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AFFINE RINGED SPACES AND SERRE'S CRITERION

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ABSTRACT. We study the notion of affine ringed space, see its meaning in topological, differentiable and algebro-geometric contexts and show how to reduce the affineness of a ringed space to that of a ringed finite space. Then, we characterize schematic finite spaces and affine schematic spaces in terms of combinatorial data. Finally, we prove Serre's criterion of affineness for schematic finite spaces. This yields, in particular, Serre's criterion of affineness on schemes.

1. Introduction. Let $(S,\mathcal{O}_S)$ be a ringed space and $A=\mathcal{O}_S(S)$. We say that $(S,\mathcal{O}_S)$ is an *affine* ringed space if it satisfies:

(1) S is acyclic, i.e., $H^i(S,\mathcal{O}_S)=0$ for any $i>0$.
(2) The global sections functor

$$\begin{aligned}\{\text{Quasi-coherent } \mathcal{O}_S\text{-modules}\} &\longrightarrow \{A\text{-modules}\}\\ \mathcal{M} &\longmapsto \Gamma(S,\mathcal{M})\end{aligned}$$

is an equivalence.

If $(S,\mathcal{O}_S)$ is a quasi-compact and quasi-separated scheme, then S is affine in the above sense if and only if S is affine in the usual sense, i.e., $S=\operatorname{Spec} A$. In the topological case, i.e., $\mathcal{O}_S$ is the constant sheaf $\mathbb{Z}$, S is affine if and only if S is homotopically trivial, see Proposition 2.8 for details. In the differentiable case, i.e., S is a separated differentiable manifold, or a differentiable space, and $\mathcal{O}_S=\mathcal{C}^\infty_S$ is the sheaf of differentiable functions, S is affine if and only if S is compact, see Proposition 2.9 and Remark 2.10.

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The concept of an affine ringed space was introduced in [**5**]. Therein, attention was focused on ringed finite spaces. Our first aim is to see how to reduce the affineness of a ringed space to that of a ringed finite space. We will be more precise. Let $(S, \mathcal{O}_S)$ be a ringed space, and let $\mathcal{U}$ be a finite covering. This covering produces a finite topological space X and a continuous map $\pi\colon S \to X$, see Example 3.3. Then, X has a ringed space structure by taking $\mathcal{O}_X = \pi_* \mathcal{O}_S$. We say that $(X, \mathcal{O}_X)$ is the ringed finite space associated to the ringed space $(S, \mathcal{O}_S)$ and the covering $\mathcal{U}$. Then, we prove the next theorem.

Theorem 1.1. *Assume that $\mathcal{U}$ is locally affine, see Definition* 3.8. *Then, $(S, \mathcal{O}_S)$ is affine if and only if the associated ringed finite space $(X, \mathcal{O}_X)$ is affine.*

This is a consequence of a deeper result (Theorem 3.9) which says that S and X have equivalent categories of quasi-coherent sheaves. The remainder of this paper deals with the study of affine ringed finite spaces, which was initiated in [**5**]. A ringed finite space is merely a ringed space $(X, \mathcal{O})$ where X is a finite topological space. Thus, X may be viewed as a preordered finite set, i.e., a finite set endowed with a relation $\leq$ which is reflexive and transitive, and the sheaf of rings $\mathcal{O}$ is equivalent to giving the following data: a ring $\mathcal{O}_p$ for each $p \in X$, and a morphism of rings

$$r_{pq}\colon \mathcal{O}_p \longrightarrow \mathcal{O}_q \quad \text{for each } p \leq q,$$

satisfying the obvious conditions:

$$r_{ql} \circ r_{pq} = r_{pl} \quad \text{for any } p \leq q \leq l,$$

and $r_{pp} = \mathrm{Id}$ for any p. By a *finite space* we mean a ringed finite space $(X, \mathcal{O})$ such that the morphisms r_{pq} are flat. We say that a finite space is *schematic* if, for any open subset

$$j\colon U \hookrightarrow X,$$

the sheaves $R^i j_* \mathcal{O}_{|U}$ are quasi-coherent. In Section 4, we characterize schematic spaces and affine schematic spaces in terms of the combinatorial data. The precise statements are given in Theorem 4.11, Theorem 4.12 and Corollary 4.13. The last section of the paper deals with Serre's characterization of affine spaces. If $(S, \mathcal{O}_S)$ is a quasi-compact and quasi-separated scheme, Serre's criterion of affineness states:

Theorem 1.2. *The following conditions are equivalent*:

(1) *S is an affine scheme.*

(2) *Any quasi-coherent module on S is acyclic.*

(3) $H^1(S, \mathfrak{p}) = 0$ *for any quasi-coherent sheaf of ideals* $\mathfrak{p}$.

We shall first see that Serre's criterion does not hold in the topological case. In particular, it does not hold for finite topological spaces. Our main result is Theorem 5.11, which states that Serre's criterion holds for schematic finite spaces. This implies, in particular, the above Serre's theorem for schemes (essentially by the reduction Theorem 3.9). Thus, Theorem 5.11 clarifies Serre's criterion on schemes in the following way: the validity of Serre's theorem for schemes is founded upon the following two properties of schemes:

(1) For any affine open subsets $V \subset U$ of a scheme S, the restriction morphism $\mathcal{O}_S(U) \to \mathcal{O}_S(V)$ is flat.

(2) For any open subset

$$j\colon U \hookrightarrow S,$$

the higher direct images $R^i j_* \mathcal{O}_U$ are quasi-coherent.

In the context of ringed finite spaces, conditions (1) and (2) exactly comprise what we have called a *schematic finite space*.

2. Generalities.

Definition 2.1. A *ringed space* is a pair $(S, \mathcal{O}_S)$, where S is a topological space, and $\mathcal{O}_S$ is a sheaf of (commutative with unit) rings on S. A morphism of ringed spaces

$$(S, \mathcal{O}_S) \longrightarrow (S', \mathcal{O}_{S'})$$

is a pair $(f, f^\#)$, where

$$f\colon S \longrightarrow S'$$

is a continuous map, and

$$f^\#\colon \mathcal{O}_{S'} \longrightarrow f_* \mathcal{O}_S$$

is a morphism of sheaves of rings, equivalently, a morphism of sheaves of rings

$$f^{-1}\mathcal{O}_{S'} \longrightarrow \mathcal{O}_S.$$

For brevity, a morphism of ringed spaces shall be denoted by

$$f\colon S \longrightarrow S'.$$

We shall denote by $(*, A)$ the ring space whose underlying topological space is a point $\{*\}$ and whose sheaf of rings is a ring A. For any ringed space $(S, \mathcal{O}_S)$, there is a natural morphism

$$\pi_S\colon S \longrightarrow (*, A),$$

with $A = \mathcal{O}_S(S)$, which is functorial on S: a morphism of ringed spaces

$$f\colon S \longrightarrow S'$$

gives a morphism

$$\mathcal{O}_{S'} \longrightarrow f_*\mathcal{O}_S,$$

and, taking global sections, a morphism of rings

$$A' = \mathcal{O}_{S'}(S') \longrightarrow A = \mathcal{O}_S(S),$$

is obtained, i.e., a morphism of ringed spaces

$$(*, A) \longrightarrow (*, A'),$$

and we have the commutative diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi_S"'] & S' \arrow[d, "\pi_{S'}"] \\ (*, A) \arrow[r] & (*, A'). \end{tikzcd}$$

Definition 2.2. Let $(S, \mathcal{O}_S)$ be a ringed space, and let $\mathcal{M}$ be an $\mathcal{O}_S$-module (a sheaf of $\mathcal{O}_S$-modules). We say that $\mathcal{M}$ is *quasi-coherent* if, for each $s \in S$, there exist an open neighborhood U of s and an exact sequence

$$\mathcal{O}_U^I \longrightarrow \mathcal{O}_U^J \longrightarrow \mathcal{M}_{|U} \longrightarrow 0$$

with I and J arbitrary sets of indices. Briefly, $\mathcal{M}$ is quasi-coherent if it is locally a cokernel of free modules.

The importance of quasi-coherent modules in algebraic geometry is clear; we shall see its meaning in other contexts.

2.1. The topological case. Every topological space S may be viewed as a ringed space, taking the constant sheaf $\mathbb{Z}$ as the sheaf of rings. We then have a functor

$$\begin{aligned}\{\text{Topological spaces}\} &\longrightarrow \{\text{Ringed spaces}\}\\ S &\longmapsto (S, \mathbb{Z}),\end{aligned}$$

which is fully faithful and has a left inverse:

$$(S, \mathcal{O}_S) \longmapsto S.$$

A sheaf of $\mathbb{Z}$-modules on S is merely a sheaf of abelian groups. If F is a locally constant sheaf of abelian groups, then it is a quasi-coherent $\mathbb{Z}$-module. The converse also holds under a local simple connectedness hypothesis:

Proposition 2.3. *Let S be a locally simply connected topological space, for example, any finite topological space, and let F be a sheaf of abelian groups on S. Then, F is a quasi-coherent $\mathbb{Z}$-module if and only if F is a locally constant sheaf.*

Proof. Given an abelian group G, we shall denote by G_S the constant sheaf G on S, that is,

$$G_S(U) = \mathrm{Hom}_{\mathrm{cont}}(U, G),$$

where G is endowed with the discrete topology.

Assuming that F is locally constant, we shall prove that it is quasi-coherent. Since it is a local question, we may assume that F is a constant sheaf G_S. Let

$$\mathbb{Z}^I \longrightarrow \mathbb{Z}^J \longrightarrow G \longrightarrow 0$$

be a resolution of G by free $\mathbb{Z}$-modules. It induces an exact sequence of sheaves

$$\mathbb{Z}_S^I \longrightarrow \mathbb{Z}_S^J \longrightarrow G_S = F \longrightarrow 0.$$

Hence, F is quasi-coherent.

Now, assume that F is a quasi-coherent $\mathbb{Z}$-module. We now prove that F is locally constant. For each $s \in S$, there exist a simply connected neighborhood U and an exact sequence (of sheaves on U)

$$\mathbb{Z}_U^I \xrightarrow{\phi} \mathbb{Z}_U^J \longrightarrow F_{|U} \longrightarrow 0.$$

Let $K = \operatorname{Ker}\phi$, $I = \operatorname{Im}\phi$. It is clear that the kernel of ϕ is a constant sheaf $K = G_U$, where G is the kernel of $\phi(U)\colon \mathbb{Z}^I \to \mathbb{Z}^J$. Moreover, for any simply connected open subset V of U, the sequence

$$0 \longrightarrow G \longrightarrow \mathbb{Z}^I \longrightarrow I(V) \longrightarrow 0$$

is exact due to $H^1(V, G_U) = 0$, since V is simply connected. If follows that I is a constant sheaf; hence, $H^1(V, I) = 0$ for any simply connected open subset V of U, and then $F_{|U}$ is constant. □

Proposition 2.3 was given in [**6**] for finite topological spaces.

2.2. The differentiable case. Let S be a separated differentiable space, for example, a differentiable manifold, with a countable basis. From the results of [**3**], we have the following:

Theorem 2.4. *The functor*

$$\pi_{S*}\colon \{\mathcal{C}_S^\infty\text{-modules}\} \longrightarrow \{A\text{-modules}\}$$

is fully faithful and $\pi_S^* \circ \pi_{S*}$ *is the identity. In particular, any* $\mathcal{C}_S^\infty$*-module is quasi-coherent.*

2.3. Affine ringed spaces. Let $(S, \mathcal{O}_S)$ be a ringed space, and let

$$\pi_S\colon (S, \mathcal{O}_S) \longrightarrow (*, A)$$

be the natural morphism, with $A = \mathcal{O}_S(S)$.

Definition 2.5. We say that S is *affine* if:

(1) S is acyclic, i.e., $H^i(S, \mathcal{O}_S) = 0$ for any $i > 0$.
(2) The direct and inverse image functors

$$\{\text{Quasi-coherent } \mathcal{O}_S\text{-modules}\} \underset{\pi_S^*}{\overset{\pi_{S*}}{\rightleftarrows}} \{A\text{-modules}\}$$

establish an equivalence between the category of A-modules and the category of quasi-coherent $\mathcal{O}_S$-modules.

In the following propositions, we see the meaning of affineness in topological, differentiable and algebro-geometric contexts.

Proposition 2.6 ([**4,** Section 7]). *Let S be a connected, locally path-connected and locally simply connected topological space and $\mathcal{O}_S = \mathbb{Z}$ the constant sheaf $\mathbb{Z}$. Then, $(S, \mathbb{Z})$ satisfies Definition* 2.5 (2) *if and only if S is simply connected.*

Proof. By Proposition 2.3, a quasi-coherent $\mathbb{Z}$-module is a locally constant sheaf of abelian groups, i.e., a representation of the fundamental group in an abelian group. Then, S satisfies Definition 2.5 (2) if and only if any locally constant sheaf of abelian groups is constant, i.e., any representation of the fundamental group in any abelian group is trivial; this is equivalent to saying that $\pi_1(S) = 0$. □

Now, to characterize affine topological spaces, we shall use the following basic result (which is essentially a consequence of Hurwitz's theorem and the universal coefficient formula).

Proposition 2.7. *Let S be a simply connected topological space. The following conditions are equivalent*:

(1) $\pi_i(S) = 0$ *for any* $i > 0$, *i.e., S is homotopically trivial.*
(2) $H_i(S, \mathbb{Z}) = 0$ *for any* $i > 0$.
(3) $H_i(S, G) = 0$ *for any* $i > 0$ *and any abelian group* G.
(4) $H^i(S, \mathbb{Z}) = 0$ *for any* $i > 0$.
(5) $H^i(S, G) = 0$ *for any* $i > 0$ *and any abelian group* G.

Proposition 2.8 (Affine topological spaces [**4,** Proposition 7.8]). *Let S be a connected, locally path-connected and locally simply connected topological space and $\mathcal{O}_S = \mathbb{Z}$ the constant sheaf $\mathbb{Z}$. Then $(S, \mathbb{Z})$ is an affine ringed space if and only if S is homotopically trivial.*

Proof. Immediate from Propositions 2.6 and 2.7. □

Proposition 2.9 (Affine differentiable spaces). *Let $(S, \mathcal{C}_S^\infty)$ be a separated differentiable space with countable basis. Then $(S, \mathcal{C}_S^\infty)$ is an affine ringed space if and only if S is compact.*

Proof. We shall need the following result, see [**3**]. We denote by $\operatorname{Spec}_r A$ the set of real points of A, i.e., the set of maximal ideals $\mathfrak{m}$ of A such that

$$\mathbb{R} \longrightarrow \frac{A}{\mathfrak{m}}$$

is an isomorphism. It is a topological space with the Zariski topology. For each $s \in S$, we have a real point

$$\mathfrak{m}_s := \{f \in A : f(s) = 0\};$$

thus, we have a map

$$S \longrightarrow \operatorname{Spec}_r A.$$

This map is a homeomorphism. In particular, if any maximal ideal of A is real, then S is homeomorphic to the maximal spectrum of A, and hence, S is compact.

The converse is also true. If S is compact, then any maximal ideal of A is real. Indeed, let $\mathfrak{m}$ be a maximal ideal of A. If $\mathfrak{m}$ is not real, then, for each $s \in S$, there exists an $f_s \in \mathfrak{m}$ such that $f_s \notin \mathfrak{m}_s$, i.e., $f_s(s) \neq 0$. Then, f_s does not vanish in a neighborhood U_s of s. The U_s cover S, and further, the $U_{s_1}, \ldots, U_{s_n}$ cover S. Then,

$$f_{s_1}^2 + \cdots + f_{s_n}^2$$

belongs to $\mathfrak{m}$ and does not vanish at any point of S, i.e., it is invertible.

We now prove the proposition. Any $\mathcal{C}_S^\infty$-module is acyclic since $\mathcal{C}_S^\infty$ is soft. Hence, by Theorem 2.4, $(S, \mathcal{C}_S^\infty)$ is affine if and only if it satisfies the following condition: π_{S*} is essentially surjective, or equivalently, $\pi_{S*} \circ \pi_S^*$ is the identity.

Assume that S is compact. Let M be an A-module. We must prove that the natural morphism

$$M \longrightarrow \pi_{S*}\pi_S^* M$$

is an isomorphism. By Theorem 2.4, this morphism is an isomorphism after taking π_S^*. Since π_S^* is an exact functor, it suffices to prove the following statement: if $\pi_S^* M = 0$, then $M = 0$. Now, $\pi_S^* M = 0$ implies that $M_s = 0$ for any $s \in S$, and further, $M_{\mathfrak{m}} = 0$ for any maximal ideal $\mathfrak{m}$ of A. Hence, $M = 0$.

Assume now that S is affine, and we will prove that it is compact. It suffices to see that any maximal ideal of A is real. Let $\mathfrak{m}$ be a maximal ideal and $A/\mathfrak{m}$ the residue field. If $\mathfrak{m}$ is not real, then $(A/\mathfrak{m})_s = 0$ for any $s \in S$, and then, $\pi_S^*(A/\mathfrak{m}) = 0$. This is not possible since S is affine. □

Remark 2.10. This proposition may lead one to think that our notion of "affineness" does not behave correctly in the differentiable case since $\mathbb{R}^n$ should be an affine ringed space. However, if Fréchet topology is considered, the desirable statement is obtained. We shall be more precise. Let $(S, \mathcal{C}_S^\infty)$ be a separated differentiable space with countable basis $A = \mathcal{C}_S^\infty(S)$. It is proven **[3]** that the functor

$$\begin{aligned} \{\mathcal{C}_S^\infty\text{-Fréchet modules}\} &\longrightarrow \{\text{Fréchet } A\text{-modules}\} \\ \mathcal{M} &\longmapsto \mathcal{M}(S) \end{aligned}$$

is an equivalence. Thus, $(S, \mathcal{C}_S^\infty)$ is affine in the Fréchet sense. In conclusion, any separated differentiable space with a countable basis is Fréchet-affine, and it is (algebraically) affine if and only if S is compact.

Proposition 2.11 (Affine schemes)**.** *Let $(S, \mathcal{O}_S)$ be a quasi-compact and quasi-separated scheme. Then, S is affine, in the sense of Definition* 2.5*, if and only if S is an affine scheme, i.e., $S = \operatorname{Spec} A$.*

Proof. In fact, condition (2) suffices to characterize affine schemes. Indeed, there is a natural morphism of schemes

$$f\colon S \longrightarrow \operatorname{Spec} A,$$

and, under condition (2), f_* and f^* yield equivalences between the categories of quasi-coherent sheaves on X and $\operatorname{Spec} A$. In particular, we obtain a bijection between quasi-coherent ideals on S and ideals of A. In particular, f is point-wise bijective and bi-continuous, i.e., a homeomorphism. Since $f_*\mathcal{O}_S = \mathcal{O}_{\operatorname{Spec} A}$, the conclusion is obtained. □

3. Reduction to ringed finite spaces. The aim of this section is to show the method of reduction of the affineness of a ringed space to the affineness of a ringed finite space, under a certain hypothesis (the existence of a locally affine finite covering). We shall first recall some elementary facts about finite topological spaces and ringed finite spaces.

3.1. Finite topological spaces.

Definition 3.1. A *finite topological space* is a topological space with a finite number of points.

Let X be a finite topological space. For each $p \in X$, we shall denote by U_p the minimum open subset containing p, i.e., the intersection of all of the open subsets containing p. The U_p form a minimal base of open subsets.

Definition 3.2. A *finite preordered set* is a finite set with a reflexive and transitive relation (denoted $\leq$), i.e., a relation satisfying $p \leq p$ for any $p \in X$ and $p \leq q \leq l \Rightarrow p \leq l$.

It is well known, Alexandroff [**1**], that an equivalence exists between finite topological spaces and finite preordered sets. If X is a finite topological space, the preorder relation is defined by:

$$p \leq q \quad \text{if and only if } U_p \supseteq U_q,$$

Conversely, if X is a finite preordered set, the topology on X is defined as $\overline{p} = \{q \in X : q \leq p\}$ is the closure of a point p.

A map

$$f\colon X \longrightarrow X',$$

between finite topological spaces is continuous if and only if it is monotone, i.e., for any $p \leq q$, $f(p) \leq f(q)$. A finite topological space is T_0, i.e., different points have different closures, if and only if the relation $\leq$ is antisymmetric, i.e., X is a partially ordered finite set (a finite poset).

Example 3.3 (Finite topological space associated to a finite covering)**.** Let S be a topological space, and let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a finite open covering of S. We consider the following equivalence relation on S. Say that $s \sim s'$ if $\mathcal{U}$ does not distinguish s and s', i.e., if we denote

$$U^s = \bigcap_{U_i \ni s} U_i,$$

then $s \sim s'$ if and only if $U^s = U^{s'}$. Let $X = S/\sim$ be the quotient set with the topology given by the following preorder:

$$[s] \leq [s'] \Longleftrightarrow U^s \supseteq U^{s'}.$$

This is a finite T_0-topological space, and the quotient map

$$\pi \colon S \longrightarrow X, \quad s \mapsto [s],$$

is continuous. For each $[s] \in X$, we have that $\pi^{-1}(U_{[s]}) = U^s$. We shall say that X is the finite topological space associated to the topological space S and the finite covering $\mathcal{U}$.

3.2. Ringed finite spaces. Let X be a finite topological space. Recall that we have a preorder relation

$$p \leq q \Longleftrightarrow U_q \subseteq U_p.$$

Giving a sheaf F of abelian groups, respectively rings, etc., on X is equivalent to giving the following data:

(1) An abelian group, respectively a ring, etc., F_p for each $p \in X$.

(2) A morphism of groups, respectively rings, etc.,

$$r_{pq} \colon F_p \longrightarrow F_q$$

for each $p \leq q$, satisfying $r_{pp} = \mathrm{Id}$ for any p, and

$$r_{qr} \circ r_{pq} = r_{pr}$$

for any $p \leq q \leq r$. The r_{pq} are called *restriction morphisms*.

Indeed, if F is a sheaf on X, then F_p is the stalk of F at p, and it coincides with the sections of F on U_p, that is,

$$F_p \text{ is the stalk of } F \text{ at } p := \text{sections of } F \text{ on } U_p := F(U_p).$$

The morphisms $F_p \to F_q$ are merely the restriction morphisms

$$F(U_p) \longrightarrow F(U_q).$$

Example 3.4. Given a group G, the constant sheaf G on X is given by the data $G_p = G$ for any $p \in X$, and $r_{pq} = \mathrm{Id}$ for any $p \leq q$.

We use the following elementary cohomological properties of finite topological spaces:

(1) For any sheaf F of abelian groups on a finite topological space X, we have:

$$H^i(U_p, F) = 0$$

for any $i > 0$ and any $p \in X$.

(2) Let

$$f\colon S \longrightarrow X$$

be a continuous map, with X a finite topological space. For any sheaf F on S, we have

$$(R^i f_* F)_p = H^i(f^{-1}(U_p), F) \quad \text{for each } p \in X.$$

Definition 3.5. A *ringed finite space* is a ringed space $(X, \mathcal{O})$ such that X is a finite topological space.

By the previous consideration, we have a ring $\mathcal{O}_p$ for each $p \in X$, and a morphism of rings

$$r_{pq}\colon \mathcal{O}_p \longrightarrow \mathcal{O}_q$$

for each $p \leq q$ such that $r_{pp} = \mathrm{Id}$ for any $p \in X$ and $r_{ql} \circ r_{pq} = r_{pl}$ for any $p \leq q \leq l$.

Giving a morphism of ringed spaces

$$(X, \mathcal{O}) \longrightarrow (X', \mathcal{O}')$$

between two ringed finite spaces, is equivalent to giving:

(1) a continuous, i.e., monotone, map

$$f\colon X \longrightarrow X';$$

(2) for each $p \in X$, a ring homomorphism

$$f_p^{\#}\colon \mathcal{O}'_{f(p)} \longrightarrow \mathcal{O}_p,$$

such that, for any $p \leq q$, the diagram (denote $p' = f(p),\ q' = f(q)$)

$$\begin{tikzcd} \mathcal{O}'_{p'} \arrow[r, "f_p^\#"] \arrow[d, "r_{p'q'}"'] & \mathcal{O}_p \arrow[d, "r_{pq}"] \\ \mathcal{O}'_{q'} \arrow[r, "f_q^\#"'] & \mathcal{O}_q \end{tikzcd}$$

is commutative.

Let $\mathcal{M}$ be a sheaf of $\mathcal{O}$-modules on a ringed finite space $(X, \mathcal{O})$. Thus, for each $p \in X$, $\mathcal{M}_p$ is an $\mathcal{O}_p$-module and, for each $p \leq q$, we have a morphism of $\mathcal{O}_p$-modules

$$\mathcal{M}_p \longrightarrow \mathcal{M}_q;$$

hence, a morphism of $\mathcal{O}_q$-modules

$$\mathcal{M}_p \otimes_{\mathcal{O}_p} \mathcal{O}_q \longrightarrow \mathcal{M}_q.$$

The next theorem may be found in **[6]**:

Theorem 3.6. *An $\mathcal{O}$-module $\mathcal{M}$ is quasi-coherent if and only if, for any $p \leq q$, the morphism*

$$\mathcal{M}_p \otimes_{\mathcal{O}_p} \mathcal{O}_q \longrightarrow \mathcal{M}_q$$

is an isomorphism.

3.3. Reduction to finite spaces.

Definition 3.7. Let $(S, \mathcal{O}_S)$ be a ringed space, and let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a finite open covering of S. Let X be the finite topological space associated to S and $\mathcal{U}$, and

$$\pi \colon S \longrightarrow X$$

the natural continuous map (Example 3.3). We then have a sheaf of rings on X, namely, $\mathcal{O} := \pi_* \mathcal{O}_S$ such that

$$\pi \colon (S, \mathcal{O}_S) \longrightarrow (X, \mathcal{O})$$

is a morphism of ringed spaces. We say that $(X, \mathcal{O})$ is the *ringed finite space associated to the ringed space S and the finite covering $\mathcal{U}$.*

Definition 3.8. Let $(S, \mathcal{O}_S)$ be a ringed space and let $\mathcal{U} = \{U_1, \dots, U_n\}$ be a finite open covering. We say that $\mathcal{U}$ is *locally affine* if, for any $s \in S$,

$$U^s := \bigcap_{U_i \ni s} U_i$$

is affine.

For example, a scheme S admits a locally affine finite covering if and only if S is quasi-compact and quasi-separated, see **[6]**.

Theorem 3.9. *Let $(S, \mathcal{O}_S)$ be a ringed space, $\mathcal{U}$ a locally affine covering and*

$$\pi\colon S \longrightarrow X$$

the associated ringed finite space. Then:

(1) *For any quasi-coherent $\mathcal{O}_S$-module $\mathcal{M}$, the direct image $\pi_*\mathcal{M}$ is a quasi-coherent module on X.*

(2) *The functors*

$$\{\textit{Quasi-coherent } \mathcal{O}_S\textit{-modules}\} \underset{\pi^*}{\overset{\pi_*}{\rightleftarrows}} \{\textit{Quasi-coherent } \mathcal{O}_X\textit{-modules}\}$$

establish an equivalence between the categories of quasi-coherent modules on S and X.

(3) $H^i(S, \mathcal{O}_S) = H^i(X, \mathcal{O}_X)$.

In particular, S is affine if and only if X is affine.

Proof. First, we need the following lemma.

Lemma 3.10. *Let $(S, \mathcal{O}_S)$ be a ringed space, U an affine open subset of S and $V \subseteq U$ an affine open subset. The natural morphism*

$$\mathcal{M}(U) \otimes_{\mathcal{O}_S(U)} \mathcal{O}_S(V) \longrightarrow \mathcal{M}(V)$$

is an isomorphism for any quasi-coherent module $\mathcal{M}$ on S.

Proof. We denote $A = \mathcal{O}_S(U)$, $B = \mathcal{O}_S(V)$, which yields a commutative diagram

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "\pi_V"'] & U \arrow[d, "\pi_U"] \\ (*, B) \arrow[r, "j"'] & (*, A). \end{tikzcd}$$

Since U is affine, $\mathcal{M}_{|U} = \pi_U^* M$, with $M = \Gamma(U, \mathcal{M})$. Analogously, since V is affine, $\mathcal{M}_{|V} = \pi_V^* N$, with $N = \Gamma(V, \mathcal{M})$. Hence,

$$\pi_V^* N = \mathcal{M}_{|V} = i^* \mathcal{M}_{|U} = i^* \pi_U^* M = \pi_V^* j^* M = \pi_V^* (M \otimes_A B),$$

and then, $N = M \otimes_A B$. $\square$

Now, we prove (1). By Theorem 3.6, we must prove that, for any $p \leq q$ in X, the natural morphism

$$(\pi_* \mathcal{M})_p \otimes_{\mathcal{O}_p} \mathcal{O}_q \longrightarrow (\pi_* \mathcal{M})_q$$

is an isomorphism. Since $\pi^{-1}(U_p)$ and $\pi^{-1}(U_q)$ are affine, the result immediately follows from Lemma 3.10.

Next, we prove (2). Let $\mathcal{M}$ be a quasi-coherent module on S, and we show that the natural morphism

$$\pi^* \pi_* \mathcal{M} \longrightarrow \mathcal{M}$$

is an isomorphism. The question is local on X, so we may assume that $X = U_p$, and then S is affine. Then, $\mathcal{M} = \pi_S^* M$ with $M = \Gamma(S, \mathcal{M})$. Since $\pi_* \mathcal{M}$ is a quasi-coherent module on X and X is affine, see Proposition 4.1, we have $\pi_* \mathcal{M} = \pi_X^* M$ due to the fact that $\Gamma(X, \pi_* \mathcal{M}) = \Gamma(S, \mathcal{M})$. We reach a conclusion since $\pi_X \circ \pi = \pi_S$. Now, let $\mathcal{N}$ be a quasi-coherent module on X, and let us prove that the natural morphism

$$\mathcal{N} \longrightarrow \pi_* \pi^* \mathcal{N}$$

is an isomorphism. The question is local on X; hence, we assume that X and S are affine. Then, $\mathcal{N} = \pi_X^* N$ and $\pi^* \mathcal{N} = \pi_S^* N$. Now, it suffices to see that

$$\mathcal{N} \longrightarrow \pi_* \pi^* \mathcal{N}$$

is an isomorphism after taking global sections. Indeed,

$$N = \Gamma(X, \mathcal{N}) \longrightarrow \Gamma(X, \pi_*\pi^*\mathcal{N}) = \Gamma(S, \pi_S^*N) = N,$$

where the last equality is due to the affineness of S.

For (3), since $\pi_*\mathcal{O}_S = \mathcal{O}_X$, it suffices to see that $R^i\pi_*\mathcal{O}_S = 0$ for $i > 0$. Now, for each $p \in X$,

$$[R^i\pi_*\mathcal{O}_S]_p = H^i(f^{-1}(U_p), \mathcal{O}_S),$$

which vanishes for $i > 0$ since $f^{-1}(U_p)$ is affine. $\square$

4. Affine schematic spaces. The aim of this section is to characterize affine schematic spaces in terms of the combinatorial data $(X, \mathcal{O})$.

4.1. Schematic spaces and schematic morphisms. The concepts of a schematic finite space and a schematic morphism were introduced in [**5**]. Briefly, schematic finite spaces and morphisms are those ringed finite spaces and morphisms which have good behavior with respect to quasi-coherent modules. We recall these concepts and some basic results due to them which shall be used later. See [**5**] for the proofs.

Proposition 4.1. *Let $(X, \mathcal{O})$ be a ringed finite space. For each $p \in X$, U_p is affine. Therefore, any ringed finite space is locally affine.*

Definition 4.2. A *finite space* is a ringed finite space $(X, \mathcal{O})$ such that, for any $p \leq q$, the morphism

$$\mathcal{O}_p \longrightarrow \mathcal{O}_q$$

is flat.

Theorem 4.3. *Let X be a finite space. The following conditions are equivalent*:

1. *X is affine.*
2. *X is acyclic, and every quasi-coherent module is generated by its global sections.*
3. *Every quasi-coherent module on X is acyclic and generated by its global sections.*

Definition 4.4. Let X be a finite space and

$$\delta\colon X \longrightarrow X \times X$$

the diagonal map. We say that X is *schematic* if $R^i\delta_*\mathcal{O}$ is quasi-coherent for any $i \geq 0$. We say that X is *semi-separated* if it is schematic and $R^i\delta_*\mathcal{O} = 0$ for any $i > 0$.

Example 4.5. Let $(S, \mathcal{O}_S)$ be a quasi-compact and quasi-separated scheme, $\mathcal{U}$ a locally affine finite covering of S and X the associated ringed finite space. Then, X is schematic. Moreover, X is semi-separated if and only if S is a semi-separated scheme (the intersection of any two affine open subschemes is affine).

For any $p, q \in X$, we denote

$$U_{pq} = U_p \cap U_q \quad \text{and} \quad \mathcal{O}_{pq} = \mathcal{O}(U_{pq}).$$

Then, X is schematic if and only if, for any $p \leq p'$, any $q \in X$ and any $i \geq 0$, the natural morphism

$$H^i(U_{pq}, \mathcal{O}) \otimes_{\mathcal{O}_p} \mathcal{O}_{p'} \longrightarrow H^i(U_{p'q}, \mathcal{O})$$

is an isomorphism. X is semi-separated if and only if the natural morphism

$$\mathcal{O}_{pq} \otimes_{\mathcal{O}_p} \mathcal{O}_{p'} \longrightarrow \mathcal{O}_{p'q}$$

is an isomorphism and U_{pq} is acyclic. In particular, if X is schematic, then for any $q, q' \geq p$, the natural morphism

$$\mathcal{O}_q \otimes_{\mathcal{O}_p} \mathcal{O}_{q'} \longrightarrow \mathcal{O}_{qq'}$$

is an isomorphism.

Any open subset of a schematic space is schematic. X is schematic if and only if U_p is schematic for any $p \in X$. An affine finite space is schematic if and only if it is semi-separated. Hence, a finite space X is schematic if and only if U_p is semi-separated for any $p \in X$. If X is an affine schematic space, then an open subset of X is affine if and only if it is acyclic.

Definition 4.6. Let $f\colon X \to Y$ be a morphism between finite spaces and

$$\Gamma\colon X \longrightarrow X \times Y$$

its graph. We say that f is *schematic* if $R^i\Gamma_*\mathcal{O}$ is quasi-coherent for all $i \geq 0$. A schematic morphism $f\colon X \to Y$ is called *affine* if $f^{-1}(U_y)$ is affine for any $y \in Y$.

If $f\colon X \to Y$ is schematic, then $R^i f_*\mathcal{M}$ is quasi-coherent for any $i \geq 0$ and any quasi-coherent module $\mathcal{M}$ on X. If X is schematic, then, for any open subset U, the inclusion

$$U \hookrightarrow X$$

is schematic. A finite space X is semi-separated if and only if the diagonal morphism

$$X \longrightarrow X \times X$$

is affine.

Affine schematic spaces have the following properties:

Proposition 4.7. *Let $(X, \mathcal{O})$ be an affine schematic space, $A = \mathcal{O}(X)$. Then*:

(1) *the natural injective morphism*

$$A \longrightarrow \prod_{p \in X} \mathcal{O}_p$$

is faithfully flat.

(2) *For any p, q in X, the natural morphism*

$$\mathcal{O}_p \otimes_A \mathcal{O}_q \longrightarrow \mathcal{O}_{pq}$$

is an isomorphism.

Our aim now is to give a type of converse of this result, i.e., to characterize affine schematic spaces in terms of their combinatorial data.

4.2. The cylinder space. Let $(X, \mathcal{O})$ be a ringed finite space, $A = \mathcal{O}(X)$.

Definition 4.8. ([**2,** subsection 2.8]). The *cylinder of* X is the ringed finite space $(\mathrm{Cyl}(X), \widetilde{\mathcal{O}})$, where

(1) The topological space $\mathrm{Cyl}(X)$ is the set

$$\{*\}\coprod X$$

with the following preorder: the restriction to X is the given preorder of X and $* < p$ for any $p \in X$. Note that X is then an open subset of $\mathrm{Cyl}(X)$.

(2) The sheaf of rings $\widetilde{\mathcal{O}}$ is given by: $\widetilde{\mathcal{O}}_* = A$, $\widetilde{\mathcal{O}}_p = \mathcal{O}_p$ for any $p \in X$. The morphism

$$\widetilde{\mathcal{O}}_* \longrightarrow \widetilde{\mathcal{O}}_p$$

is the natural morphism

$$A \longrightarrow \mathcal{O}_p \quad \text{and} \quad \widetilde{\mathcal{O}}_{|X} = \mathcal{O}.$$

By definition, $*$ is a minimum of $\mathrm{Cyl}(X)$; hence, $\mathrm{Cyl}(X) = U_*$. In particular, $\mathrm{Cyl}(X)$ is affine.

Example 4.9. Let S be a scheme, $\mathcal{U}$ a locally affine covering and X the associated finite space. X is schematic, but in general $\mathrm{Cyl}(X)$ is not. It is simple to prove that $\mathrm{Cyl}(X)$ is schematic if and only if S is a quasi-affine scheme, i.e., an open subset of an affine scheme.

Theorem 4.10. *Let $(X, \mathcal{O})$ be a ringed finite space, $A = \mathcal{O}(X)$. Then, X is an affine schematic space if and only if it satisfies*:

(1)
$$A \longrightarrow \prod_{p\in X} \mathcal{O}_p$$
is faithfully flat.

(2) $\mathrm{Cyl}(X)$ *is schematic.*

Proof. Assume that X is affine and schematic. Then, it satisfies Proposition 4.7 (1). We now show that $\mathrm{Cyl}(X)$ is schematic. It is clear that it is a finite space. Towards our conclusion, we must prove that, for any $\widetilde{p}, \widetilde{q} \in \mathrm{Cyl}(X)$ and any $\widetilde{p} \leq \widetilde{l}$, the natural morphism

$$H^i(U_{\tilde{p}\tilde{q}}, \widetilde{\mathcal{O}}) \otimes_{\widetilde{\mathcal{O}}_{\tilde{p}}} \widetilde{\mathcal{O}}_{\tilde{l}} \longrightarrow H^i(U_{\tilde{l}\tilde{q}}, \widetilde{\mathcal{O}})$$

is an isomorphism. If $\widetilde{p}, \widetilde{q} \in X$, it is an isomorphism since X is schematic. If $\widetilde{q} = *$, both sides are zero for $i > 0$ and an isomorphism for $i = 0$ by Proposition 4.7. Finally, if $\widetilde{p} = *$, both members are zero for $i > 0$ (the right hand is zero since U_{pq} is acyclic and X is semi-separated), and it is an isomorphism for $i = 0$ by Proposition 4.7.

Assume now that X satisfies (1) and (2). Then, X is schematic since it is an open subset of $\mathrm{Cyl}(X)$. It remains to prove that X is affine. Since $\mathrm{Cyl}(X)$ is affine, it suffices to see that X is acyclic. Let $j\colon X \hookrightarrow \mathrm{Cyl}(X)$. Since j is schematic, $R^i j_*\mathcal{O}$ is quasi-coherent. Hence, for any $p \in X$,

$$H^i(X, \mathcal{O}) \otimes_A \mathcal{O}_p = (R^i j_*\mathcal{O})_* \otimes_{\widetilde{\mathcal{O}}_*} \widetilde{\mathcal{O}}_p = (R^i j_*\mathcal{O})_p = H^i(U_p, \mathcal{O}) = 0$$

for $i > 0$. From condition (1), $H^i(X, \mathcal{O}) = 0$ for $i > 0$. □

In view of Theorem 4.10, in order to characterize affine schematic spaces in terms of their combinatorial data, it suffices to characterize when the cylinder is schematic. The next theorem characterizes when a ringed finite space is schematic.

Theorem 4.11. *Let $(X, \mathcal{O})$ be a ringed finite space. The following conditions are equivalent*:

(1) *X is a schematic finite space.*

(2) *For any $p \in X$ and any $q, q' \in U_p$ the natural morphism*

$$\mathcal{O}_q \otimes_{\mathcal{O}_p} \mathcal{O}_{q'} \longrightarrow \prod_{t \geq q, q'} \mathcal{O}_t$$

is faithfully flat.

(3) *For any $p \in X$ and any $q, q' \in U_p$ the natural morphism*

$$\mathcal{O}_q \otimes_{\mathcal{O}_p} \mathcal{O}_{q'} \longrightarrow \mathcal{O}_{qq'}$$

is an isomorphism and the morphism

$$\mathcal{O}_{qq'} \longrightarrow \prod_{t \geq q, q'} \mathcal{O}_t$$

is faithfully flat.

Proof.

(3) $\Rightarrow$ (2) is immediate. Moreover, if

$$\mathcal{O}_q \otimes_{\mathcal{O}_p} \mathcal{O}_{q'} \longrightarrow \prod_{t \geq q, q'} \mathcal{O}_t$$

is faithfully flat, then

$$\mathcal{O}_p \longrightarrow \mathcal{O}_t$$

is flat for any $p \leq t$ (take $q = q' = p$). Hence, we assume that X is a finite space.

(1) $\Rightarrow$ (3). If X is schematic, then

$$\mathcal{O}_q \otimes_{\mathcal{O}_p} \mathcal{O}_{q'} \longrightarrow \mathcal{O}_{qq'}$$

is an isomorphism and $U_{qq'}$ is affine since U_p is semi-separated; thus,

$$\mathcal{O}_{qq'} \longrightarrow \prod_{t \geq q, q'} \mathcal{O}_t$$

is faithfully flat by Proposition 4.7.

(3) $\Rightarrow$ (1). We proceed by induction on $\#X$, the case $\#X = 1$ being obvious. Since the question is local, we may assume that $X = U_p$, and we must prove that $U_{qq'}$ is acyclic. If either $q = p$ or $q' = p$, it is immediate. Hence, let us assume $q, q' > p$. Since

$$\mathcal{O}_{qq'} \longrightarrow \prod_{t \geq q, q'} \mathcal{O}_t$$

is faithfully flat, it suffices to see that

$$H^i(U_{qq'}, \mathcal{O}) \otimes_{\mathcal{O}_{qq'}} \mathcal{O}_t = 0$$

for any $t \in U_{qq'}$ and $i > 0$. Let

$$\delta \colon U_{qq'} \longrightarrow U_q \times_A U_{q'}$$

be the diagonal morphism ($A = \mathcal{O}_p$). By induction, U_q and $U_{q'}$ are schematic; thus, δ is schematic. Then, $R^i\delta_*\mathcal{O}_{U_{qq'}}$ is quasi-coherent and

$$(R^i\delta_*\mathcal{O}_{U_{qq'}})_{(q,q')} \otimes_{\mathcal{O}_{(q,q')}} \mathcal{O}_{(t,t)} = (R^i\delta_*\mathcal{O}_{U_{qq'}})_{(t,t)} = H^i(U_t, \mathcal{O}) = 0$$

for $i > 0$. Now, taking into account the equalities

$$\mathcal{O}_{(q,q')} = \mathcal{O}_q \otimes_A \mathcal{O}_{q'} = \mathcal{O}_{qq'},$$
$$\mathcal{O}_{(t,t)} = \mathcal{O}_t, (R^i\delta_*\mathcal{O}_{U_{qq'}})_{(q,q')} = H^i(U_{qq'}, \mathcal{O}),$$

we obtain

$$H^i(U_{qq'}, \mathcal{O}) \otimes_{\mathcal{O}_{qq'}} \mathcal{O}_t = 0,$$

as desired.

Finally, we see that (2) $\Rightarrow$ (3). We proceed by induction on $\#X$, the case $\#X = 1$ being obvious. We must prove that

$$\mathcal{O}_q \otimes_{\mathcal{O}_p} \mathcal{O}_{q'} \longrightarrow \mathcal{O}_{qq'}$$

is an isomorphism. If either $q = p$ or $q' = p$, it is immediate; thus, assume that $q, q' > p$. Since $\mathcal{O}$ is a sheaf we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{qq'} \longrightarrow \prod_{t \in U_{qq'}} \mathcal{O}_t \longrightarrow \prod_{t,t' \in U_{qq'}} \mathcal{O}_{tt'}.$$

On the other hand, we denote

$$B = \mathcal{O}_q \otimes_{\mathcal{O}_p} \mathcal{O}_{q'} \quad \text{and} \quad C = \prod_{t \in U_{qq'}} \mathcal{O}_t.$$

By hypothesis, $B \to C$ is faithfully flat; hence, we have an exact sequence

$$0 \longrightarrow B \longrightarrow C \longrightarrow C \otimes_B C.$$

For the conclusion, it is enough to see that

$$C \otimes_B C = \prod_{t,t' \in U_{qq'}} \mathcal{O}_{tt'}.$$

This is the same as proving that $\mathcal{O}_t \otimes_B \mathcal{O}_{t'} = \mathcal{O}_{tt'}$. We have natural morphisms

$$\mathcal{O}_t \otimes_{\mathcal{O}_q} \mathcal{O}_{t'} \longrightarrow \mathcal{O}_t \otimes_B \mathcal{O}_{t'} \longrightarrow \mathcal{O}_{tt'},$$

whose composition is an isomorphism by induction ($\#U_q < \#X$). Since

$$\mathcal{O}_t \otimes_{\mathcal{O}_q} \mathcal{O}_{t'} \longrightarrow \mathcal{O}_t \otimes_B \mathcal{O}_{t'}$$

is surjective, we are done. $\square$

Now, we can characterize affine schematic spaces.

Theorem 4.12. *Let $(X, \mathcal{O})$ be a ringed finite space $A = \mathcal{O}(X)$. The following conditions are equivalent:*

(1) *$(X, \mathcal{O})$ is an affine schematic space.*
(2) *$(X, \mathcal{O})$ satisfies*
  (2a)
$$A \longrightarrow \prod_{p \in X} \mathcal{O}_p$$
  is faithfully flat.
  (2b) *For any $p, q \in X$, the natural morphism*
$$\mathcal{O}_p \otimes_A \mathcal{O}_q \longrightarrow \mathcal{O}_{pq}$$
  is an isomorphism.
  (2c) *For any $p, q \in X$, the natural morphism*
$$\mathcal{O}_{pq} \longrightarrow \prod_{t \geq p,q} \mathcal{O}_t$$
  is faithfully flat.
(3) *$(X, \mathcal{O})$ satisfies*
  (3a)
$$A \longrightarrow \prod_{p \in X} \mathcal{O}_p$$
  is faithfully flat.
  (3b) *For any $p, q \in X$, the natural morphism*
$$\mathcal{O}_p \otimes_A \mathcal{O}_q \longrightarrow \prod_{t \geq p,q} \mathcal{O}_t$$
  is faithfully flat.

Proof.

(1) $\Rightarrow$ (2). If X is affine and schematic, it satisfies (2a) and (2b) by Proposition 4.7. Since X is semi-separated, U_{pq} is affine; hence, (2c) follows from Proposition 4.7.

(2) $\Rightarrow$ (3) is immediate. For the conclusion, we must show that (3) $\Rightarrow$ (1).

From condition (3b) it follows that, for any $p \in X$ and any $q, q' \in U_p$, the morphism

$$\mathcal{O}_q \otimes_{\mathcal{O}_p} \mathcal{O}_{q'} \longrightarrow \prod_{t \geq p,q} \mathcal{O}_t$$

is faithfully flat. Indeed, we have morphisms

$$\mathcal{O}_q \otimes_A \mathcal{O}_{q'} \longrightarrow \mathcal{O}_q \otimes_{\mathcal{O}_p} \mathcal{O}_{q'} \longrightarrow \prod_{t \geq q,q'} \mathcal{O}_t,$$

whose composition is faithfully flat and the first morphism is surjective. This implies that the second morphism is faithfully flat. Moreover, condition (3b) in the case $p = q$ states that

$$\mathcal{O}_p \otimes_A \mathcal{O}_p \longrightarrow \prod_{t \in U_p} \mathcal{O}_t$$

is faithfully flat. Since this morphism factors through the epimorphism

$$\mathcal{O}_p \otimes_A \mathcal{O}_p \longrightarrow \mathcal{O}_p,$$

we conclude that $\mathcal{O}_p \to \mathcal{O}_t$ are flat. It is now easy to see that $\mathrm{Cyl}(X)$ satisfies Theorem 4.11 (3); hence, $\mathrm{Cyl}(X)$ is schematic. We conclude that X is affine and schematic by Theorem 4.10. $\square$

Corollary 4.13. *Let X be a semi-separated finite space, $A = \mathcal{O}(X)$. Then, X is affine if and only if:*

(1)

$$A \longrightarrow \prod_{p \in X} \mathcal{O}_p$$

is faithfully flat.

(2) *For any* $p, q \in X$, *the natural morphism*

$$\mathcal{O}_p \otimes_A \mathcal{O}_q \longrightarrow \mathcal{O}_{pq}$$

is an isomorphism.

Proof. If X is affine, then it satisfies conditions (1) and (2) by Theorem 4.12. Conversely, assume that X is semi-separated and satisfies (1) and (2), and we shall prove that X is affine. By Theorem 4.12, it suffices to see that

$$\mathcal{O}_{pq} \longrightarrow \prod_{t \geq p,q} \mathcal{O}_t$$

is faithfully flat. However, U_{pq} is affine since X is semi-separated; thus, we conclude by Proposition 4.7. $\square$

5. Serre's criterion of affineness. Let $(S, \mathcal{O}_S)$ be a quasi-compact and quasi-separated scheme. We have the following cohomological characterization of affineness.

Theorem 5.1. *The following conditions are equivalent*:

(1) S *is an affine scheme.*
(2) S *is Serre-affine, i.e., any quasi-coherent module on* S *is acyclic.*
(3) $H^1(S, \mathfrak{p}) = 0$ *for any quasi-coherent sheaf of ideals* $\mathfrak{p}$.

The aim of this section is to generalize Serre's characterization of affine schemes to schematic finite spaces. We will first show that Serre's criterion does not hold in the topological context.

Definition 5.2. We say that a ringed space $(S, \mathcal{O}_S)$ is *Serre-affine* if any quasi-coherent module on S is acyclic.

Theorem 5.3. *Let* S *be a locally simply connected topological space, and let us assume that there exists an integer* d *such that* $H^i(S, \mathcal{M}) = 0$ *for* $i > d$ *and any quasi-coherent module* $\mathcal{M}$ *on* S, *for example, any finite topological space. Let*

$$\pi : \overline{S} \longrightarrow S$$

be a finite cover. Then, S *is Serre-affine if and only if so is* $\overline{S}$.

Proof. Assume that S is Serre-affine. For any quasi-coherent module $\mathcal{M}$ on $\overline{S}$, we have

$$H^i(\overline{S}, \mathcal{M}) = H^i(S, \pi_*\mathcal{M})$$

since $R^i\pi_*\mathcal{M} = 0$ for $i > 0$. Now, $\pi_*\mathcal{M}$ is quasi-coherent; hence,

$$H^i(S, \pi_*\mathcal{M}) = 0 \quad \text{for } i > 0$$

since S is Serre-affine. Then, $\overline{S}$ is Serre-affine.

We assume that $\overline{S}$ is Serre-affine. Let $\mathcal{M}$ be a quasi-coherent module on S and

$$\pi_*\pi^*\mathcal{M} \longrightarrow \mathcal{M},$$

the trace morphism, which is surjective. Then, we have an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \pi_*\pi^*\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow 0,$$

and $\mathcal{K}$ is quasi-coherent. Since $H^{d+1}(S, \mathcal{K}) = 0$,

$$H^d(S, \pi_*\pi^*\mathcal{M}) \longrightarrow H^d(S, \mathcal{M})$$

is surjective. However,

$$H^d(S, \pi_*\pi^*\mathcal{M}) = H^d(\overline{S}, \pi^*\mathcal{M}) = 0;$$

thus, $H^d(S, \mathcal{M}) = 0$, that is, we have proved that $H^d(S, \mathcal{M}) = 0$ for *any* quasi-coherent module $\mathcal{M}$ on S. Hence, the long exact sequence of cohomology proves that $H^{d-1}(S, \mathcal{M}) = 0$, that is, we have proved that $H^{d-1}(S, \mathcal{M}) = 0$ for any quasi-coherent module $\mathcal{M}$ on S. Proceeding in this way, we reach our conclusion. □

Corollary 5.4. *Let S be a topological space under the hypothesis of Theorem* 5.3. *Let*

$$\pi\colon \widetilde{S} \longrightarrow S$$

be the universal cover. Assume that π is finite, i.e., the fundamental group of S is finite. Then, S is Serre-affine if and only if $\widetilde{S}$ is homotopically trivial.

Proof. By Theorem 5.3, S is Serre-affine if and only if $\widetilde{S}$ is Serre-affine. Since $\widetilde{S}$ is simply connected, we reach our conclusion by Proposition 2.7. □

Proposition 5.5. *Let S be a connected and locally simply connected topological space. Then, $H^1(S,\mathfrak{p}) = 0$ for any quasi-coherent sheaf of ideals $\mathfrak{p}$ if and only if $H^1(S,\mathbb{Z}) = 0$.*

Proof. A quasi-coherent sheaf of ideals $\mathfrak{p}$ is a locally constant subsheaf of abelian groups of the constant sheaf $\mathbb{Z}$. It is easy to see that $\mathfrak{p}$ must be a constant sheaf; hence, it is isomorphic to $\mathbb{Z}$. □

These propositions state, in particular, that Serre's criterion does not hold in the topological case; as a positive counterpart, Proposition 2.7 states that, if S is simply connected, then S is affine if and only if S is Serre-affine. Regarding the differentiable case, let S be a separated differentiable manifold with countable basis. Any $\mathcal{C}^\infty_S$-module is acyclic; hence, S is Serre-affine. On the other hand, S is affine if and only if S is compact (Proposition 2.9). Thus, Serre's criterion fails in the differentiable context. If we consider Fréchet topology, as in Remark 2.10, then S is Fréchet- and Serre-affine.

Our aim now is to prove that Serre's criterion of affineness holds for schematic finite spaces.

5.1. Removable points.

Definition 5.6. A *weak equivalence* is a schematic affine morphism

$$f\colon X \longrightarrow Y$$

satisfying $f_*\mathcal{O}_X = \mathcal{O}_Y$.

It is proven in [**5**] that, if $f\colon X \to Y$ is a weak equivalence, then f_* and f^* yield equivalences between the categories of quasi-coherent modules on X and Y; moreover,

$$H^i(X,\mathcal{M}) = H^i(Y, f_*\mathcal{M})$$

for any quasi-coherent module $\mathcal{M}$ on X. Consequently, X is affine if and only if so is Y.

Definition 5.7. Let X be a schematic finite space, and let $p \in X$ be a closed point. We say that p is *removable* if

$$\mathcal{O}_p \longrightarrow \prod_{q>p} \mathcal{O}_q$$

is faithfully flat.

Examples 5.8. In order to illustrate the role that removable points play, we show a couple of examples in the algebro-geometric case. We omit the proofs since these results are unnecessary for the rest of the paper. Let S be a scheme, $\mathcal{U}$ a locally affine covering and X the associated finite space. Then,

(1) X has no removable points. If $p_1, \dots, p_n$ are the closed points of X, there exists a quasi-coherent ideal $\mathfrak{p}$ such that the support of $\mathcal{O}/\mathfrak{p}$ is $\{p_1, \dots, p_n\}$.

(2) Assume that S is quasi-affine, i.e., an open subset of an affine scheme. Then, $\mathrm{Cyl}(X)$ is schematic, see Remark 4.9, and $*$ is the unique closed point. Then, $*$ is removable if and only if S is affine. If $*$ is removable, then there is no quasi-coherent ideal $\mathfrak{p}$ on $\mathrm{Cyl}(X)$ such that the support of $\mathcal{O}_{\mathrm{Cyl}(X)}/\mathfrak{p}$ is $*$.

Proposition 5.9. *Let p be a closed point of a schematic finite space X. Let*

$$j\colon X - p \hookrightarrow X$$

be the natural inclusion. Then, p is removable if and only if j is a weak equivalence.

Proof. Assume that p is a removable point. We first show that

$$j_*\mathcal{O}_{X-p} = \mathcal{O}_X.$$

The natural morphism $\mathcal{O}_X \to j_*\mathcal{O}_{X-p}$ is an isomorphism after taking the fiber at any point $q \neq p$ since $j^{-1}(U_q) = U_q$ in this case. If we take the fiber at p, then it is also an isomorphism. Indeed, since

$$\mathcal{O}_p \longrightarrow \prod_{q>p} \mathcal{O}_q$$

is faithfully flat, it suffices to see that it is an isomorphism after tensoring by $\otimes_{\mathcal{O}_p}\mathcal{O}_q$ for any $q > p$, i.e., the morphism

$$\mathcal{O}_q \longrightarrow (j_*\mathcal{O}_{X-p})_p \otimes_{\mathcal{O}_p} \mathcal{O}_q$$

is an isomorphism. Since $j_*\mathcal{O}_{X-p}$ is quasi-coherent,

$$(j_*\mathcal{O}_{X-p})_p \otimes_{\mathcal{O}_p} \mathcal{O}_q = (j_*\mathcal{O}_{X-p})_q = \mathcal{O}_q.$$

In order to conclude that j is a weak equivalence, we must see that $j^{-1}(U_q)$ is affine. If $q \neq p$, then it is immediate. If $q = p$, then we must prove that $U_p - p$ is affine. Since U_p is schematic and affine, it suffices to show that $U_p - p$ is acyclic. Now, $R^i j_*\mathcal{O}_{X-p}$ is quasi-coherent; thus, for any $q > p$,

$$\begin{aligned} H^i(U_p - p, \mathcal{O}) \otimes_{\mathcal{O}_p} \mathcal{O}_q &= (R^i j_*\mathcal{O}_{X-p})_p \otimes_{\mathcal{O}_p} \mathcal{O}_q = (R^i j_*\mathcal{O}_{X-p})_q \\ &= H^i(U_q, \mathcal{O}) = 0 \quad \text{for } i > 0. \end{aligned}$$

Hence, $H^i(U_p - p, \mathcal{O}) = 0$ since p is a removable point.

Assume now that j is a weak equivalence. Then, $U_p - p = j^{-1}(U_p)$ is affine and $\mathcal{O}(U_p - p) = \mathcal{O}_p$ since $j_*\mathcal{O}_{X-p} = \mathcal{O}_X$. By Proposition 4.7,

$$\mathcal{O}_p \longrightarrow \prod_{q>p} \mathcal{O}_q$$

is faithfully flat. □

Corollary 5.10. *Let p be a removable point of X. Then, X is affine if and only if $X - p$ is affine.*

Theorem 5.11. *Let X be a schematic finite space. Then, the following conditions are equivalent:*

(1) *X is affine.*
(2) *Every quasi-coherent module on X is acyclic.*
(3) *For any quasi-coherent sheaf of ideals $\mathfrak{p}$, we have $H^1(X, \mathfrak{p}) = 0$.*

Proof.

(1) $\Rightarrow$ (2) by Theorem 4.3, and (2) obviously implies (3).

It remains to prove that (3) implies (1). Let $A = \mathcal{O}(X)$ and $Y = \operatorname{Spec} A$. For each $y \in Y$, let A_y be the local ring at y. Let X_y be the ringed finite space whose underlying topological space is X and whose sheaf of rings is $\mathcal{O} \otimes_A A_y$, in other words, it is the fibered product of X and $(*, A_y)$ over $(*, A)$. It is easy to see that X is affine, respectively, X satisfies (3), if and only if X_y is affine, respectively, satisfies (3), for any $y \in Y$, in other words, we may assume that A is a local ring. Now, by

Corollary 5.10, we assume that X has no removable points. Then, the result follows from the next proposition. □

Proposition 5.12. *Let X be a schematic space with no removable points and such that $A = \mathcal{O}(X)$ is a local ring. If $H^1(X, \mathfrak{p}) = 0$ for any quasi-coherent sheaf of ideals $\mathfrak{p}$, then X has a unique closed point p, i.e., $X = U_p$.*

Proof. Let $p_1, \dots, p_n$ be the closed points of X. Since p_i is not a removable point, there exists a prime ideal $\mathfrak{q}_i$ of $\mathcal{O}_{p_i}$ such that $\mathfrak{q}_i \cdot \mathcal{O}_q = \mathcal{O}_q$ for any $q > p_i$. Let $\mathfrak{p}$ be the sheaf of ideals on X given by

$$\mathfrak{p}_{p_i} = \mathfrak{q}_i, \qquad \mathfrak{p}_q = \mathcal{O}_q \quad \text{for } q \neq p_1, \dots, p_n$$

By construction, $\mathfrak{p}$ is quasi-coherent and $\mathcal{O}/\mathfrak{p}$ is supported at $p_1, \dots, p_n$. Since $H^1(X, \mathfrak{p}) = 0$, we have an epimorphism

$$A \longrightarrow \Gamma(X, \mathcal{O}/\mathfrak{p}) = \mathcal{O}_{p_1}/\mathfrak{q}_1 \times \cdots \times \mathcal{O}_{p_1}/\mathfrak{q}_1.$$

Since A is a local ring, it must be $n = 1$. □

Remark 5.13. Let S be a quasi-compact and quasi-separated scheme, $\mathcal{U}$ a locally affine finite covering and $\pi \colon S \to X$ the associated ringed finite space (which is a schematic finite space). The functors π_* and π^* yield equivalences between the categories of quasi-coherent modules on S and X. Moreover, $H^i(S, \mathcal{M}) = H^i(X, \pi_* \mathcal{M})$ for any quasi-coherent module on S, and $\mathfrak{p}$ is a quasi-coherent sheaf of ideals on S if and only if $\pi_* \mathfrak{p}$ is a quasi-coherent sheaf of ideals on X. Hence, Serre's criterion on X yields the usual Serre's criterion on S.

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