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ALGEBRAIC PROPERTIES OF SPANNING SIMPLICIAL COMPLEXES

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ABSTRACT. In this paper, we study some algebraic properties of the spanning simplicial complex $\Delta_s(G)$ associated to a multigraph G. It is proved that, for any finite multigraph G, $\Delta_s(G)$ is a pure vertex decomposable simplicial complex and therefore shellable and Cohen-Macaulay. As a consequence, we deduce that, for any multigraph G, the quotient ring $R/I_c(G)$ is Cohen-Macaulay, where

$$I_c(G) = (x_{i_1} \cdots x_{i_k} \mid \{x_{i_1}, \ldots, x_{i_k}\} \text{ is the edge set of a cycle in } G).$$

Also, some homological invariants of the Stanley-Reisner ring of $\Delta_s(G)$, such as projective dimension and regularity, are investigated.

Introduction. Simplicial complexes are widely used structures which have many applications in algebraic topology and commutative algebra. In particular, in order to characterize monomial quotient rings with a desired property, the simplicial complex is a very strong tool considering the Stanley-Reisner correspondence between simplicial complexes and monomial ideals. Characterizing simplicial complexes, which have properties like vertex decomposability, shellability and Cohen-Macaulayness, is a main topic in combinatorial commutative algebra. It is rather hopeless to give a full classification of simplicial complexes with each of these properties. In this regard, finding classes of simplicial complexes with a desired property has been considered by many researchers, cf., **[5, 6, 12, 16]**.

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Let $G = (V(G), E(G))$ be a multigraph with vertex set $V(G)$ and edge set $E(G)$. A growing number of ways exist for associating a simplicial complex to a graph or a multigraph G. The best known is the independence complex of G whose faces are those subsets of $V(G)$ which contain no edge. Recently, the spanning simplicial complex associated to a simple connected graph G was defined **[1]**, and for unicyclic graphs, the spanning simplicial complex is studied. For a finite simple connected graph G, the spanning simplicial complex associated to G, denoted $\Delta_s(G)$, is one whose facets are the edge sets of all of the spanning trees of G. In **[1]**, it is shown that the spanning simplicial complex associated to a unicyclic graph is shifted, and therefore shellable, and some invariants of its Stanley-Reisner ring are computed. Also, in **[7, 17]**, the spanning simplicial complex of r-cyclic graphs is studied.

In this paper, we study the spanning simplicial complex associated to a finite connected multigraph G. Since any spanning tree of a finite connected multigraph G has $|V(G)|-1$ edges, $\Delta_s(G)$ is a pure simplicial complex of dimension $|V(G)| - 2$.

The paper proceeds as follows. In Section 1, we give some preliminaries which are needed in the rest of this note.

Section 2 is devoted to the study of the spanning simplicial complex. As the main result, it is proven that, for any finite multigraph G, $\Delta_s(G)$ is a pure vertex decomposable simplicial complex, and hence, it is shellable and Cohen-Macaulay. It will be seen that, for the ideal

$$I_c(G) = (x_{i_1} \cdots x_{i_k} \mid \{x_{i_1}, \ldots, x_{i_k}\} \text{ is the edge set of a cycle in } G),$$

which is the Stanley-Reisner ideal associated to $\Delta_s(G)$, the quotient ring $R/I_c(G)$ is Cohen-Macaulay, where G is a finite connected multigraph with the edge set $\{x_1, \ldots, x_m\}$, K is a field and $R = K[x_1, \ldots, x_m]$.

In Section 3, some homological invariants of the Stanley-Reisner ring $K[\Delta_s(G)]$ are studied. In Theorem 3.1, some inductive formulas for projective dimension, regularity and graded Betti numbers of $K[\Delta_s(G)]$ (K is a field) are given and, in Theorem 3.2, the precise value of the projective dimension is given in terms of the invariants of G. In fact, it is shown that

$$\mathrm{pd}(K[\Delta_s(G)]) = \mathrm{ht}(I_{\Delta_s(G)}) = |E(G)| - |V(G)| + 1.$$

Moreover, for some classes of multigraphs, the regularity of $K[\Delta_s(G)]$ is explained by combinatorial information from G, see Corollary 3.4.

Throughout this paper, we assume that $G = (V(G), E(G))$ is a finite connected multigraph and K is a field.

1. Preliminaries. In this section, we recall some preliminaries needed in the sequel. We begin with the definition of a multigraph.

Definition 1.1. A finite *multigraph* $G = (V(G), E(G))$ consists of two finite sets $V(G)$ and $E(G)$ called *vertices* and *edges* of G, respectively, where each edge is associated to a set consisting of either one or two vertices called its *endpoints*. An edge with only one endpoint is called a *loop*, and two distinct edges with the same set of endpoints are said to be *parallel*. A multigraph without loops and parallel edges is called a *simple graph* or briefly, a *graph*.

Note that a loop of a multigraph forms a cycle of length 1, and any two parallel edges comprise a cycle of length 2.

A *spanning tree* of a finite connected multigraph G, is a subgraph T of G such that T is a tree (connected and cycle-free) and $V(T) = V(G)$. Therefore, any spanning tree of G has $|V(G)| - 1$ edges.

An *r-cyclic multigraph* $G_{t_1,\ldots,t_r}$ with a common edge is a connected graph with

$$\sum_{i=1}^{r}(t_i - 2) + 2 \quad \text{vertices} \qquad \text{and} \qquad \sum_{i=1}^{r}(t_i - 1) + 1 \quad \text{edges}$$

obtained by joining r cyclic graphs $G_{t_1}, \ldots, G_{t_r}$ with a common edge, where G_{t_i} denotes the cyclic graph with t_i vertices. The reader is referred to **[14]** for more details in the context of graph theory.

Now, we recall the definition of a vertex decomposable simplicial complex. To this aim, we need to recall definitions of the link and the deletion of a face in Δ. For a simplicial complex Δ and $F \in \Delta$, the *link* of F in Δ is defined as

$$\mathrm{lk}_\Delta(F) = \{G \in \Delta \mid G \cap F = \emptyset,\ G \cup F \in \Delta\},$$

and the *deletion* of F is the simplicial complex

$$\mathrm{del}_\Delta(F) = \{G \in \Delta \mid G \cap F = \emptyset\}.$$

Definition 1.2. A simplicial complex Δ is called *vertex decomposable* if Δ is a simplex, or Δ contains a vertex x such that

(i) both $\mathrm{del}_\Delta(x)$ and $\mathrm{lk}_\Delta(x)$ are vertex decomposable, and
(ii) every facet of $\mathrm{del}_\Delta(x)$ is a facet of Δ.

A vertex x which satisfies condition (ii) is called a *shedding vertex* of Δ.

For a $\mathbb{Z}$-graded R-module M, the *Castelnuovo-Mumford regularity*, or briefly, *regularity*, of M is defined as

$$\mathrm{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\},$$

and the *projective dimension* of M is defined as

$$\mathrm{pd}(M) = \max\{i \mid \beta_{i,j}(M) \neq 0 \text{ for some } j\},$$

where $\beta_{i,j}(M)$ is the (i,j)th graded Betti number of M.

Definition 1.3. A monomial ideal I in the ring $R = K[x_1, \dots, x_n]$ has *linear quotients* if there exists an ordering $f_1, \dots, f_m$ on the minimal generators of I such that the colon ideal $(f_1, \dots, f_{i-1}) :_R (f_i)$ is generated by a subset of $\{x_1, \dots, x_n\}$ for all $2 \leq i \leq m$. Also, monomial ideal I generated by monomials of degree d has a *linear resolution* if $\beta_{i,j}(I) = 0$ for all $j \neq i + d$.

For a squarefree monomial ideal $I = (x_{11} \cdots x_{1n_1}, \dots, x_{t1} \cdots x_{tn_t})$, the *Alexander dual ideal* of I, denoted $I^\vee$, is defined as

$$I^\vee = (x_{11}, \dots, x_{1n_1}) \cap \cdots \cap (x_{t1}, \dots, x_{tn_t}).$$

For a simplicial complex Δ, $\mathcal{F}(\Delta)$ denotes the set of facets of Δ and, if $\mathcal{F}(\Delta) = \{F_1, \dots, F_k\}$, then we write $\Delta = \langle F_1, \dots, F_k \rangle$.

For a simplicial complex Δ with the vertex set $X = \{x_1, \dots, x_n\}$, the *Alexander dual simplicial complex* $\Delta^\vee$ associated to Δ is defined as

$$\Delta^\vee = \{X \setminus F \mid F \notin \Delta\}.$$

For a subset $C \subseteq X$, by x^C we mean the monomial $\prod_{x \in C} x$ in the ring $K[x_1, \ldots, x_n]$. It may be seen that

$$(I_\Delta)^\vee = (x^{F^c} \mid F \in \mathcal{F}(\Delta)),$$

where I_Δ is the Stanley-Reisner ideal associated to Δ and $F^c = X \setminus F$. Moreover, it is easily seen that $(I_\Delta)^\vee = I_{\Delta^\vee}$.

2. Spanning simplicial complexes are Cohen-Macaulay. We begin this section with a definition of the spanning simplicial complex, as follows.

Definition 2.1 ([**1,** Definition 2.5])**.** Let G be a finite, connected multigraph, and let $s(G) = \{E_1, \ldots, E_s\}$ be the set of edge sets of all possible spanning trees of G. The *spanning simplicial complex* associated to G, denoted $\Delta_s(G)$, is a simplicial complex with the vertex set $E(G)$ such that the elements of $s(G)$ are its facets, in other words,

$$\Delta_s(G) = \langle E_1, \ldots, E_s \rangle.$$

Remark 2.2. Since a spanning tree of a multigraph has no loop, $\Delta_s(G)$ is similar to the spanning simplicial complex associated with a multigraph which is obtained by removing all of the loops in G. Thus, we may assume that G is a multigraph with no loop.

Example 2.3. Let G be the graph with the edge set $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ depicted in Figure 1. Then,

$$\begin{aligned}\Delta_s(G) = \langle &\{e_1, e_2, e_3, e_6\}, \{e_1, e_2, e_4, e_6\}, \{e_1, e_3, e_5, e_6\}, \\ &\{e_1, e_4, e_5, e_6\}, \{e_1, e_3, e_4, e_6\}, \{e_2, e_3, e_4, e_6\}, \\ &\{e_2, e_3, e_5, e_6\}, \{e_2, e_4, e_5, e_6\}\rangle.\end{aligned}$$

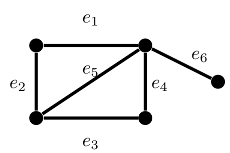

FIGURE 1.

Remark 2.4 ([**14**])**.** If G is a multigraph, then it is well known that any of its spanning trees has $|V(G)|-1$ edges, and thus, $\Delta_s(G)$ is always a pure simplicial complex of dimension $|V(G)|-2$.

Assume that G is a multigraph and e is an edge of G. The *deletion of G by e*, denoted $G-e$, is a multigraph obtained from G by removing the edge e. Also, the *contraction of G by e*, denoted G/e, is a multigraph obtained from G by removing the edge e, while simultaneously merging the two vertices previously joined by e. If e is an edge with endpoints x and y, then we denote $V(G/e)$ by $(V(G)\setminus\{x,y\})\cup\{w\}$, where w is the vertex obtained by merging x and y. The edge set of G/e is precisely explained in the following remarks.

Remark 2.5.

(i) Let e be an edge of the multigraph G with endpoints x and y and $w\in V(G/e)$ the vertex obtained by merging x and y. There is a bijection $f: E(G-e)\to E(G/e)$ such that, for any $e'\in E(G-e)$ with endpoints u and v, where $\{x,y\}\cap\{u,v\}=\emptyset$, $f(e')=e'$, for any $e'\in E(G-e)$ where one of its endpoints is $u\neq x,y$ and another is x or y, $f(e')$ is incidental to u and w and, if $e'\in E(G-e)$ has endpoints x and y, then $f(e')$ is a loop in the vertex w. Thus, hereafter, we may consider $E(G/e)=E(G-e)$, that is, $E(G/e)$ may be considered as the subset $E(G)\setminus\{e\}$ of $E(G)$.

(ii) Let G be a multigraph with at least one cycle and e an edge of a cycle in G. Assume that $\{T_1,\ldots,T_s\}$ is the set of all spanning trees of G such that, for each $1\le i\le r$, $e\notin E(T_i)$ and, for each $r+1\le i\le s$, $e\in E(T_i)$. Then, by the deletion-contraction formula, $\{T_1,\ldots,T_r\}$ is the set of all spanning trees of $G-e$, and $\{T_{r+1}/e,\ldots,T_s/e\}$ is the set of all spanning trees of G/e.

(iii) Let G be a multigraph with at least one cycle and e an edge of a cycle in G. Let $\mathcal{F}(\Delta_s(G))=\{F_1,\ldots,F_s\}$ be such that $e\notin F_i$ for any $1\le i\le r$, and $e\in F_i$ for any $r+1\le i\le s$. In light of (ii), $\mathcal{F}(\Delta_s(G-e))=\{F_1,\ldots,F_r\}$ and $F_{r+1}\setminus\{e\},\ldots,F_s\setminus\{e\}$ correspond to the facets of $\Delta_s(G/e)$. Thus,

$$\mathcal{F}(\Delta_s(G))=\mathcal{F}(\Delta_s(G-e))\cup\mathcal{F}(\Delta_s(G/e)*\langle\{e\}\rangle).$$

Lemma 2.6. *Let G be a multigraph with at least one cycle and e an edge of a cycle in G. Then,*

$$\mathrm{lk}_{\Delta_s(G)}(e) = \Delta_s(G/e),$$

and

$$\mathrm{del}_{\Delta_s(G)}(e) = \Delta_s(G - e).$$

Proof. Assume that $\Delta_s(G) = \langle F_1, \ldots, F_s \rangle$ is such that $e \notin F_i$ for any $1 \le i \le r$ and $e \in F_i$ for any $r+1 \le i \le s$. By Remark 2.5 (iii), $\Delta_s(G-e) = \langle F_1, \ldots, F_r \rangle$ and $\Delta_s(G/e) = \langle F_{r+1} \setminus \{e\}, \ldots, F_s \setminus \{e\} \rangle$. It is clear that

$$\mathrm{lk}_{\Delta_s(G)}(e) = \Delta_s(G/e)$$

and

$$\mathrm{del}_{\Delta_s(G)}(e) = \langle F_1, \ldots, F_r \rangle \cup \langle F_{r+1} \setminus \{e\}, \ldots, F_s \setminus \{e\} \rangle.$$

Let $r+1 \le i \le s$ be such that F_i is the edge set of a spanning tree T of G which contains e. We will show that $F_i \setminus \{e\}$ is contained in the edge set of some spanning tree of $G - e$. Consider the subgraph H of T which is obtained by deleting the edge e. Let A and B be the connected components of H such that every component contains an endpoint of e. Since e belongs to a cycle C in G, there is an edge e' in C with endpoints in A and B. It is easily seen that $e' \notin F_i$ since, otherwise, T would contain a cycle. Due to the choice of e',

$$F_\ell = (F_i \setminus \{e\}) \cup \{e'\}$$

is the edge set of a spanning tree of G. Since $e \notin F_\ell$, we have $1 \le \ell \le r$, and hence, $F_i \setminus \{e\} \subseteq F_\ell$. This shows that $\mathrm{del}_{\Delta_s(G)}(e) = \Delta_s(G-e)$, as required. $\square$

The next theorem is one of the main results of this paper, which introduces a large class of pure vertex decomposable, and hence, Cohen-Macaulay, simplicial complexes.

Theorem 2.7. *The spanning simplicial complex of a multigraph is vertex decomposable.*

Proof. If G is a tree, then $\Delta_s(G)$ is a simplex, and there is nothing to prove. Therefore, assume that G is not a tree. We proceed by induction on $|E(G)|$. Assume inductively that the result has been proved for smaller values of $|E(G)|$. Since G is a connected multigraph which is not a tree, G has an edge, say e, such that e is an edge of some cycle in G. By the inductive hypothesis, $\Delta_s(G-e)$ and $\Delta_s(G/e)$ are vertex decomposable. Hence, Remark 2.5 (iii) and Lemma 2.6 complete the proof. $\square$

It is known that every vertex decomposable simplicial complex is shellable, see [**4,** Theorem 11.3]. In addition, every pure shellable simplicial complex is Cohen-Macaulay, see [**13,** Theorem 5.3.18]. These facts, together with Remark 2.4 and Theorem 2.7, imply the following result.

Corollary 2.8 (cf., [**1,** Corollary 3.9]). *The spanning simplicial complex of a multigraph is always pure shellable, and hence, Cohen-Macaulay.*

Hereafter, we assume that G is a multigraph with the edge set $\{x_1,\dots,x_m\}$. Also, we consider the same notion x_i for the edges of G and indeterminates of the polynomial ring $R=K[x_1,\dots,x_m]$.

Corollary 2.9. *Let G be a connected multigraph with the edge set $E(G)=\{x_1,\dots,x_m\}$, $R=K[x_1,\dots,x_m]$ the polynomial ring over a field K and*

$$I_c(G)=(x_{i_1}\cdots x_{i_k}\mid \{x_{i_1},\dots,x_{i_k}\} \text{ is the edge set of a cycle in } G).$$

Then, $R/I_c(G)$ is a Cohen-Macaulay ring.

Proof. We show that $I_c(G)$ is equal to the Stanley-Reisner ideal associated to $\Delta_s(G)$. It is clear that $I_c(G)\subseteq I_{\Delta_s(G)}$.

For the reverse containment, note that the spanning tree of a connected multigraph can also be defined as a maximal set of edges of G which contains no cycle. Therefore, every subgraph without cycles, by adding edges, can be extended to a maximal subgraph which is a spanning tree of G. Hence, if $x_{i_1}\cdots x_{i_k}$ is a generator of $I_{\Delta_s(G)}$, then $\{x_{i_1},\dots,x_{i_k}\}$ contains the edge set of a cycle in G, and thus,

$x_{i_1} \cdots x_{i_k} \in I_c(G)$ as required. Therefore,

$$\begin{aligned} I_{\Delta_s(G)} &= (x_{i_1} \cdots x_{i_k} \mid \{x_{i_1}, \ldots, x_{i_k}\} \text{ is the edge set of a cycle in } G) \\ &= I_c(G). \end{aligned}$$

The result now follows from Corollary 2.8. □

Example 2.10. Assume that G is the graph depicted in Figure 1, and set $E(G) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Then,

$$I_c(G) = (x_1x_2x_5,\, x_3x_4x_5,\, x_1x_2x_3x_4),$$

and, in view of Corollary 2.9, $R/I_c(G)$ is a Cohen-Macaulay ring.

Remark 2.11. If $\Delta_s(G) = \langle F_1, \ldots, F_s \rangle$, then

$$I_{\Delta_s(G)^\vee} = (x^{E(G)\setminus F_i} \mid 1 \leq i \leq s).$$

Thus, it is generated by monomials $x_{i_1} \cdots x_{i_k}$ such that $\{x_{i_1}, \ldots, x_{i_k}\}$ is a subset of $E(G)$ which should be removed from G to make it into a spanning tree. Clearly, the cardinality of each of these subsets of edges equals

$$|E(G)| - |F_i| = |E(G)| - |V(G)| + 1.$$

This number is a graph-theoretic invariant of G, called the *circuit rank* of G. Hence, $I_{\Delta_s(G)^\vee}$ can be generated by monomials with the same degree of circuit rank of G.

In the next corollary, we introduce a class of ideals with linear quotients and linear resolution. Toward this aim, firstly we recall the following definition.

Definition 2.12 ([**8,** Definition 2.1])**.** A monomial ideal I in $R = K[X]$ is called *vertex splittable* if it can be obtained by the following recursive procedure:

(i) if u is a monomial and $I = (u)$, $I = (0)$ or $I = R$, then I is a vertex splittable ideal.

(ii) If there is a variable $x \in X$ and vertex splittable ideals I_1 and I_2 of $K[X \setminus \{x\}]$ such that $I = xI_1 + I_2$, $I_2 \subseteq I_1$ and $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(xI_1)$ and $\mathcal{G}(I_2)$, then I is a vertex splittable ideal.

Corollary 2.13. *For any multigraph G, $I_{\Delta_s(G)^\vee}$ is vertex splittable and then has linear quotients and a linear resolution.*

Proof. In light of Theorem 2.7, $\Delta_s(G)$ is vertex decomposable. Therefore, [**8,** Theorems 2.3, 2.4] ensure that $I_{\Delta_s(G)^\vee}$ is vertex splittable and has linear quotients. By Remark 2.11, $I_{\Delta_s(G)^\vee}$ is generated by monomials of the same degree. Hence, [**5,** Lemma 5.2] implies that $I_{\Delta_s(G)^\vee}$ has linear resolution, as desired. □

By the multigraph gained in Example 2.3 we present an ideal which has linear resolution and quotients as follows.

Example 2.14. Assume that G is the multigraph in Example 2.3. Then,

$$I_{\Delta_s(G)^\vee} = (x_4x_5,\, x_3x_5,\, x_2x_4,\, x_2x_3,\, x_2x_5,\, x_1x_5,\, x_1x_4,\, x_1x_3).$$

Further, in view of Corollary 2.13, I is vertex splittable, has linear quotients and a linear resolution.

3. Regularity and projective dimension of the Stanley-Reisner ring of a spanning simplicial complex. This section is devoted to characterizing some invariants of the Stanley-Reisner ring $K[\Delta_s(G)]$ in terms of the invariants of G.

We begin with the next theorem which immediately follows from Lemma 2.6 and [**8,** Corollary 2.11]. Note that, if G is a tree, then $I_{\Delta_s(G)} = 0$; thus, we consider connected multigraphs with at least one cycle.

Theorem 3.1. *Let G be a multigraph with at least one cycle and e an edge of a cycle in G. Set $\Delta_1 = \Delta_s(G - e)$ and $\Delta_2 = \Delta_s(G/e)$. Then, we have the following statements*:

(i) $\beta_{i,j}(I_{\Delta_s(G)^\vee}) = \beta_{i,j-1}(I_{\Delta_1^\vee}) + \beta_{i,j}(I_{\Delta_2^\vee}) + \beta_{i-1,j-1}(I_{\Delta_2^\vee})$.
(ii) $\mathrm{pd}(K[\Delta_s(G)]) = \max\{\mathrm{pd}(K[\Delta_1]) + 1,\, \mathrm{pd}(K[\Delta_2])\}$.
(iii) $\mathrm{reg}(K[\Delta_s(G)]) = \max\{\mathrm{reg}(K[\Delta_1]),\, \mathrm{reg}(K[\Delta_2]) + 1\}$.

The next theorem, another main result of our paper, ties together graph theory and commutative algebra, in some sense.

Theorem 3.2. *Assume that G is a connected multigraph. Then, we have*

$$\begin{aligned}\dim(K[\Delta_s(G)]) &= |V(G)| - 1,\\ \operatorname{pd}(K[\Delta_s(G)]) &= \operatorname{bigheight}(I_{\Delta_s(G)})\\ &= \operatorname{ht}(I_{\Delta_s(G)})\\ &= |E(G)| - |V(G)| + 1.\end{aligned}$$

Proof. It is well known that $\dim(K[\Delta_s(G)]) = \dim(\Delta_s(G)) + 1$. Therefore, Remark 2.4 ensures that $\dim(K[\Delta_s(G)]) = |V(G)| - 1$, as desired.

In view of Corollary 2.8, $K[\Delta_s(G)]$ is Cohen-Macaulay. Therefore, $I_{\Delta_s(G)}$ is unmixed and

$$\operatorname{pd}(K[\Delta_s(G)]) = \operatorname{bigheight}(I_{\Delta_s(G)}) = \operatorname{ht}(I_{\Delta_s(G)}),$$

by **[9,** Corollary 3.33]. Also,

$$\operatorname{ht}(I_{\Delta_s(G)}) = |E(G)| - |V(G)| + 1,$$

which completes the proof. □

In graph theory, the circuit rank, cyclomatic number or nullity of a graph G is the minimum number r of edges to remove all its cycles from G, making it into a forest. It may easily be seen that $r = |E(G)| - |V(G)| + c$, where c is the number of connected components of G. The set of edges, the removal of which leaves G acyclic, may be found as the complement of a spanning forest of G. The minimum cardinality of such a set is, in fact, the circuit rank of G. The cyclomatic number is also the dimension of the cycle space of G, see [**2**].

Topologically, G may be viewed as an example of a one-dimensional simplicial complex, and its cyclomatic number is the rank of the first homology group of this complex, see [**10**]. Also, the circuit rank controls the number of ears in an ear decomposition of a graph, see [**15**]. Hence, Theorem 3.2 investigates the circuit rank of a connected multigraph from an algebraic point of view. In fact, it shows that the circuit rank of a connected multigraph G is also the projective dimension of the Stanley-Reisner ring of the spanning simplicial complex associated to G.

The next corollary, which immediately follows from Theorem 3.2, may be valuable in turn.

Corollary 3.3. *Assume that G is a connected multigraph, $E(G) = \{x_1, \ldots, x_m\}$ and $R = K[x_1, \ldots, x_m]$, where K is a field.*

(i) *G is a tree if and only if $K[\Delta_s(G)]$ is a projective (free) R-module.*

(ii) *If G has exactly r cycles such that no two cycles share a common edge, i.e., all cycles in G are induced, then*

$$\mathrm{pd}(K[\Delta_s(G)]) = r.$$

(iii) *If G is an r-cyclic multigraph with a common edge, then $\mathrm{pd}(K[\Delta_s(G)]) = r$.*

(iv) *Let G be an r-cyclic multigraph with a common edge and trees rooted on its vertices, that is, there is an r-cyclic graph H and trees $T_1, \ldots, T_k$, such that*

$$G = H \cup T_1 \cup \cdots \cup T_k,$$

where, for each distinct integer $1 \leq i,\ j \leq k$,

$$V(T_i) \cap V(T_j) = \emptyset \quad \text{and} \quad |V(T_i) \cap V(H)| = 1.$$

Then, $\mathrm{pd}(K[\Delta_s(G)]) = r$.

Proof.

(i) A connected multigraph G is a tree if and only if $|E(G)| = |V(G)| - 1$ and, by Theorem 3.2, it holds if and only if $\mathrm{pd}(K[\Delta_s(G)]) = 0$. Also, if G is a tree, then $I_{\Delta_s(G)} = 0$; thus, $K[\Delta_s(G)] = R$ is a free R-module.

(ii) It may easily be seen that, in this case, $|E(G)| = |V(G)| + (r-1)$. Therefore, Theorem 3.2 shows that $\mathrm{pd}(K[\Delta_s(G)]) = r$.

(iii) If G is an r-cyclic multigraph with a common edge $G_{t_1,\ldots,t_r}$, then we have

$$V(G) = \sum_{i=1}^{r}(t_i - 2) + 2 \quad \text{and} \quad E(G) = \sum_{i=1}^{r}(t_i - 1) + 1.$$

Thus, by means of Theorem 3.2, we have $\mathrm{pd}(K[\Delta_s(G)]) = r$, as desired.

(iv) In view of Theorem 3.2, if G is an r-cyclic multigraph with a common edge and trees rooted on its vertices, then

$$\mathrm{pd}(K[\Delta_s(G)]) = \mathrm{pd}(K[\Delta_s(G_{t_1,\ldots,t_r})]).$$

Therefore, the result follows from (iii). $\square$

We conclude this paper with the following result concerning the regularity of the Stanley-Reisner ring of the spanning simplicial complex associated to special multigraphs.

Corollary 3.4.

(i) *Assume that G is a connected multigraph which has exactly r cycles $C_1, \ldots, C_r$ such that no two cycles share a common edge. Then,*

$$\mathrm{reg}(K[\Delta_s(G)]) = \sum_{i=1}^{r} |E(C_i)| - r.$$

(ii) *If G is an r-cyclic multigraph $G_{t_1,\ldots,t_r}$ with a common edge and* (*possible*) *trees rooted on its vertices, then*

$$\mathrm{reg}(K[\Delta_s(G)]) = \sum_{i=1}^{r}(t_i - 2) + 1 = |E(G_{t_1,\ldots,t_r})| - r.$$

Proof.

(i) If $r = 0$, then G is a tree and $I_{\Delta_s(G)} = 0$. Thus,

$$\mathrm{reg}(K[\Delta_s(G)]) = 0 = \sum_{i=1}^{r} |E(C_i)| - r.$$

Therefore, we assume that $r \geq 1$ and use induction on $|E(G)|$. Since G has at least one cycle and no loop, then $|E(G)| \geq 2$. If $|E(G)| = 2$, then G is a cycle of length 2. Let $E(G) = \{x_1, x_2\}$. Thus, $I_{\Delta_s(G)} = (x_1x_2)$, and then, $\mathrm{reg}(K[\Delta_s(G)]) = 1$. Also, $|E(C_1)| - 1 = 2 - 1 = 1$. Therefore,

$$\mathrm{reg}(K[\Delta_s(G)]) = |E(C_1)| - 1.$$

Assume that $|E(G)| > 2$, and let $e \in E(C_r)$, $\Delta_1 = \Delta_s(G - e)$ and $\Delta_2 = \Delta_s(G/e)$. Then, by Theorem 3.1,

$$\mathrm{reg}(K[\Delta_s(G)]) = \max\{\mathrm{reg}(K[\Delta_1]), \mathrm{reg}(K[\Delta_2]) + 1\}.$$

Note that $G-e$ has exactly $r-1$ cycles $C_1,\dots,C_{r-1}$. Then,

$$\operatorname{reg}(K[\Delta_1]) = \sum_{i=1}^{r-1} |E(C_i)| - (r-1).$$

In addition, G/e has exactly r cycles $C_1,\dots,C_{r-1},C_r/e$. Thus,

$$\operatorname{reg}(K[\Delta_2]) = \sum_{i=1}^{r-1} |E(C_i)| + |E(C_r/e)| - r.$$

Since $|E(C_r/e)| = |E(C_r)|-1$, we have

$$\operatorname{reg}(K[\Delta_2]) = \sum_{i=1}^{r} |E(C_i)| - r - 1.$$

Therefore,

$$\operatorname{reg}(K[\Delta_s(G)]) = \max\left\{\sum_{i=1}^{r-1} |E(C_i)| - (r-1), \sum_{i=1}^{r} |E(C_i)| - r\right\}.$$

The result now follows from the obvious inequality

$$\sum_{i=1}^{r-1} |E(C_i)| - (r-1) \le \sum_{i=1}^{r} |E(C_i)| - r.$$

(ii) Next, note that, if G is an r-cyclic multigraph $G_{t_1,\dots,t_r}$ with a common edge and trees possibly rooted on its vertices, then $\operatorname{reg}(K[\Delta_s(G)]) = \operatorname{reg}(K[\Delta_s(G_{t_1,\dots,t_r})])$. Thus, we can assume that G is an r-cyclic multigraph $G_{t_1,\dots,t_r}$ with a common edge. We prove the assertion by induction on $|E(G)|$. If $|E(G)|=2$, then G is a cycle graph of length 2 and, as was shown in (i), $\operatorname{reg}(K[\Delta_s(G)]) = 1 = (t_1-2)+1$.

Assume that $|E(G)|>2$, and the result is true for all r'-cyclic multigraphs G' with $|E(G')| < |E(G_{t_1,\dots,t_r})|$. If $r=1$, then G is a cycle graph; thus, by (i), $\operatorname{reg}(K[\Delta_s(G)]) = |E(G)|-1 = (t_1-1)$. Also, if $t_1=\cdots=t_r=2$, then $\Delta_s(G)$ is a zero-dimensional simplicial complex and

$$\operatorname{reg}(K[\Delta_s(G)]) = 1 = \sum_{i=1}^{r}(t_i-2)+1.$$

Therefore, without loss of generality, assume that $r > 1$ and $t_r > 2$. Let e be an edge on the cycle G_{t_r}, which is not the common edge. Then, we have

$$\operatorname{reg}(K[\Delta_s(G-e)]) = \operatorname{reg}(K[\Delta_s(G_{t_1,\dots,t_{r-1}})])$$

and

$$\Delta_s(G/e) = \Delta_s(G_{t_1,\dots,t_{r-1},t_r-1}).$$

Hence, by Theorem 3.1, we have

$$\begin{aligned}\operatorname{reg}(K[\Delta_s(G)]) = \max\{&\operatorname{reg}(K[\Delta_s(G_{t_1,\dots,t_{r-1}})]),\\ &\operatorname{reg}(K[\Delta_s(G_{t_1,\dots,t_{r-1},t_r-1})]) + 1\}.\end{aligned}$$

By the induction hypothesis, we have

$$\operatorname{reg}(K[\Delta_s(G_{t_1,\dots,t_{r-1}})]) = \sum_{i=1}^{r-1}(t_i-2)+1 < \sum_{i=1}^{r}(t_i-2)+1,$$

since $t_r > 2$, and

$$\begin{aligned}\operatorname{reg}(K[\Delta_s(G_{t_1,\dots,t_{r-1},t_r-1})]) &= \sum_{i=1}^{r-1}(t_i-2)+(t_r-1-2)+1\\ &= \sum_{i=1}^{r}(t_i-2).\end{aligned}$$

Thus,

$$\operatorname{reg}(K[\Delta_s(G)]) = \sum_{i=1}^{r}(t_i-2)+1. \quad \square$$

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REFERENCES

1. I. Anwar, Z. Raza and A. Kashif, *Spanning simplicial complexes of uni-cyclic graphs*, Alg. Colloq. **22** (2015), 707–710.

2. C. Berge, *The theory of graphs*, Dover Publications, Inc., Mineola, NY, 2001.

3. ———, *Graphs and hypergraphs*, North-Holland Math. Libr. **6**, Elsevier Publishing Co., Inc., New York, 1976.

4. A. Björner and M.L. Wachs, *Shellable nonpure complexes and posets*, II, Trans. Amer. Math. Soc. **349** (1997), 3945–3975.

5. S. Faridi, *Simplicial trees are sequentially Cohen-Macaulay*, J. Pure Appl. Alg. **190** (2004), 121–136.

6. H.T. Hà, S. Morey and R.H. Villarreal, *Cohen-Macaulay admissible clutters*, J. Comm. Alg. **1** (2009), 463–480.

7. A. Kashif, I. Anwar and Z. Raza, *On the algebraic study of spanning simplicial complexes of r-cyclic graphs $G_{n,r}$*, ARS Combin. **115** (2014), 89–99.

8. S. Moradi and F. Khosh-Ahang, *On vertex decomposable simplicial complexes and their Alexander duals*, Math. Scandinavia **118** (2016), 43–56.

9. S. Morey and R.H. Villarreal, *Edge ideals*: *Algebraic and combinatorial properties*, Progr. Comm. Alg. **1**, de Gruyter, Berlin, 2012.

10. J.P. Serre, *Trees*, Springer Mono. Math., Springer-Verlag, Berlin, 2003.

11. R.P. Stanley, *Combinatorics and commutative algebra*, Progr. Math. **41**, Birkhäuser, Inc., Boston, 1996.

12. A. Van Tuyl, *Sequentially Cohen-Macaulay bipartite graphs*: *Vertex decomposability and regularity*, Arch. Math. (Basel) **93** (2009), 451–459.

13. R.H. Villarreal, *Monomial algebras*, Mono. Res. Notes Math., CRC Press, Boca Raton, 2015.

14. D.B. West, *Introduction to graph theory*, Prentice Hall, Inc., Upper Saddle River, NJ, 1996.

15. H. Whitney, *Non-separable and planar graphs*, Trans. Amer. Math. Soc. **34** (1932), 339–362.

16. R. Woodroofe, *Vertex decomposable graphs and obstructions to shellability*, Proc. Amer. Math. Soc. **137** (2009), 3235–3246.

17. G. Zhu, F. Shi and Y. Geng, *Spanning simplicial complexes of n-cyclic graphs with a common edge*, Int. Electr. J. Alg. **15** (2014), 132–144.

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