)(\psi(A+B)-\psi(A)-\psi(B))^*=0.$$

Adding these equations, we have

$$(\psi(A+B) - \psi(A) - \psi(B))\psi(P_1) = 0.$$

Multiplying the above equation by $\psi(P_1)$ from the right side and using equation (2.15), we have

$$(\psi(A+B) - \psi(A) - \psi(B))P_1 = 0. \tag{2.16}$$

Similarly, we can show that $\psi(P_2)^2 = P_2$ and

$$(\psi(A+B) - \psi(A) - \psi(B))P_2 = 0. \tag{2.17}$$

Adding equations (2.16) and (2.17), we have

$$\psi(A+B) = \psi(A) + \psi(B). \qquad \square$$

Step 2. $\psi(I)^2 = \psi(I)\psi(I)^* = I$ and $\psi(P_i) = \psi(I)P_i = P_i\psi(I)$ for $i = 1, 2$.

Proof. First, we show that equations (2.3) and (2.4) hold for $P = I$. Letting $P = P_1$ and $P = P_2$ in equation (2.3), respectively, we have

$$\psi(A) \bullet \psi(P_1) = A \bullet P_1$$

and

$$\psi(A) \bullet \psi(P_2) = A \bullet P_2$$

for all $A \in \mathcal{A}$. Adding these two equations, the equation

$$\psi(A) \bullet (\psi(P_1) + \psi(P_2)) = A \bullet (P_1 + P_2)$$

is inferred, and, from the additivity of ψ, we have

$$\psi(A) \bullet \psi(I) = A \bullet I. \tag{2.18}$$

In a similar way, we have

$$[\psi(A), \psi(I)]_* = [A, I]_*. \tag{2.19}$$

Let $A = I$ in equations (2.18) and (2.19). With their aid, we can write

$$\psi(I)^2 + \psi(I)\psi(I)^* = 2I$$
$$\psi(I)^2 - \psi(I)\psi(I)^* = 0.$$

Hence,

$$\psi(I)^2 = \psi(I)\psi(I)^* = I.$$

Letting $A = I$ and $P = P_i$ for $i = 1, 2$ in equations (2.3) and (2.4) we have

$$\psi(I)\psi(P_i) + \psi(P_i)\psi(I)^* = 2P_i$$

and

$$\psi(I)\psi(P_i) - \psi(P_i)\psi(I)^* = 0.$$

These equations yield

$$\psi(I)\psi(P_i) = P_i.$$

Multiplying this equation with $\psi(I)$ from the left side, and from $\psi(I)^2 = I$, we have

$$\psi(P_i) = \psi(I)P_i.$$

Similarly to obtaining $A = P_i$ for $i = 1, 2$ in equations (2.18) and (2.19), we can obtain

$$\psi(P_i) = P_i\psi(I). \qquad \square$$

Step 3. There exists a $Z \in \mathcal{A}$ with $Z^2 = I$ such that $\psi(T) = TZ$ for all $T \in \mathcal{A}$.

Proof. From equation (2.3) and the fact that $\psi(P_i) = \psi(I)P_i$ for $i = 1, 2$, we have

$$\begin{aligned} TP_1 + P_1T^* &= \psi(T)\psi(P_1) + \psi(P_1)\psi(T)^* \\ &= \psi(T)\psi(I)P_1 + \psi(I)P_1\psi(T)^* \end{aligned}$$

and

$$\begin{aligned} TP_2 + P_2T^* &= \psi(T)\psi(P_2) + \psi(P_2)\psi(T)^* \\ &= \psi(T)\psi(I)P_2 + \psi(I)P_2\psi(T)^* \end{aligned}$$

for every $T \in \mathcal{A}$. Adding these two equations, we have

$$T + T^* = \psi(T)\psi(I) + \psi(I)\psi(T)^*.$$

In addition, from equation (2.4), we can similarly obtain

$$T - T^* = \psi(T)\psi(I) - \psi(I)\psi(T)^*.$$

Adding these two latter equations, we can write

$$T = \psi(T)\psi(I).$$

Multiplying this equation with $\psi(I)$ from the right side and the fact that $\psi(I)^2 = I$, we have

$$\psi(T) = T\psi(I).$$

Therefore, by obtaining $Z = \psi(I)$, we have $Z^2 = I$ and $\psi(T) = TZ$ for all $T \in \mathcal{A}$.

This completes the proof of Theorem 2.2. $\square$

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