

THE C^* -ALGEBRA GENERATED BY IRREDUCIBLE TOEPLITZ AND COMPOSITION OPERATORS

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ABSTRACT. We describe the C^* -algebra generated by an irreducible Toeplitz operator T_ψ , with continuous symbol ψ on the unit circle \mathbb{T} , and finitely many composition operators on the Hardy space H^2 induced by certain linear fractional self-maps of the unit disc, modulo the ideal of compact operators $K(H^2)$. For specific automorphism-induced composition operators and certain types of irreducible Toeplitz operators, we show that the above C^* -algebra is not isomorphic to that generated by the shift and composition operators.

1. Introduction. The Hardy space $H^2 = H^2(\mathbb{D})$ is the collection of all analytical functions f on the open unit disk \mathbb{D} satisfying the norm condition

$$\|f\|^2 := \lim_{r \rightarrow 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

For any analytic self-map φ of the open unit disk \mathbb{D} , a bounded composition operator on H^2 is defined by

$$C_\varphi : H^2 \rightarrow H^2, \quad C_\varphi(f) = f \circ \varphi.$$

If $f \in H^2$, then the radial limit $f(e^{i\theta}) := \lim_{r \rightarrow 1^-} f(re^{i\theta})$ exists almost everywhere on the unit circle \mathbb{T} . Hence, we can consider H^2 as a subspace of $L^2(\mathbb{T})$. Let ϕ be a bounded measurable function on \mathbb{T} and P_{H^2} the orthogonal projection of $L^2(\mathbb{T})$ (associated with normalized arc-length measure on \mathbb{T}) onto H^2 . The Toeplitz operator T_ϕ is defined on H^2 by $T_\phi f = P_{H^2}(\phi f)$ for all $f \in H^2$. Coburn [4, 5] showed that the unital C^* -algebra $C^*(T_z)$ generated by the unilateral shift operator T_z contains compact operators on H^2 as an ideal, and every element

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$a \in C^*(T_z)$ has a unique representation $a = T_\phi + k$ for some $\phi \in C(\mathbb{T})$ and $k \in \mathfrak{K} := K(H^2)$. He showed that $C^*(T_z)/\mathfrak{K}$ is *-isomorphic to $C(\mathbb{T})$ and determines the essential spectrum of Toeplitz operators with continuous symbol.

Recently, the unital C^* -algebra generated by the shift operator T_z and the composition operator C_φ for a linear fractional self-map φ of \mathbb{D} have been studied. For a linear fractional self-map φ of \mathbb{D} , if $\|\varphi\|_\infty < 1$, then C_φ is a compact operator on H^2 [16]. Therefore, those linear fractional self-maps φ which satisfy $\|\varphi\|_\infty = 1$ should be considered. If, moreover, φ is an automorphism of \mathbb{D} , then $C^*(T_z, C_\varphi)/\mathfrak{K}$ is *-isomorphic to the crossed products $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ or $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}_m$ [7, 8]. When φ is not an automorphism, there are three different cases:

- (i) φ has only one fixed point γ which is on the unit circle \mathbb{T} , i.e., φ is a parabolic map [15]. In this case, $C^*(T_z, C_\varphi)/\mathfrak{K}$ is a commutative C^* -algebra and *-isomorphic to the minimal unitization of $C_\gamma(\mathbb{T}) \oplus C_0([0, 1])$, where $C_\gamma(\mathbb{T})$ is the set of functions in $C(\mathbb{T})$ vanishing at $\gamma \in \mathbb{T}$ and $C_0([0, 1])$ is the set of all $f \in C([0, 1])$ vanishing at zero.
- (ii) φ has a fixed point $\gamma \in \mathbb{T}$ and fixes another point in $\mathbb{C} \cup \{\infty\}$, equivalently, φ has a fixed point $\gamma \in \mathbb{T}$ and $\varphi'(\gamma) \neq 1$ [15]. In this case, $C^*(T_z, C_\varphi)/\mathfrak{K}$ is *-isomorphic to the minimal unitization of $C_\gamma(\mathbb{T}) \oplus (C_0([0, 1]) \rtimes \mathbb{Z})$.
- (iii) φ fixes no point of \mathbb{T} , but there exist distinct points $\gamma, \eta \in \mathbb{T}$ with $\varphi(\gamma) = \eta$ [9]. In this case, $C^*(T_z, C_\varphi)/\mathfrak{K}$ is the C^* -subalgebra \mathcal{D} of $C(\mathbb{T}) \oplus M_2(C([0, 1]))$ defined by

$$\mathcal{D} = \left\{ (f, V) \in C(\mathbb{T}) \oplus M_2(C([0, 1])) : V(0) = \begin{bmatrix} f(\gamma) & 0 \\ 0 & f(\eta) \end{bmatrix} \right\}.$$

The study of C^* -algebras generated by Toeplitz and composition operators may be employed to calculate essential spectrum of certain combinations of composition operators and their adjoints with Toeplitz operators [9, 10]. In addition, these algebras are interesting on their own, as they provide examples of nuclear C^* -algebras (this follows from the above observations).

This paper generalizes some of the above results. The generalization is two fold. We replace the shift operator by an irreducible Toeplitz operator with continuous symbols on the unit circle and a single composition operator with finitely many composition operators on the

Hardy space H^2 induced by certain linear fractional self-maps of the unit disk.

The paper is organized as follows. In Section 2, we review basic facts and known results which are used later in the paper. In Section 3, we find the C^* -algebras $C^*(T_\psi, C_{\varphi_1}, \dots, C_{\varphi_n})/\mathfrak{K}$ and $C^*(T_\psi, C_\varphi)/\mathfrak{K}$ and obtain more general results in the above cases, where $\varphi_1, \dots, \varphi_n$ are as in case (ii), T_ψ is an irreducible Toeplitz operator with continuous symbols on \mathbb{T} and φ is as in cases (i) or (iii). When φ is an automorphism, the composition operator C_φ often generates the unilateral shift operator T_z . In Section 4, we investigate the C^* -algebra generated by a composition operator induced by a rotation and an irreducible Toeplitz operator with a symbol whose range is invariant under this rotation.

2. Preliminaries. In this section, we review some of the known results which are used in the next sections. Here, we use $C_0([0, 1])$ to denote the set of functions in $C([0, 1])$ vanishing at zero and $[T]$ to denote the coset of operator $T \in B(H^2)$ in the Calkin algebra $B(H^2)/K(H^2)$.

A linear fractional self-map ρ of \mathbb{D} with fixed point $\gamma \in \mathbb{T}$ is *parabolic* if and only if $\rho'(\gamma) = 1$. In this case, ρ is conjugate to translation on the right half plane $\Omega := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ via the conformal map $\alpha : z \mapsto (\gamma + z)/(\gamma - z)$ of \mathbb{D} onto Ω . Therefore, $\alpha \circ \rho \circ \alpha^{-1}$ is the translation map $z \mapsto z + a$ for some $a \in \mathbb{C}$ with non-negative real part. We denote the map ρ by $\rho_{\gamma,a}$. This is an automorphism of \mathbb{D} if and only if $\operatorname{Re} a = 0$. For $\gamma \in \mathbb{T}$, the set $\mathbb{P}_\gamma := \{C_{\rho_{\gamma,a}} : \operatorname{Re} a > 0\}$ consists of all composition operators induced by non automorphic parabolic self-maps of \mathbb{D} fixing γ . If φ is a non automorphic parabolic self-map of \mathbb{D} , then C_φ is irreducible [12], and $C_\varphi^* C_\varphi - C_\varphi C_\varphi^*$ is a non-zero compact operator (C_φ is essentially normal) [1]. Therefore, the unital C^* -algebra $C^*(\mathbb{P}_\gamma)$ is irreducible and $C^*(\mathbb{P}_\gamma) \cap \mathfrak{K} \neq \{0\}$. By [13, Theorem 2.4.9], $C^*(\mathbb{P}_\gamma)$ contains all compact operators on H^2 . Since the elements of \mathbb{P}_γ satisfy $\rho_{\gamma,a} \circ \rho_{\gamma,b} = \rho_{\gamma,a+b}$, $C^*(\mathbb{P}_\gamma)/\mathfrak{K}$ is a unital commutative C^* -algebra.

The next theorem completely describes this C^* -algebra.

Theorem 2.1. [11, Theorem 3, Corollary 2]. *There is a unique $*$ -isomorphism $\Sigma : C([0, 1]) \rightarrow C^*(\mathbb{P}_\gamma)/\mathfrak{K}$ such that $\Sigma(x^a) = [C_{\rho_{\gamma,a}}]$ for $\operatorname{Re} a > 0$. Moreover if ρ is a parabolic non automorphic self-map of \mathbb{D} fixing γ , then $C^*(C_\rho) = C^*(\mathbb{P}_\gamma)$.*

For a linear fractional self-map φ of \mathbb{D} ,

$$U_\varphi = C_\varphi(C_\varphi^*C_\varphi)^{-1/2}$$

is the partial isometry in the polar decomposition of C_φ . Since both C_φ and C_φ^* are injective, U_φ is a unitary operator.

The following results are useful in finding the C^* -algebra generated by Toeplitz operators and composition operators induced by linear fractional self-maps of \mathbb{D} .

Theorem 2.2. [9, Section 4]. *Suppose that φ is a linear fractional non automorphic self-map of \mathbb{D} sending $\gamma \in \mathbb{T}$ to $\eta \in \mathbb{T}$. For every $f \in C(\mathbb{T})$, there exist compact operators k and k' such that*

$$T_f C_\varphi = f(\gamma)C_\varphi + k, \quad C_\varphi T_f = f(\eta)C_\varphi + k'.$$

Theorem 2.3. [7]. *Let φ_1 and φ_2 be automorphisms of \mathbb{D} and $f \in C(\mathbb{T})$. If U_{φ_1} and U_{φ_2} are unitary parts of the polar decomposition of C_{φ_1} and C_{φ_2} , respectively, then the operators $U_{\varphi_1}U_{\varphi_2} - U_{\varphi_2 \circ \varphi_1}$ and $U_{\varphi_1}T_fU_{\varphi_1}^* - T_{f \circ \varphi_1}$ are compact.*

For $\gamma \in \mathbb{T}$ and a positive real number t , following [15], we consider the automorphism $\Psi_{\gamma,t}$ of \mathbb{D} defined by

$$\Psi_{\gamma,t}(z) = \frac{(1+t)z + (1-t)\gamma}{(1-t)\bar{\gamma}z + (1+t)},$$

which fixes γ and satisfies $\Psi'_{\gamma,t}(\gamma) = t$. The set $\{\Psi_{\gamma,t} : t > 0\}$ is an abelian group as $\Psi_{\gamma,t_1} \circ \Psi_{\gamma,t_2} = \Psi_{\gamma,t_1 t_2}$ for all $t_1, t_2 > 0$.

If $\varphi, \varphi_1, \dots, \varphi_n$ are linear fractional non automorphic self-maps of \mathbb{D} fixing $\gamma \in \mathbb{T}$, by [15, equation (4.6)]

$$\begin{aligned} (2.1) \quad C^*(C_{\varphi_1}, \dots, C_{\varphi_1}, \mathfrak{K}) &= C^*(\{C_{\rho_{\gamma,a}} U_{\Psi_{\gamma,\varphi'_1(\gamma)}^{m_1} \dots \varphi'_n(\gamma)^{m_n}} : \\ &\quad \text{Re } a > 0, (m_1, \dots, m_n) \in \mathbb{Z}^n \}). \end{aligned}$$

In particular,

$$(2.2) \quad C^*(C_\varphi, \mathfrak{K}) = C^*(\{C_{\rho_{\gamma,a}} U_{\Psi_{\gamma,\varphi'(\gamma)}^n} : \text{Re } a > 0, n \in \mathbb{Z}\}).$$

Theorem 2.4. [15, Theorem 4.4]. *Let G be a collection of automorphisms of \mathbb{D} that fix $\gamma \in \mathbb{T}$. If G is an abelian group and $\eta'(\gamma) \neq 1$ for all $\eta \in G \setminus \{\text{id}\}$, then $C^*(\{[C_{\rho_{\gamma,a}} U_\eta] : \text{Re } a > 0, \eta \in G\})$ is $*$ -isomorphic to the minimal unitization of $C_0([0, 1]) \rtimes_\alpha G$ where the action $\alpha : G \rightarrow \text{Aut}(C_0([0, 1]))$ is defined by $\alpha_\eta(f)(x) = f(x\eta'(\gamma))$ for $\eta \in G, f \in C_0([0, 1])$ and $x \in [0, 1]$.*

By (2.1), (2.2) and Theorem 2.4, the C^* -algebras $C^*(C_{\varphi_1}, \dots, C_{\varphi_n}, \mathfrak{K})/\mathfrak{K}$ and $C^*(C_\varphi, \mathfrak{K})/\mathfrak{K}$ are determined as follows.

Corollary 2.5. [15, Theorems 4.6, 4.7]. *Let $\varphi, \varphi_1, \dots, \varphi_n$ be linear fractional non automorphic self-maps of \mathbb{D} that fix $\gamma \in \mathbb{T}, \varphi'(\gamma) \neq 1$ and $\ln \varphi'_1(\gamma), \dots, \ln \varphi'_n(\gamma)$ linearly independent over \mathbb{Z} . Define the actions $\alpha : \mathbb{Z} \rightarrow \text{Aut}(C_0([0, 1]))$ and $\alpha' : \mathbb{Z}^n \rightarrow \text{Aut}(C_0([0, 1]))$ by $\alpha_n(f)(x) = f(x\varphi'(\gamma)^n)$ and $\alpha'_{(m_1, \dots, m_n)}(f)(x) = f(x\varphi'_1(\gamma)^{m_1} \dots \varphi'_n(\gamma)^{m_n})$, respectively, for $f \in C_0([0, 1]), n \in \mathbb{Z}, (m_1, \dots, m_n) \in \mathbb{Z}^n$ and $x \in [0, 1]$. Then $C^*(C_\varphi, \mathfrak{K})/\mathfrak{K}$ and $C^*(C_{\varphi_1}, \dots, C_{\varphi_n}, \mathfrak{K})/\mathfrak{K}$ are $*$ -isomorphic to the minimal unitizations of $C_0([0, 1]) \rtimes_\alpha \mathbb{Z}$ and $C_0([0, 1]) \rtimes_{\alpha'} \mathbb{Z}^n$, respectively.*

3. Irreducible Toeplitz operators. Let X be a compact Hausdorff space and \mathcal{A} a C^* -subalgebra of $C(X)$ containing the constants. For $x, y \in X$, set $x \sim y$ if and only if $f(x) = f(y)$ for all f in \mathcal{A} . Then \sim is an equivalence relation on X . Let $[x]$ denote the equivalence class of x , let $[X]$ be the quotient space and equip $[X]$ with the weak topology induced by \mathcal{A} . Let X/\sim be the quotient space equipped with the quotient topology. Then \mathcal{A} is $*$ -isomorphic to $C([X])$ and a C^* -subalgebra of $C(X/\sim)$ via $f \mapsto \tilde{f}$ where $\tilde{f}([x]) := f(x)$ for $x \in X$. Note that $[X]$ is always Hausdorff, and it is homeomorphic to X/\sim if and only if the latter is Hausdorff [5].

If D is an irreducible C^* -subalgebra of $C^*(T_z)$, then it contains a nonzero compact operator [5]. Hence, D contains all of compact operators on H^2 [13, Theorem 2.4.9]. We set

$$D_0 = \{f \in C(\mathbb{T}) : T_f \in D\}.$$

Theorem 3.1. [5, Theorems 2, 5]. *If D is an irreducible C^* -subalgebra of $C^*(T_z)$, then D_0 is a C^* -subalgebra of $C(\mathbb{T})$ and D/\mathfrak{K} is*

**-isomorphic to $C([\mathbb{T}])$, where $[\mathbb{T}]$ is the quotient with respect to the equivalence relation induced by D_0 .*

Let T_ψ be an irreducible Toeplitz operator, i.e., the only closed vector subspaces of H^2 are 0 and H^2 which reduce for T_ψ , with continuous symbol ψ , see [2, 4, 5, 14]. Then, $D = C^*(T_\psi)$ is irreducible, and by [5, Theorem 6], D_0 is the C^* -subalgebra of $C(\mathbb{T})$ generated by ψ . Hence, by Theorem 3.1, D/\mathfrak{K} is *-isomorphic to $C([\mathbb{T}])$ where $[\mathbb{T}]$ is the quotient with respect to the equivalence relation induced by ψ , that is, $x \sim y$ if and only if $\psi(x) = \psi(y)$.

Note that T_z is irreducible and there are other irreducible Toeplitz operators, for example, see [14, Examples 1, 2]. If $D = C^*(T_\psi) = C^*(T_z)$ for some continuous function ψ , then $D_0 = C(\mathbb{T})$ is generated by ψ . By the Stone-Weierstrass theorem, ψ must be one-to-one on the unit circle. Therefore, we are interested in the case that ψ is not one-to-one on \mathbb{T} .

Quertermous [15] showed that, if φ is a linear fractional non automorphic self-map of \mathbb{D} fixing $\gamma \in \mathbb{T}$, then $C^*(T_z, C_\varphi)/\mathfrak{K}$ is *-isomorphic to the minimal unitization of $C_\gamma(\mathbb{T}) \oplus C_0([0, 1])$, where $C_\gamma(\mathbb{T})$ is the set of all $f \in C(\mathbb{T})$ vanishing at γ . We extend this result to an arbitrary irreducible Toeplitz operator with continuous symbol on the unit circle, instead of the shift operator, and finitely many composition operators induced by linear fractional non automorphic self-maps of \mathbb{D} with a common fixed point on the unit circle. Our approach is similar to that of Quertermous [15]. As in the previous section, we use the notation $[T]$ for the coset of T in the Calkin algebra. Let t_1, \dots, t_n be nonzero positive real numbers, $\gamma \in \mathbb{T}$ and Σ the map defined in Theorem 2.1. Consider

$$\mathcal{N}_{\gamma, t_1, \dots, t_n} = \{\Sigma(g)[U_{\Psi_{\gamma, t_1^{m_1} \dots t_n^{m_n}}}] : g \in C_0([0, 1]), (m_1, \dots, m_n) \in \mathbb{Z}^n\},$$

and let $\mathcal{A}_{\gamma, t_1, \dots, t_n}$ the non-unital C^* -algebra generated by $\mathcal{N}_{\gamma, t_1, \dots, t_n}$. By Theorem 2.1 and the fact that $\Psi_{\gamma, 1}$ is the identity map of \mathbb{D} , $\mathcal{A}_{\gamma, 1, \dots, 1} \cong C_0([0, 1])$.

If $\varphi_1, \dots, \varphi_n$ are linear fractional non automorphic self-maps of \mathbb{D} fixing $\gamma \in \mathbb{T}$, and T_ψ is an irreducible Toeplitz operator with continuous symbol ψ on \mathbb{T} , then

$$(3.1) \quad C^{*}(T_{\psi}, C_{\varphi_1}, \dots, C_{\varphi_n})/\mathfrak{K} = C^{*}(\{[T_{\phi}] : \phi \in C^{*}(\psi)\} \cup \mathcal{A}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)}),$$

by (4.6) and [15, Proof of Theorem 4.4]. Moreover, if $\ln \varphi'_1(\gamma), \dots, \ln \varphi'_n(\gamma)$ are linearly independent over \mathbb{Z} , then Corollary 2.5 implies

$$(3.2) \quad \mathcal{A}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)} \cong C_0([0, 1]) \rtimes_{\alpha'} \mathbb{Z}^n.$$

Let T_{ψ} be an irreducible Toeplitz operator and $[\mathbb{T}]$ the quotient space with respect to the equivalence relation induced by ψ . Set

$$C_{[\gamma]}([\mathbb{T}]) := \{f \in C([\mathbb{T}]) : f([\gamma]) = 0\},$$

and let $\mathcal{B}_{\gamma, t_1, \dots, t_n}$ be the minimal unitization of $C_{[\gamma]}([\mathbb{T}]) \oplus \mathcal{A}_{\gamma, t_1, \dots, t_n}$.

The next lemma is necessary for what follows.

Lemma 3.2. [15, Lemma 6.3]. *If $\gamma \in \mathbb{T}$, $f \in C(\mathbb{T})$ and $A \in \mathcal{A}_{\gamma, t_1, \dots, t_n}$, then*

$$[T_f]A = f(\gamma)A = A[T_f].$$

Moreover, if $[T_f] + A = [0]$, then $f \equiv 0$ and $A = 0$.

Theorem 3.3. *If T_{ψ} is an irreducible Toeplitz operator on Hardy space H^2 with symbol $\psi \in C(\mathbb{T})$ and $\varphi_1, \dots, \varphi_n$ are linear fractional non automorphic self-maps of \mathbb{D} fixing $\gamma \in \mathbb{T}$, then $C^{*}(T_{\psi}, C_{\varphi_1}, \dots, C_{\varphi_n})/\mathfrak{K}$ is $*$ -isomorphic to $\mathcal{B}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)}$.*

Proof. For $t_1, \dots, t_n > 0$, set

$$\mathcal{C}_{\gamma, t_1, \dots, t_n, \psi} := \{[T_{\phi}] + A : \phi \in C^{*}(\psi), A \in \mathcal{A}_{\gamma, t_1, \dots, t_n}\}.$$

By (3.1),

$$C^{*}(T_{\psi}, C_{\varphi_1}, \dots, C_{\varphi_n})/\mathfrak{K} = C^{*}(\mathcal{C}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma), \psi}).$$

We show that $\mathcal{C}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma), \psi}$ is a C^{*} -algebra. It is clear that $\mathcal{C}_{\gamma, t_1, \dots, t_n, \psi}$ is closed when taking the linear combination and adjoint. On the other hand, by Lemma 3.2 and the fact that, for $\phi_1, \phi_2 \in C^{*}(\psi)$, $[T_{\phi_1}][T_{\phi_2}] = [T_{\phi_1\phi_2}]$ (since $T_{\phi_1\phi_2} - T_{\phi_1}T_{\phi_2}$ is a compact operator), $\mathcal{C}_{\gamma, t_1, \dots, t_n, \psi}$ is also closed under multiplication. Hence, $\mathcal{C}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma), \psi}$ is a dense $*$ -subalgebra of $C^{*}(T_{\psi}, C_{\varphi_1}, \dots, C_{\varphi_n})/\mathfrak{K}$. Similar to [15,

Proof of Theorem 6.4], we define the map

$$\mathcal{F} : \mathcal{B}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)} \longrightarrow \mathcal{C}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma), \psi}$$

by

$$\mathcal{F}((\tilde{\phi} - \tilde{\phi}([\gamma]), A) + \tilde{\phi}([\gamma])I) = [T_\phi] + A,$$

for $\phi \in C^*(\psi)$ and $A \in \mathcal{A}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)}$. By Lemma 3.2, \mathcal{F} is an injective $*$ -homomorphism and its image is $\mathcal{C}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma), \psi} = C^*(T_\psi, C_{\varphi_1}, \dots, C_{\varphi_n})/\mathfrak{K}$. \square

The following results are straightforward consequences of Theorems 2.1, 3.3, equation (3.2) and Corollary 2.5.

Corollary 3.4. *If T_ψ is irreducible with symbol ψ in $C(\mathbb{T})$ and ρ is a parabolic non automorphic self-map of \mathbb{D} fixing $\gamma \in \mathbb{T}$, then $C^*(T_\psi, C_\rho)/\mathfrak{K}$ is $*$ -isomorphic to the minimal unitization of $C_{[\gamma]}([\mathbb{T}]) \oplus C_0([0, 1])$.*

Corollary 3.5. *If T_ψ is irreducible with symbol ψ in $C(\mathbb{T})$ and φ is a linear fractional non automorphic self-map of \mathbb{D} fixing $\gamma \in \mathbb{T}$ such that $\varphi'(\gamma) \neq 1$, then $C^*(T_\psi, C_\varphi)/\mathfrak{K}$ is $*$ -isomorphic to the minimal unitization of $C_{[\gamma]}([\mathbb{T}]) \oplus (C_0([0, 1]) \rtimes_\alpha \mathbb{Z})$, where the action α is defined as in Corollary 2.5.*

Corollary 3.6. *If T_ψ is irreducible with symbol ψ in $C(\mathbb{T})$ and $\varphi_1, \dots, \varphi_n$ are linear fractional non automorphic self-maps of \mathbb{D} fixing $\gamma \in \mathbb{T}$ such that $\ln \varphi'_1(\gamma), \dots, \ln \varphi'_n(\gamma)$ are linearly independent over \mathbb{Z} , then $C^*(T_\psi, C_{\varphi_1}, \dots, C_{\varphi_n})/\mathfrak{K}$ is $*$ -isomorphic to the minimal unitization of $C_{[\gamma]}([\mathbb{T}]) \oplus (C_0([0, 1]) \rtimes_{\alpha'} \mathbb{Z}^n)$, where the action α' is defined as in Corollary 2.5.*

Now consider the case where φ is a linear fractional non automorphic self-map of \mathbb{D} such that $\varphi(\gamma) = \eta$ for some $\gamma \neq \eta \in \mathbb{T}$. Kriete, MacCluer and Moorhouse investigated this case in [9]. We summarize their results as follows.

Theorem 3.7. [9] *Let φ be a linear fractional non automorphic self-map of \mathbb{D} with $\varphi(\gamma) = \eta$ for some distinct points $\gamma, \eta \in \mathbb{T}$. Then,*

for every $a \in C^*(T_z, C_\varphi)/\mathfrak{K}$, there is a unique $\omega \in C(\mathbb{T})$ and unique functions f, g, h and k in $C_0([0, 1])$ such that

$$a = [T_\omega] + f([C_\varphi^* C_\varphi]) + g([C_\varphi C_\varphi^*]) + [U_\varphi]h([C_\varphi^* C_\varphi]) + [U_\varphi^*]k([C_\varphi C_\varphi^*]).$$

Moreover, the map $\Phi : C^*(T_z, C_\varphi)/\mathfrak{K} \rightarrow C(\mathbb{T}) \oplus M_2(C([0, 1]))$ defined by

$$\Phi(a) = \left(\omega, \begin{bmatrix} \omega(\gamma) + g & h \\ k & \omega(\eta) + f \end{bmatrix} \right)$$

is a $*$ -isomorphism of $C^*(T_z, C_\varphi)/\mathfrak{K}$ onto the following C^* -subalgebra of $C(\mathbb{T}) \oplus M_2(C([0, 1]))$,

$$\mathcal{D} = \left\{ (\omega, V) \in C(\mathbb{T}) \oplus M_2(C([0, 1])) : V(0) = \begin{bmatrix} \omega(\gamma) & 0 \\ 0 & \omega(\eta) \end{bmatrix} \right\}.$$

The shift operator is replaced by an arbitrary irreducible Toeplitz operator T_ψ with continuous symbol.

Theorem 3.8. *Let φ be a linear fractional non automorphic self-map of \mathbb{D} such that $\varphi(\gamma) = \eta$ for distinct points $\gamma, \eta \in \mathbb{T}$ and T_ψ irreducible with continuous symbol ψ on \mathbb{T} . Then, every element b in $C^*(T_\psi, C_\varphi)/\mathfrak{K}$ has a unique representation of the form*

$$b = [T_\omega] + f([C_\varphi^* C_\varphi]) + g([C_\varphi C_\varphi^*]) + [U_\varphi]h([C_\varphi^* C_\varphi]) + [U_\varphi^*]k([C_\varphi C_\varphi^*]),$$

where $\omega \in C^*(\psi)$ and f, g, h and k are in $C_0([0, 1])$. Moreover, $C^*(T_\psi, C_\varphi)/\mathfrak{K}$ is $*$ -isomorphic to the C^* -subalgebra \mathcal{D} of $C(\mathbb{T}) \oplus M_2(C([0, 1]))$, defined by

$$\mathcal{D} = \left\{ (f, S) \in C(\mathbb{T}) \oplus M_2(C([0, 1])) : S(0) = \begin{bmatrix} f([\gamma]) & 0 \\ 0 & f([\eta]) \end{bmatrix} \right\}.$$

Proof. Since $C^*(T_\psi, C_\varphi)/\mathfrak{K}$ is a C^* -subalgebra of $C^*(T_z, C_\varphi)/\mathfrak{K}$, by Theorem 3.7, for every element $b \in C^*(T_\psi, C_\varphi)/\mathfrak{K}$ there is a unique $\omega \in C(\mathbb{T})$ and unique functions f, g, h and k in $C_0([0, 1])$ such that

$$b = [T_\omega] + f([C_\varphi^* C_\varphi]) + g([C_\varphi C_\varphi^*]) + [U_\varphi]h([C_\varphi^* C_\varphi]) + [U_\varphi^*]k([C_\varphi C_\varphi^*]).$$

We show that $\omega \in C^*(\psi)$. By Theorem 2.2, for each $\varepsilon > 0$, there is an element $b_\varepsilon \in C^*(T_\psi, C_\varphi)/\mathfrak{K}$ such that $\|b_\varepsilon - b\| < \varepsilon$ and $b_\varepsilon = p([T_\psi], [T_\psi^*]) + q([C_\varphi], [C_\varphi^*])$, for some polynomials p and q . It is

straightforward to show that $p([T_\psi], [T_\psi^*]) = [T_{p(\psi, \bar{\psi})}]$. Hence, by Theorem 3.7, there are unique functions $f_\varepsilon, g_\varepsilon, h_\varepsilon$ and k_ε in $C_0([0, 1])$ such that

$$b_\varepsilon - b = [T_{\omega - p(\psi, \bar{\psi})}] + f_\varepsilon([C_\varphi^* C_\varphi]) + g_\varepsilon([C_\varphi C_\varphi^*]) + [U_\varphi] h_\varepsilon([C_\varphi^* C_\varphi]) + [U_\varphi^*] k_\varepsilon([C_\varphi C_\varphi^*])$$

and

$$\left\| \begin{pmatrix} \omega - p(\psi, \bar{\psi}), & \left[\begin{matrix} \omega(\gamma) - p(\psi, \bar{\psi})(\gamma) + g_\varepsilon & h_\varepsilon \\ k_\varepsilon & \omega(\eta) - p(\psi, \bar{\psi})(\eta) + f_\varepsilon \end{matrix} \right] \end{pmatrix} \right\| = \|b_\varepsilon - b\| < \varepsilon.$$

Thus, $\|\omega - p(\psi, \bar{\psi})\| < \varepsilon$ and $\omega \in C^*(\psi)$. By Theorem 3.7, $C^*(T_\psi, C_\varphi)/\mathfrak{K}$ is $*$ -isomorphic to the following C^* -subalgebra of $C^*(\psi) \oplus M_2(C([0, 1]))$

$$\mathcal{C} = \left\{ (\omega, S) \in C^*(\psi) \oplus M_2(C([0, 1])) : S(0) = \begin{bmatrix} \omega(\gamma) & 0 \\ 0 & \omega(\eta) \end{bmatrix} \right\}.$$

Hence, by Theorem 3.1, $C^*(T_\psi, C_\varphi)/\mathfrak{K}$ is $*$ -isomorphic to

$$\mathcal{D} = \left\{ (f, S) \in C([\mathbb{T}]) \oplus M_2(C([0, 1])) : S(0) = \begin{bmatrix} f([\gamma]) & 0 \\ 0 & f([\eta]) \end{bmatrix} \right\}. \quad \square$$

4. Composition operators with automorphic symbols. A self-map φ of the unit disk \mathbb{D} is an automorphism if φ is a one-to-one holomorphic map of \mathbb{D} onto \mathbb{D} . We denote the class of automorphisms of \mathbb{D} by $\text{Aut}(\mathbb{D})$. A well-known consequence of the Schwarz lemma shows that every element $\varphi \in \text{Aut}(\mathbb{D})$ has the form

$$(4.1) \quad \varphi(z) = \omega \frac{s - z}{1 - \bar{s}z},$$

for some $\omega \in \mathbb{T}$, where $s = \varphi^{-1}(0) \in \mathbb{D}$.

A *Fuchsian group* Γ is a discrete group of automorphisms of \mathbb{D} . Fix a point $z_0 \in \mathbb{D}$. The limit set of Γ is the set of limit points of the orbit $\{\varphi(z_0) : \varphi \in \Gamma\}$ in \mathbb{D} . This is a closed subset of the unit circle and does not depend on the choice of z_0 . The limit set of a Fuchsian group has either 0, 1, 2, or infinitely many elements. When the limit set is infinite, it is perfect and nowhere dense (and hence uncountable) on

the entire unit circle. A Fuchsian group Γ is called *non-elementary* if its limit set is infinite.

Jury [7] describes the C^* -algebra $C^*(\{C_\varphi : \varphi \in \Gamma\})/\mathfrak{K}$ when Γ is a non-elementary Fuchsian group. A basic point in the proof is that the non-elementary condition on Γ guarantees that the C^* -algebra $C^*(\{C_\varphi : \varphi \in \Gamma\})$ contains the unilateral shift T_z . The result may be extended from the non-elementary case to an slightly more general situation as follows.

Theorem 4.1. *Let Γ be a Fuchsian group. If the limit set of Γ contains at least two linearly independent points, then $C^*(\{C_\varphi : \varphi \in \Gamma\})$ contains the unilateral shift operator, and there is an exact sequence*

$$0 \longrightarrow \mathfrak{K} \longrightarrow C^*(\{C_\varphi : \varphi \in \Gamma\}) \longrightarrow C(\mathbb{T}) \rtimes \Gamma \longrightarrow 0.$$

Proof. If z_1 and z_2 are linearly independent points of the limit set, then two elements of the orbit $\{\varphi(0) : \varphi \in \Gamma\}$ may be chosen near z_1, z_2 that are linearly independent. By the proofs of [7, Proposition 3.4] and [8, Theorem 2.6], $C^*(\{C_\varphi : \varphi \in \Gamma\})$ contains the unilateral shift T_z . On the other hand, since the action of a Fuchsian group is amenable and topologically free, by a similar argument to that of [7, Theorem 3.1], the exactness of the above sequence follows. \square

For example, let φ be an automorphism of the form (4.1). If ω is not real ($\omega \neq \pm 1$) and $s \neq 0$, then $T_z \in C^*(C_\varphi)$. Indeed, a simple calculation shows that

$$\varphi(0) = \omega s, \quad \varphi^2(0) = \omega s \frac{1 - \omega}{1 - |s|^2 \omega}$$

and, if $\omega = x + iy$ ($y \neq 0$),

$$\frac{1 - \omega}{1 - |s|^2 \omega} = \frac{(1 + |s|^2)(1 - x) + i(|s|^2 y - y)}{|1 - |s|^2 \omega|^2}.$$

Therefore, $(1 - \omega)/(1 - |s|^2 \omega)$ is not real and $\varphi(0)$ and $\varphi^2(0)$ are linearly independent over \mathbb{R} . Hence, $T_z \in C^*(C_\varphi)$. If ω is real (1 or -1) or $s = 0$, then all $\varphi^n(0)$ s are linearly dependent.

Jury [8] found the C^* -algebra $C^*(T_z, C_\varphi)/\mathfrak{K}$, for $\varphi \in \text{Aut}(\mathbb{D})$ as a crossed product C^* -algebra. We do the same when the shift operator is replaced by a general irreducible Toeplitz operator T_ψ . The above

example shows that, if $\varphi \in \text{Aut}(\mathbb{D})$ is of the form (4.1) for some non-real $\omega \in \mathbb{T}$ and non-zero $s \in \mathbb{D}$, then the quotient $C^*(T_z, C_\varphi)/\mathfrak{K} = C^*(C_\varphi)/\mathfrak{K}$ does not change, if T_z is replaced with T_ψ . Here, we check the case $s = 0$.

Theorem 4.2. *Let φ be a rotational automorphism $\varphi(z) = \omega z$ for some $\omega \in \mathbb{T}$. If T_ψ is irreducible with continuous symbol ψ on \mathbb{T} and $\varphi(\psi(\mathbb{T})) = \psi(\mathbb{T})$, then there is an exact sequence of C^* -algebras*

$$0 \longrightarrow \mathfrak{K} \longrightarrow C^*(T_\psi, C_\varphi) \longrightarrow C(\psi(\mathbb{T})) \rtimes_\varphi \mathbb{Z} \longrightarrow 0,$$

if φ has infinite order. In the case that φ has finite order q , in the exact sequence, \mathbb{Z} is replaced by the finite cyclic group $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$.

Proof. Let the rotation φ have infinite order. The image $X = \psi(\mathbb{T})$ of ψ is a compact Hausdorff space, and $\varphi^n(X) = X$, for all $n \in \mathbb{Z}$. Now, \mathbb{Z} acts on X by

$$\beta : \mathbb{Z} \longrightarrow \text{Home}(X); \quad n \longmapsto \beta_n, \quad \beta_n(x) = \varphi^n(x),$$

for $n \in \mathbb{Z}$ and $x \in X$. This induces an action of \mathbb{Z} on $C(X)$ given by

$$\alpha : \mathbb{Z} \longrightarrow \text{Aut}(C(X)); \quad \alpha_n(f)(x) = f(\varphi^{-n}(x)).$$

Since every unitary composition operator on $H^2(\mathbb{D})$ is induced by a rotation, the C^* -algebra $C^*(T_\psi, C_\varphi)/\mathfrak{K}$ is generated by $C^*(T_\psi)/\mathfrak{K} \cong C(X)$ and unitary $[C_{\varphi^n}]$, for all integers n . On the other hand, Theorem 2.3 shows that the unitary representation $n \rightarrow [C_{\varphi^{-n}}]$ satisfies the covariance relation $[C_{\varphi^{-n}}]f[C_{\varphi^{-n}}^*] = \alpha_n(f)$. Hence, there is a surjective $*$ -homomorphism from the full crossed product $C(X) \rtimes_\varphi \mathbb{Z}$ to $C^*(T_\psi, C_\varphi)/\mathfrak{K}$. Since the action of the amenable group \mathbb{Z} on compact Hausdorff space X is amenable and topologically free, i.e., for each $n \in \mathbb{Z}$, if $n \neq 0$, the set of points that is fixed by φ^n has empty interior, similar to the proof of [8, Theorem 2.1], the above $*$ -homomorphism is also injective, and hence, an isometry. In the case that the order φ is finite, the proof is similar, with \mathbb{Z} replaced by a finite cyclic group. \square

It may be conjectured whether, for a rotational automorphism $\varphi(z) = \omega z$, there is a function ψ which satisfies the hypothesis of Theorem 4.2. As a concrete example, let us consider $\varphi(z) = ze^{2\pi i/3}$. A sufficient condition for irreducibility of Toeplitz operator T_ψ is given in [14, Theorem 1]: if the restriction of a function $\psi \in H^2$ to a Borel

subset $S \subseteq \mathbb{T}$, with Lebesgue measure on the unit circle, is one-to-one and the sets $\psi(S)$ and $\psi(\mathbb{T} \setminus S)$ are disjoint, then T_ψ is irreducible. By the Riemann mapping theorem, for example, see [6], there exists a biholomorphic (bijective and holomorphic) map ψ from the unit disk \mathbb{D} onto the simply connected set

$$\mathbb{D} - \left(\left[\frac{1}{2}, 1 \right) \cup \left\{ r e^{2\pi i/3} : \frac{1}{2} \leq r < 1 \right\} \cup \left\{ r e^{4\pi i/3} : \frac{1}{2} \leq r < 1 \right\} \right)$$

in the complex plane. A result of Carathéodory in [3] states that ψ continuously extends to the closure of the unit disk. Moreover, one can choose ψ , see Figure 1, such that

$$\begin{aligned} \psi(1) &= \frac{1}{2}, \psi(e^{2\pi i/3}) = \frac{1}{2}e^{2\pi i/3}, \psi(e^{4\pi i/3}) = \frac{1}{2}e^{4\pi i/3}, \\ \psi(e^{\pi i/3}) &= e^{\pi i/3}, \psi(-1) = -1, \psi(e^{5\pi i/3}) = e^{5\pi i/3} \end{aligned}$$

and

$$\begin{aligned} \psi(e^{2\pi i/15}) &= \psi(e^{28\pi i/15}) = 1, \psi(e^{8\pi i/15}) = \psi(e^{4\pi i/3}) = e^{2\pi i/3}, \\ \psi(e^{6\pi i/5}) &= \psi(e^{22\pi i/15}) = e^{4\pi i/3}. \end{aligned}$$

The restriction of ψ to \mathbb{T} , also denoted ψ , is an element of H^2 , one-to-one on

$$S = \left\{ e^{i\theta} : \theta \in \left(\frac{2\pi}{15}, \frac{8\pi}{15} \right) \cup \left(\frac{4\pi}{5}, \frac{6\pi}{5} \right) \cup \left(\frac{22\pi}{15}, \frac{28\pi}{15} \right) \right\},$$

$\psi(S)$ and $\psi(\mathbb{T} \setminus S)$ are disjoint and the image of ψ is invariant under φ , that is, $\varphi(\psi(\mathbb{T})) = \psi(\mathbb{T})$. Moreover, since ψ is not one-to-one on \mathbb{T} , $C^*(T_\psi) \neq C^*(T_z)$.

More generally, if the automorphism φ is of the form

$$(4.2) \quad \varphi(z) = z e^{2\pi i/q\pi},$$

where p and q are relatively prime integers with q positive, then, by a similar construction, there is a function ψ that satisfies the conditions of the above corollary, is not one-to-one on the unit circle, and

$$\psi(\mathbb{T}) = \mathbb{T} \cup \bigcup_{n=0}^{q-1} \varphi^n \left(\left[\frac{1}{2}, 1 \right) \right),$$

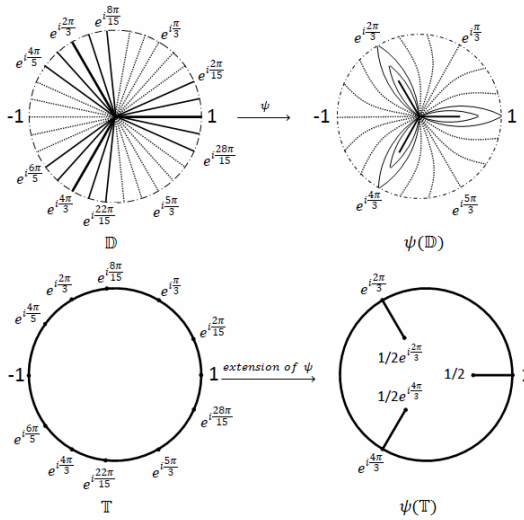


FIGURE 1.

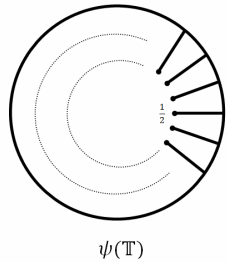


FIGURE 2.

see Figure 2. In order to illustrate that our construction may provide new examples, using Theorem 4.2 and certain facts on crossed products of C^* -algebras, we show that, for rational rotations (4.2) and the constructed continuous functions ψ on \mathbb{T} , $C^*(T_\psi, C_\varphi)/\mathcal{K}$ is not equal or even isomorphic to $C^*(T_z, C_\varphi)/\mathcal{K}$. This is easy of course when $p = 0$, that is, $\varphi(z) = z$ on \mathbb{T} and the action of \mathbb{Z} on \mathbb{T} and $\psi(\mathbb{T})$ is trivial. In this case, the C^* -algebras are isomorphic to $C(\psi(\mathbb{T}))$ and $C(\mathbb{T})$,

respectively. The spectra of these C^* -algebras are $\psi(\mathbb{T}) = \mathbb{T} \cup [1/2, 1)$ and \mathbb{T} , which are not homeomorphic.

For a more general example, we need some preparation to show that the crossed products are non isomorphic (the reader is referred to [17] for more details). Let G be a topological group acting on a topological space X from left. The orbit of $x \in X$ is the set $G \cdot x = \{s \cdot x : s \in G\}$. The stability group at x is $G_x := \{s \in G : s \cdot x = x\}$. The action is called free if $G_x = \{e\}$ for all $x \in G$. The set of orbits is denoted by $G \backslash X$ and is called the *orbit space*. It is equipped with the largest topology, making the natural quotient map $p : X \rightarrow G \backslash X$ continuous.

In the case where the automorphism φ is of the form (4.2) since the action of finite group \mathbb{Z}_q is free on the compact spaces \mathbb{T} and $\psi(\mathbb{T})$, using the same idea as in the proof of [17, Proposition 2.52], the spectra of the C^* -algebras $C(\psi(\mathbb{T})) \rtimes_{\varphi} \mathbb{Z}_q$ and $C(\mathbb{T}) \rtimes_{\varphi} \mathbb{Z}_q$ are $\mathbb{Z}_q \backslash \psi(\mathbb{T})$ and $\mathbb{Z}_q \backslash \mathbb{T}$, respectively. It is easy to see that $\mathbb{Z}_q \backslash \psi(\mathbb{T})$ is homeomorphic to $\mathbb{T} \cup [1/2, 1)$ and $\mathbb{Z}_q \backslash \mathbb{T}$ is homeomorphic to \mathbb{T} . Therefore, the spectra of these C^* -algebras are not homeomorphic, and thus, they could not be isomorphic.

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