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INEQUALITIES AND BI-LIPSCHITZ CONDITIONS FOR THE TRIANGULAR RATIO METRIC

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ABSTRACT. Let $G \subsetneq \mathbb{R}^n$ be a domain, and let d_1 and d_2 be two metrics on G. We compare the geometries defined by the two metrics to each other for several pairs of metrics. The metrics we study include the distance ratio metric, the triangular ratio metric and the visual angle metric. Finally, we apply our results to study Lipschitz maps with respect to these metrics.

1. Introduction. Several metrics play an important role in geometric function theory and in the study of quasiconformal maps in the plane and space [**7, 9, 10, 20**]. One of the key topics studied is uniform continuity of quasiconformal mappings with respect to metrics. Many authors have proved that these maps are either Lipschitz or Hölder continuous with respect to hyperbolic type metrics [**9, 21, 24**]. Ferrand studied [**16**] the reverse question: does Lipschitz continuity imply quasiconformality? A negative answer was given [**6**] in the case of a conformally invariant metric introduced by Ferrand [**16**].

Our goal here is to continue this research and to study similar questions for some other metrics, which, in our terminology, are of "hyperbolic type." While this term does not have a precise meaning, it refers to the fact that these metrics share some properties of the hyperbolic metric: the boundary has a strong influence on the value of the distance between points. In particular, we are interested in the visual angle metric introduced and studied recently in [**14**] and the triangular ratio metric from [**4, 14**]. The triangular ratio metric is

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defined as follows for a domain $G \subsetneq \mathbb{R}^n$ and $x, y \in G$:

$$s_G(x,y) = \sup_{z \in \partial G} \frac{|x-y|}{|x-z|+|z-y|} \in [0,1]. \tag{1.1}$$

The visual angle metric is defined by

$$v_G(x,y) = \sup\{\measuredangle(x,z,y) : z \in \partial G\}, \quad x,y \in G, \tag{1.2}$$

for domains $G \subsetneq \mathbb{R}^n$, $n \geq 2$, such that ∂G is not a proper subset of a line, see [**14,** Lemma 2.8]. Here, the notation $\measuredangle(x,z,y)$ means the angle in the range $[0,\pi]$ between the segments $[x,z]$ and $[y,z]$.

This paper is organized as follows. In Section 2, we give some preliminary results and prove various inequalities between the above metrics which will be applied later on. It is easy to see that domains G exist with isolated boundary points such that the metrics s_G and v_G are not comparable, see also, [**11,** Remark 2.18]. Then, we introduce, in Section 3, two conditions on domains G under which s_G and v_G are comparable. The first condition applies to domains G which satisfy that ∂G is "locally uniformly nonlinear," see Theorem 3.3, whereas the second condition, similar to the so-called porosity condition, applies to domains satisfying the "exterior ball condition," see Theorem 3.8. In Section 4, we show, motivated in part by Väisälä's work [**21**], that bilipschitz maps with respect to the triangular ratio metric distance ratio metric and quasihyperbolic metric are quasiconformal. Finally, applying the results of Section 2, we prove that quasiregular mappings $f : \mathbb{B}^n \to \mathbb{B}^n$ are Hölder continuous with respect to the metric $s_{\mathbb{B}^n}$.

2. Preliminary results. We introduce some terminology and notation, following [**20**]. For $x \in \mathbb{R}^n$ and $r > 0$, let

$$B^n(x,r) = \{z \in \mathbb{R}^n : |x-z| < r\},$$
$$S^{n-1}(x,r) = \{z \in \mathbb{R}^n : |x-z| = r\},$$

denote the ball and sphere, respectively, centered at x with radius r. The abbreviations $B^n(r) = B^n(0,r)$, $S^{n-1}(r) = S^{n-1}(0,r)$, $\mathbb{B}^2 = B^n(1)$ and $S^{n-1} = S^{n-1}(1)$ will be used frequently. Dimensions $B(x,r)$ and $S(x,r)$ are sometimes omitted.

2.1. Hyperbolic metric. The hyperbolic metrics $\rho_{\mathbb{H}^n}$ and $\rho_{\mathbb{B}^n}$ of the upper half plane $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ and of the unit ball $\mathbb{B}^n = \{z \in \mathbb{R}^n : |z| < 1\}$ may be defined as weighted metrics with the weight functions $w_{\mathbb{H}^n}(x) = 1/x_n$ and $w_{\mathbb{B}^n}(x) = 2/(1-|x|^2)$, respectively. Explicitly, by [**3,** page 35] we have, for $x, y \in \mathbb{H}^n$,

$$\text{ch}\rho_{\mathbb{H}^n}(x,y) = 1 + \frac{|x-y|^2}{2x_n y_n}, \tag{2.1}$$

and, by [**3,** page 40] for $x, y \in \mathbb{B}^n$,

$$\text{sh}\frac{\rho_{\mathbb{B}^n}(x,y)}{2} = \frac{|x-y|}{\sqrt{1-|x|^2}\sqrt{1-|y|^2}}\,. \tag{2.2}$$

From (2.2), we easily obtain

$$\text{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{2} = \frac{|x-y|}{\sqrt{|x-y|^2+(1-|x|^2)(1-|y|^2)}}\,.$$

For both $\mathbb{B}^n$ and $\mathbb{H}^n$ the hyperbolic metric can be defined by using absolute ratios, c.f., e.g., [**24,** (2.21)]. Due to the Möbius invariance of the absolute ratio, we may thus define, for every Möbius transformation h, the hyperbolic metric in $h(\mathbb{B}^n)$. This metric will be denoted by $\rho_{h(\mathbb{B}^n)}$.

2.2. Distance ratio metric. For a proper open subset $G \subset \mathbb{R}^n$ and for all $x, y \in G$, the distance ratio metric j_G is defined as

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x,\partial G), d(y,\partial G)\}}\right).$$

This metric was introduced by Gehring and Osgood [**9**] in a slightly different form and in the above form in [**23**]. If confusion seems unlikely, then we write $d(x) = d(x,\partial G)$. In addition to j_G, we study the metric

$$j_G^*(x,y) = \text{th}\frac{j_G(x,y)}{2}.$$

Because j_G is a metric, it follows easily, see [**1,** 7.42 (1)], that j_G^* is a metric, too. Moreover, by [**1,** Lemma 7.56] and [**24,** Lemma 2.41 (2)], if $G \in \{\mathbb{B}^n, \mathbb{H}^n\}$, then

$$j_G(x,y) \le \rho_G(x,y) \le 2j_G(x,y), \tag{2.3}$$

for all $x, y \in G$.

2.3. Quasihyperbolic metric. Let G be a proper subdomain of $\mathbb{R}^n$. For all $x, y \in G$, the quasihyperbolic metric k_G is defined as

$$k_G(x,y) = \inf_{\gamma} \int_{\gamma} \frac{1}{d(z, \partial G)} \, |dz|,$$

where the infimum is taken over all rectifiable arcs γ joining x to y in G **[10]**. From [**10,** Lemma 2.1], it follows that

$$j_G(x,y) \le k_G(x,y), \tag{2.4}$$

for all $x, y \in G$. It is easy to see that $k_{\mathbb{H}^n} \equiv \rho_{\mathbb{H}^n}$ and that, for all $x, y \in \mathbb{B}^n$,

$$\rho_{\mathbb{B}^n}(x,y) \le 2k_{\mathbb{B}^n}(x,y) \le 2\rho_{\mathbb{B}^n}(x,y). \tag{2.5}$$

2.4. Point pair function. We define, for $x, y \in G \subsetneq \mathbb{R}^n$, the point pair function

$$p_G(x,y) = \frac{|x-y|}{\sqrt{|x-y|^2 + 4\, d(x)\, d(y)}} \, .$$

This point pair function was introduced in **[4]** where it turned out to be a very useful function in the study of the triangular ratio metric. However, there are domains G such that p_G is not a metric.

Lemma 2.1. *Let G be a proper subdomain of $\mathbb{R}^n$. If $x, y \in G$, then*

$$j_G^*(x,y) = \frac{|x-y|}{|x-y| + 2\min\{d(x), d(y)\}}$$

and

$$j_G^*(x,y) \le s_G(x,y) \le \frac{e^{j_G(x,y)} - 1}{2} \, .$$

The first inequality is sharp for $G = \mathbb{R}^n \setminus \{0\}$.

Proof. By symmetry, we may assume that $d(x) \le d(y)$. For $x, y \in G$, let $z \in \partial G$ be a point satisfying $d(x) = |x - z|$. For the equality claim, we see that

$$\begin{aligned} \frac{|x-y|}{|x-y| + 2d(x)} &= \frac{|x-y|/d(x)}{|x-y|/d(x) + 2} = \frac{e^{j_G(x,y)} - 1}{e^{j_G(x,y)} + 1} \\ &= \frac{e^{j_G(x,y)/2} - e^{-j_G(x,y)/2}}{e^{j_G(x,y)/2} + e^{-j_G(x,y)/2}} = j_G^*(x,y). \end{aligned}$$

For the first inequality, we observe that, by the triangle inequality,

$$s_G(x,y) \geq \frac{|x-y|}{|x-z|+|z-y|} \geq \frac{|x-y|}{|x-y|+2d(x)} = j_G^*(x,y).$$

The sharpness of the first inequality when $G = \mathbb{R}^n \setminus \{0\}$ follows if we choose $x = 1$ and $y = t > 1$. Then, $s_G(x,y) = (t-1)/(t+1) = j_G^*(x,y)$. For the second inequality, note that

$$s_G(x,y) \leq \frac{|x-y|}{d(x)+d(y)} \leq \frac{|x-y|}{2\sqrt{d(x)\,d(y)}} \leq \frac{|x-y|}{2d(x)} = \frac{e^{j_G(x,y)}-1}{2}. \quad \square$$

Lemma 2.2. *Let G be a proper subdomain of $\mathbb{R}^n$. Then, for all $x,y \in G$, we have*

$$s_G(x,y) \leq 2j_G^*(x,y).$$

This inequality is sharp when the domain is $G = \mathbb{R}^n \setminus \{0\}$.

Proof. We first consider the points $x,y \in G$ satisfying $e^{j_G(x,y)} \geq 3$. The definition of j_G readily yields

$$2j_G^*(x,y) = \frac{2(e^{j_G(x,y)}-1)}{e^{j_G(x,y)}+1} \geq 1 \geq s_G(x,y).$$

We next suppose that $e^{j_G(x,y)} < 3$. In this case, it is clear that

$$2j_G^*(x,y) \geq \frac{e^{j_G(x,y)}-1}{2},$$

which, together with Lemma 2.1, implies the desired inequality.

The sharpness of the inequality can easily be verified by investigating the domain $G = \mathbb{R}^n \setminus \{0\}$. For any $x \in G$, selecting $y = -x$ gives $s_G(x,y) = 1$ and $j_G^*(x,y) = 1/2$. $\square$

Lemma 2.3. *If G is a proper subdomain of $\mathbb{R}^n$, then, for all $x,y \in G$,*

$$j_G^*(x,y) \leq p_G(x,y) \leq \frac{w}{\sqrt{w^2+1}} \leq \sqrt{2}j_G^*(x,y),$$

with $w = (e^{j_G(x,y)}-1)/2$. The first and last bounds are sharp when the domain is $G = \mathbb{R}^n \setminus \{0\}$.

Proof. Without loss of generality, we may suppose that $d(x) \le d(y)$. Then, by Lemma 2.1, the first inequality is equivalent to

$$\frac{|x-y|}{|x-y|+2d(x)} \le \frac{|x-y|}{\sqrt{|x-y|^2+4d(x)\,d(y)}}\,.$$

This, in turn, follows easily from the inequality $d(y) \le |x-y| + d(x)$.

For the second inequality, observe that, with $w = (e^{j_G(x,y)}-1)/2$,

$$\begin{aligned} p_G(x,y) &= \frac{|x-y|}{2d(x)\sqrt{(|x-y|/(2d(x)))^2+d(y)/d(x)}} = \frac{w}{\sqrt{w^2+d(y)/d(x)}} \\ &\le \frac{w}{\sqrt{w^2+1}} \le \frac{1+w}{\sqrt{w^2+1}} j_G^*(x,y) \le \sqrt{2} j_G^*(x,y). \end{aligned}$$

In order to see the sharpness of the first inequality in $G = \mathbb{R}^n \setminus \{0\}$, if we choose $y = 1/x$, $x > 1$, then

$$j_G^*\left(x, \frac{1}{x}\right) = \frac{x^2-1}{x^2-1+2} = p_G\left(x, \frac{1}{x}\right).$$

For the sharpness of the last inequality, again in $G = \mathbb{R}^n \setminus \{0\}$, we choose $y = -x$. Then

$$p_G(x,-x) = \frac{1}{\sqrt{2}}, \qquad j_G^*(x,-x) = \frac{1}{2}\,. \qquad \square$$

Proposition 2.4. *If G is a bounded domain of $\mathbb{R}^n$, then, for all $x, y \in G$,*

$$j_G^*(x,y) \ge \frac{|x-y|}{d(G)}\,.$$

Proof. Fix $x, y \in G$ and a line L through x, y. Then, there are points $x_1, y_1 \in L \cap \partial G$ such that x_1, x, y, y_1 are in this order on L, and hence,

$$\begin{aligned} d(G) &\ge |x_1 - y_1| = |x_1 - x| + |x-y| + |y-y_1| \\ &\ge |x-y| + 2\min\{d(x), d(y)\}. \end{aligned} \tag{2.6}$$

The proof follows from Lemma 2.1. $\square$

Lemma 2.5. *Let G be a proper subdomain of $\mathbb{R}^n$. Then, for all $x, y \in G$,*

(i) $(1/\sqrt{2})p_G(x,y) \le s_G(x,y) \le 2p_G(x,y)$,

(ii) $s_G(x,y) \le (p_G(x,y))/(1-p_G(x,y))$.

Proof. By symmetry, we may suppose that $d(x) \le d(y)$.

(i) The lower bound follows from [**4,** Lemma 3.4 (2)]. For the upper bound, observe that, by Lemma 2.2,

$$s_G(x,y) \le \frac{2|x-y|}{|x-y|+2d(x)} \le \frac{2|x-y|}{\sqrt{|x-y|^2+4d(x)\,d(y)}} = 2p_G(x,y),$$

where the second inequality follows from the inequality $d(y) \le d(x) + |x-y|$.

(ii) The first inequality in Lemma 2.3 may be written as

$$\frac{w}{1+w} \le p_G(x,y), \quad w = \frac{e^{j_G(x,y)}-1}{2}\,.$$

Since $s_G(x,y) \le w$ by Lemma 2.1, inequality (ii) follows from the above inequality. $\square$

In [**4,** 3.23], it was proved that $\mathrm{th}(\rho_{\mathbb{B}^n}(x,y)/2) \le 2s_{\mathbb{B}^n}(x,y)$ for all $x,y \in \mathbb{B}^n$. We next apply Lemma 2.1 to improve this upper bound. Note that, by [**4,** (2.4), 3.4], we have, for all $x,y \in \mathbb{H}^n$,

$$s_{\mathbb{H}^n}(x,y) = p_{\mathbb{H}^n}(x,y) = \mathrm{th}\frac{\rho_{\mathbb{H}^n}(x,y)}{2}\,. \tag{2.7}$$

Lemma 2.6. *For $x,y \in \mathbb{B}^n$, we have*

$$\mathrm{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{4} \le s_{\mathbb{B}^n}(x,y) \le p_{\mathbb{B}^n}(x,y) \le \mathrm{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{2} \le 2\mathrm{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{4}\,.$$

Proof. For the first inequality, by Lemma 2.1 and (2.3), we have

$$s_{\mathbb{B}^n}(x,y) \ge j^*_{\mathbb{B}^n}(x,y) = \mathrm{th}\frac{j_{\mathbb{B}^n}(x,y)}{2} \ge \mathrm{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{4}\,.$$

The second and the third inequalities follow from [**4,** Lemma 3.4 (1), Lemma 3.8]. For the last inequality, by [**24,** 2.29 (1)],

$$\mathrm{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{4} = \frac{\mathrm{th}(\rho_{\mathbb{B}^n}(x,y)/2)}{1+\sqrt{1-\mathrm{th}^2(\rho_{\mathbb{B}^n}(x,y)/2)}}\,.$$

Therefore,

$$2\mathrm{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{4} \geq \mathrm{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{2}\,,$$

since $1+\sqrt{1-\mathrm{th}^2(\rho_{\mathbb{B}^n}(x,y)/2)} \leq 2$. $\square$

Lemma 2.7.

(i) *Let G be a proper subdomain of $\mathbb{R}^n$. If $x, y \in G$, then*

$$\mathrm{th}\frac{j_G(x,y)}{2} \leq p_G(x,y) \leq \mathrm{th} j_G(x,y).$$

(ii) *If $G \subset \mathbb{R}^n$ is a convex domain, $x, y \in G$ and $m = \min\{d(x), d(y)\}$, then*

$$\mathrm{th}(j_G(x,y)/2) \leq s_G(x,y) \leq \frac{|x-y|}{\sqrt{|x-y|^2+4m^2}} \leq \mathrm{th} j_G(x,y).$$

Proof.

(i) For the second inequality, by symmetry, we may assume that $d(x) \leq d(y)$. Writing $|x-y| = b$,

$$p_G(x,y) = \frac{b}{\sqrt{b^2+4\,d(x)\,d(y)}} \leq \frac{b}{\sqrt{b^2+4\,d(x)^2}}\,,$$

we have

$$\mathrm{th} j_G(x,y) = \frac{e^{2j_G(x,y)}-1}{e^{2j_G(x,y)}+1} = \frac{(1+b/d(x))^2-1}{(1+b/d(x))^2+1} = \frac{b^2+2b\,d(x)}{b^2+2b\,d(x)+2d(x)^2}\,.$$

Denote $t = d(x)$. Then, the inequality

$$p_G(x,y) \leq \mathrm{th} j_G(x,y),$$

is equivalent to

$$\frac{b}{\sqrt{b^2+4t^2}} \leq \frac{b^2+2bt+2t^2}{b^2+2bt}\,,$$

and this is equivalent to $4b^2t^3(2b+3t) \geq 0$, which is true since $t = d(x) > 0$. The first inequality follows from Lemma 2.3.

(ii) Fix $x, y \in G$. Because G is convex, by [**4,** Lemma 3.4 (1)] and the proof of (i), we have

$$s_G(x,y) \le p_G(x,y) \le \frac{|x-y|}{\sqrt{|x-y|^2+4m^2}} \le \operatorname{th} j_G(x,y).$$

The first inequality follows from Lemma 2.3. □

Lemma 2.8. *For a convex domain $G \subsetneq \mathbb{R}^n$ and all $x, y \in G$, we have $v_G(x,y) \ge s_G(x,y) \ge j_G^*(x,y)$.*

Proof. By [**11,** Lemma 2.16], $s_G(x,y) \le v_G(x,y)$; thus, the result directly follows from Lemma 2.3. □

Theorem 2.9. *For a convex domain $G \subsetneq \mathbb{R}^n$ and all $x, y \in G$, we have*

(i) $s_G(x,y) \le \sqrt{2}\, j_G^*(x,y)$, *and*
(ii) $v_G(x,y) \ge \dfrac{1}{\sqrt{2}}\, p_G(x,y)$.

Proof.

(i) This inequality follows from Lemma 2.3 and [**4,** Lemma 3.4 (1)].
(ii) By Lemmas 2.3 and 2.8, we have

$$v_G(x,y) \ge j_G^*(x,y) \ge \frac{1}{\sqrt{2}}\, p_G(x,y). \qquad \square$$

The next theorem shows that the constant $1/\sqrt{2}$ in Theorem 2.9 (ii) can be improved for the case of a half space or a ball to be 1. The sharp constant in the case of a convex domain will be given in Remark 2.11.

Theorem 2.10. *Let G be a half space or a ball in the Euclidean space $\mathbb{R}^n$. Then, for all $x, y \in G$,*

$$v_G(x,y) \ge p_G(x,y).$$

Proof. Since both the visual angle metric v_G and the point pair function p_G are invariant under the similarities of domain G, we may assume that domain G is the upper half space $\mathbb{H}^n$ or the unit ball $\mathbb{B}^n$. We first consider the case of $G = \mathbb{H}^n$. By the left-hand side inequality of [**14,** Theorem 3.19] and the well-known Shafer inequality

$\arctan t \geq 3t/(1+2\sqrt{1+t^2})$ for $t>0$, see **[2, 19]**, we have that

$$v_{\mathbb{H}^n}(x,y) \geq \arctan\left(\operatorname{sh}\frac{\rho_{\mathbb{H}^n}(x,y)}{2}\right) \geq \frac{3\operatorname{sh}(\rho_{\mathbb{H}^n}(x,y)/2)}{1+2\sqrt{1+\operatorname{sh}^2(\rho_{\mathbb{H}^n}(x,y)/2)}} =: A.$$

By (2.1), we have that

$$\operatorname{sh}\left(\frac{\rho_{\mathbb{H}^n}(x,y)}{2}\right) = \sqrt{\frac{\operatorname{ch}\rho_{\mathbb{H}^n}(x,y)-1}{2}} = \frac{|x-y|}{2\sqrt{d(x)d(y)}},$$

and hence,

$$A = \frac{3|x-y|}{\sqrt{4d(x)d(y)}+2\sqrt{|x-y|^2+4d(x)d(y)}} \geq \frac{|x-y|}{\sqrt{|x-y|^2+4d(x)d(y)}}$$
$$= p_G(x,y).$$

For the case of $G=\mathbb{B}^n$, we use the inequality

$$\arctan\left(\operatorname{sh}\frac{\rho_{\mathbb{B}^n}(x,y)}{2}\right) \leq v_{\mathbb{B}^n}(x,y),$$

see **[14,** Theorem 3.11**]**. The same argument as in the case of the upper half space gives the proof for $v_{\mathbb{B}^n}(x,y) \geq p_{\mathbb{B}^n}(x,y)$. $\square$

Remark 2.11. For a general convex domain $G\subset\mathbb{R}^n$, the inequality $v_G \geq p_G$ may not hold. Consider the strip domain

$$S = \{(x,y)\in\mathbb{R}^2 : -\infty < x < \infty, -1 < y < 1\}$$

and two points $a=(0,t)$, $b=(0,-t)$ for $0<t<1$. Then, it is easy to see that

$$p_S(a,b) = \frac{t}{\sqrt{t^2+(1-t)^2}} \quad \text{and} \quad v_S(a,b) = \arcsin t.$$

We see that

$$C := \inf_{t\in(0,1)} \frac{v_S(a,b)}{p_S(a,b)} = 0.73707\cdots > 1/\sqrt{2} = 0.707107\ldots.$$

Actually, it can be proven that, in general, for a convex domain G, we have that

$$v_G \geq C p_G, \quad C = 0.73707\ldots. \tag{2.8}$$

Let $t = e^{j_G(x,y)} - 1$. To this end, we apply the inequality [**25,** Theorem 4.1], which states that, for a convex domain G and $x, y \in G$,

$$v_G(x,y) \geq \arcsin \frac{t}{t+2}\,.$$

On the other hand, it is easy to see that

$$p_G(x,y) \leq \frac{t}{\sqrt{t^2+4}}\,.$$

Hence, we have that

$$\frac{v_G(x,y)}{p_G(x,y)} \geq \frac{\arcsin(t/(t+2))}{t/\sqrt{t^2+4}} = \frac{\arcsin s}{s/\sqrt{s^2+(1-s)^2}} \geq C,$$

where $s = t/(t+2)$. The above example of the strip domain shows that constant C is the best possible. Thus, inequality (2.8) improves Theorem 2.9 (ii).

Lemma 2.12. *Let G be a proper subdomain of $\mathbb{R}^n$, $z \in G$, and let $\lambda \in (0,1)$. Then, for all $x, y \in B(z, \lambda d(z))$,*

$$k_{B(z,d(z))}(x,y) \leq \frac{1+\lambda}{1-\lambda}\, k_G(x,y).$$

Proof. By the definition

$$k_G(x,y) = \int_{J_G} \frac{|du|}{d(u,\partial G)}\,,$$

where J_G is the geodesic segment of the metric k_G joining x and y in G and due to the fact that $x, y \in B(z, \lambda\, d(z))$, it follows from [**17,** Theorem 2.2] that $J_G \subset B(z, \lambda\, d(z))$, and hence, for all $u \in J_G$, $d(u,\partial G) \leq (1+\lambda)\, d(z)$. Further,

$$\begin{aligned} k_{B(z,d(z))}(x,y) &\leq \int_{J_G} \frac{|du|}{d(u, S^{n-1}(z,d(z)))} \\ &= \int_{J_G} \frac{|du|}{d(z)-|u-z|} \leq \int_{J_G} \frac{|du|}{(1-\lambda)d(z)} \\ &\leq \int_{J_G} \frac{|du|}{(1-\lambda)/(1+\lambda)\, d(u,\partial G)} = \frac{1+\lambda}{1-\lambda}\, k_G(x,y). \qquad \square \end{aligned}$$

Theorem 2.13. *Let $G \subset \mathbb{R}^n$, $x, y \in G$, and $\lambda \in (0,1)$. Then,*

$$s_G(x,y) \le c\,\mathrm{th}\left(\frac{1+\lambda}{1-\lambda} k_G(x,y)\right); \quad c = \frac{1}{\mathrm{th}((1+\lambda)/(1-\lambda)\log(1+\lambda))}.$$

Proof. We divide the proof into two cases.

Case 1. $x, y \in B(z, \lambda d(z))$ for some $z \in G$. By domain monotonicity, Lemma 2.6, (2.3), [**24,** 3.4] and Lemma 2.12,

$$\begin{aligned} s_G(x,y) &\le s_{B(z,d(z))}(x,y) \le \mathrm{th}\left(\frac{\rho_{B(z,d(z))}(x,y)}{2}\right) \\ &\le \mathrm{th}\,(j_{B(z,d(z))}(x,y)) \le \mathrm{th}\,(k_{B(z,d(z))}(x,y)) \\ &\le \mathrm{th}\left(\frac{1+\lambda}{1-\lambda} k_G(x,y)\right). \end{aligned}$$

Case 2. Case 1 is not true. Then, choosing $z = x$, we see that $y \notin B(x, \lambda d(x))$, and hence, by (2.4), $k_G(x,y) \ge \log(1+\lambda)$ because $|x-y| \ge \lambda \min\{d(x), d(y)\}$, and

$$s_G(x,y) \le c\,\mathrm{th}\left(\frac{1+\lambda}{1-\lambda}\log(1+\lambda)\right) \le c\,\mathrm{th}\left(\frac{1+\lambda}{1-\lambda} k_G(x,y)\right)$$

holds if $c = 1/\mathrm{th}((1+\lambda)/(1-\lambda)\log(1+\lambda))$. $\square$

Remark 2.14. A uniform domain $G \subset \mathbb{R}^n$ is a domain with the following comparison property between the quasihyperbolic metric and the distance ratio metric: there exists a constant $C > 1$ such that, for all $x, y \in G$,

$$j_G(x,y) \le k_G(x,y) \le C j_G(x,y).$$

Recall that the lower bound holds for all domains G by (2.4). In [**9,** Theorem 2], a similar characterization of uniform domains was given but with the expression $a j_G(x,y) + b$ on the right hand side. It was pointed out in [**23,** 2.50 (2)] that the pair of constants (a,b) can be replaced by $(c,0)$ where c depends only upon (a,b). Hence, this comparison property and the above results yield numerous new inequalities between the quasihyperbolic metric and the triangular ratio metric or the visual angle metric in uniform domains. The class of uniform domains is very wide; for instance, quasidisks in $\mathbb{R}^2$ are such domains [**8**].

3. Comparison results between the triangular ratio metric and the visual angle metric. In this section, we introduce two conditions on domains G for which s_G and v_G are comparable. The first condition applies to domains G which satisfy that ∂G is "locally uniformly nonlinear," see Theorem 3.3, whereas the second condition applies to domains satisfying the "exterior ball condition."

Very recently, after the submission of this paper, we found another proof of Theorem 3.1, see [**11,** Lemma 2.11].

Theorem 3.1. *If $G \subset \mathbb{R}^n$ is a domain, then, for all $x, y \in G$,*

$$s_G(x,y) \geq \sin \frac{v_G(x,y)}{2}\,.$$

Proof. Let $w_0 \in \partial G$ be a point such that $v_G(x,y) = \measuredangle(x, w_0, y)$. Let E be the envelope of the pair (x,y) which defines $v_G(x,y)$, see [**14,** 2.9]. Clearly,

$$s_G(x,y) \geq \frac{|x-y|}{|x-w_0|+|w_0-y|} \geq \inf_{w\in\partial E} \frac{|x-y|}{|x-w|+|w-y|}\,.$$

We need to obtain the maximum of $|x-w|+|w-y|$ when $w \in \partial E$. It is easy to check that the radius of the boundary circular arcs of envelope E is $R = |x-y|/(2\sin v_G(x,y))$. For $w \in \partial E$, let θ be the central angle formed by the points y, w and the center. We see that

$$\begin{aligned}|x-w|+|w-y| &= 2R\sin\frac{\theta}{2} + 2R\cos\left(v_G(x,y) - \frac{\pi-\theta}{2}\right)\\ &= 2R\sin\frac{\theta}{2} + 2R\sin\left(v_G(x,y) + \frac{\theta}{2}\right)\\ &\equiv f(\theta),\end{aligned}$$

and

$$\max(f(\theta)) = f(\pi - v_G(x,y)) = 4R\cos\frac{v_G(x,y)}{2}\,.$$

Therefore,

$$\begin{aligned}s_G(x,y) &\geq \frac{|x-y|}{4R\cos(v_G(x,y)/2)}\\ &= \frac{|x-y|}{4|x-y|/(2\sin v_G(x,y))\cdot\cos(v_G(x,y)/2)} = \sin\frac{v_G(x,y)}{2}\,. \square\end{aligned}$$

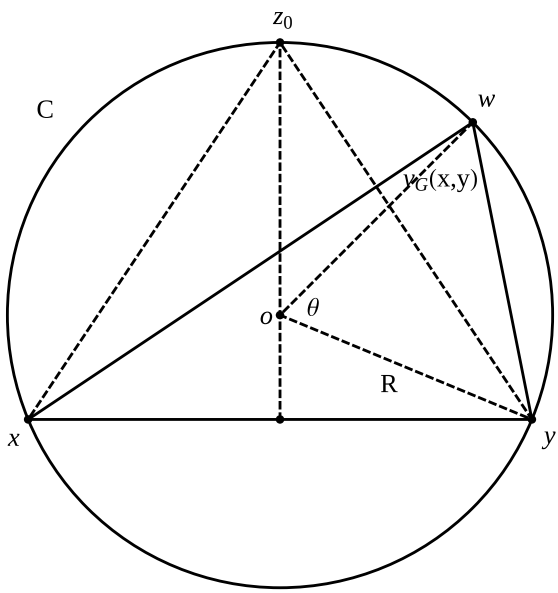

FIGURE 1. Proof of Theorem 3.1.

In general, it is not true that v_G has a lower bound in terms of s_G. For instance, this fails for $G = \mathbb{B}^2 \setminus \{0\}$, [**11,** Remark 2.18]. The nonlinearity condition in the next theorem is similar to the thickness condition in [**22**], and it ensures a lower bound for v_G in terms of s_G. For the case $n = 2$, an example of a domain satisfying the nonlinearity condition is that of the snowflake domain.

Definition 3.2. Suppose that $G \subset \mathbb{R}^2$ is a domain. We say that ∂G satisfies the nonlinearity condition, if there exists a $\delta \in (0,1)$ such that, for every $z \in \partial G$ and, for every $r \in (0, d(G))$ and every line L with $L \cap B(z,r) \neq \emptyset$, there exists

$$w \in B(z,r) \cap \partial G \setminus \bigcup_{y \in L} B(y, \delta r).$$

Theorem 3.3. *Let $G \subset \mathbb{R}^2$ be a domain such that ∂G satisfies the nonlinearity condition. If $x, y \in G$ and $s_G(x, y) < 1$, then*

$$v_G(x, y) > \arctan\left(\frac{\delta}{6}\, s_G(x, y)\right).$$

Proof. Fix $x, y \in G$. We may assume that $d(x) \le d(y)$. Choose $z_0 \in \partial G$ such that $|x - z_0| = d(x)$. Let $r = d(x) + |x - y|$. Then, $B(z_0, r) \subset B(m, t)$ and $m = (x + y)/2$ for $t = 2r$. By the nonlinearity condition, as in Figure 2,

$$\begin{aligned} v_G(x, y) &\ge \measuredangle(x, w, y) = \alpha, \\ w &= m + te^{i\theta}\left(\frac{y - x}{|y - x|}\right), \\ \theta &= \arcsin\frac{\delta r}{t}. \end{aligned}$$

Writing $w_1 = m + t(y - x)/|y - x|$, $\beta = \measuredangle(w, y, w_1)$ and $\gamma = \measuredangle(w, x, w_1)$, we see that

$$\tan\beta = \frac{\delta r}{\sqrt{4r^2 - \delta^2 r^2} - |x - y|/2}$$

and

$$\tan\gamma = \frac{\delta r}{\sqrt{4r^2 - \delta^2 r^2} + |x - y|/2},$$

and hence,

$$\tan\alpha = \tan\left(\beta - \gamma\right) = \frac{\delta r|x - y|}{4r^2 - |x - y|^2/4}.$$

Therefore,

$$\begin{aligned} v_G(x, y) \ge \alpha &= \arctan\frac{\delta r|x - y|}{4r^2 - |x - y|^2/4} \\ &= \arctan\frac{\delta(d(x) + |x - y|)|x - y|}{4(d(x) + |x - y|)^2 - |x - y|^2/4} \\ &= \arctan\frac{\delta(1 + |x - y|/d(x))|x - y|/d(x)}{4(1 + |x - y|/d(x))^2 - (|x - y|/2d(x))^2}. \end{aligned}$$

Then, $s_G(x,y) \leq |x-y|/(2d(x))$. A simple calculation shows that the function $f(t) = (1+t)t/(4(1+t)^2 - (t/2)^2)$ is increasing for $t > 0$, since

$$f'(t) = 2\left(\frac{1}{(4+3t)^2} + \frac{1}{(4+5t)^2}\right) > 0.$$

On the other hand, $g(t) = f(t)/t$ is decreasing for $t > 0$. Hence, for $0 < t \leq 2$,

$$g(t) \geq g(2) = \frac{3}{35} > \frac{1}{12} \quad \text{and} \quad f(t) \geq \frac{1}{12}\,t.$$

Therefore,

$$\begin{aligned}\arctan \frac{\delta(1+|x-y|/d(x))|x-y|/d(x)}{4(1+|x-y|/d(x))^2-(|x-y|/2d(x))^2} &= \arctan(f(|x-y|/d(x))\delta)\\ &\geq \arctan(f(2s_G(x,y))\delta) \geq \arctan\left(\frac{\delta}{6}s_G(x,y)\right),\end{aligned}$$

and the proof is complete. $\square$

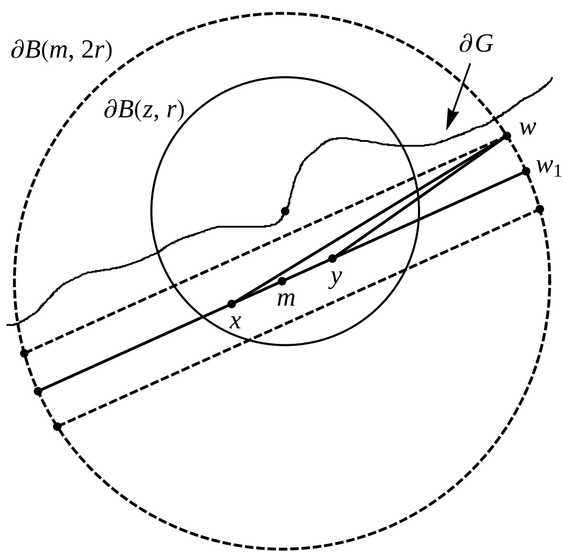

FIGURE 2. Proof of Theorem 3.3.

Lemma 3.4. *Let $G \subset \mathbb{R}^n$ be a proper subdomain of $\mathbb{R}^n$, $x \in G$ and $y \in B^n(x, d(x))$. Then,*

$$\sin(v_G(x,y)) \le \sup_{w \in \partial G} \frac{|x-y|}{|x-w|} = \frac{|x-y|}{d(x)}.$$

Proof. Fix $x \in G$ and $y \in B^n(x, d(x))$. For each $w \in \partial G$, we have, by elementary geometry,

$$\measuredangle(x-w, y-w) \le \theta; \quad \sin\theta = \frac{|x-y|}{|x-w|}.$$

Taking the supremum over all $w \in \partial G$, we obtain

$$\sin(v_G(x,y)) \le \sup_{w \in \partial G} \frac{|x-y|}{|x-w|} = \frac{|x-y|}{d(x)}. \qquad \square$$

Theorem 3.5. *Let G be a proper subdomain of $\mathbb{R}^2$. For $x, y \in G$,*

$$s_G(x,y) \le \frac{|x-y|/d(x)}{1 + \cos(v_G(x,y)) + \sqrt{(|x-y|/d(x))^2 - \sin^2(v_G(x,y))}}.$$

Proof. We may assume that $d(x) \le d(y)$. We first consider the case of $\partial G \cap [x,y] \ne \emptyset$. It is clear in this case that $s_G(x,y) = 1$ and $v_G(x,y) = \pi$, and the desired inequality holds as an equality.

Next, we assume that $\partial G \cap [x,y] = \emptyset$. Let E be the interior of the envelope which defines the visual angle metric between x and y. Then,

$$D = B^2(x, d(x)) \cup B^2(y, d(x))) \cup E$$

is a subdomain of G. Let $w_0 \in \partial D \cap S^1(x, d(x)) \cap \partial E$. By use of the law of the cosine in the triangle $\triangle xyw_0$, we obtain

$$\begin{aligned} |x-w_0| + |w_0-y| = {} & (1 + \cos(v_G(x,y)))\, d(x) \\ & + \sqrt{|x-y|^2 - d(x)^2 \sin^2(v_G(x,y))}. \end{aligned}$$

A simple geometric observation gives

$$s_D(x,y) = \frac{|x-y|}{|x-w_0| + |w_0-y|}$$

$$= \frac{|x-y|/d(x)}{1+\cos(v_G(x,y))+\sqrt{(|x-y|/d(x))^2-\sin^2(v_G(x,y))}}.$$

Then, the domain monotonicity of the s-metric yields the desired inequality $s_G(x,y) \le s_D(x,y)$. □

Remark 3.6.

(i) If $|x-y|/d(x) > 1$, then the square root in Theorem 3.5 is clearly well defined. In the case $|x-y|/d(x) \le 1$, it follows from Lemma 3.4 that the square root is well defined, too.

(ii) The inequalities in Theorem 3.5 are sharp in the following sense: If $v_G(x,y) = 0$, then $s_G(x,y) \le |x-y|/(|x-y|+2d(x))$ which, together with Lemma 2.1, actually gives $s_G(x,y) = |x-y|/(|x-y|+2d(x))$. If $s_G(x,y) = 1$, then the inequality actually gives $v_G(x,y) = \pi$.

Definition 3.7. Let $\delta \in (0,1/2)$. We say that a domain $G \subset \mathbb{R}^n$ satisfies condition $H(\delta)$ if, for every $z \in \partial G$ and all $r \in (0, d(G)/2)$, there exists a $w \in B^n(z,r) \cap (\mathbb{R}^n \setminus G)$ such that $B^n(w,\delta r) \subset B^n(z,r) \cap (\mathbb{R}^n \setminus G)$.

Note that condition $H(\delta)$ excludes those domains whose boundaries have zero angle cusps directed into the domain. For instance, the domain $B^2 \setminus [0,1]$ does not satisfy condition $H(\delta)$. A similar condition has also been studied in **[13, 18]**, and sometimes this condition is referred to as the *porosity condition.* For instance, domains with smooth boundaries are in the class $H(\delta)$.

Theorem 3.8. *Let $G \subset \mathbb{R}^2$ be a domain satisfying condition $H(\delta)$. Then, for all $x, y \in G$, we have*

$$\sin v_G(x,y) \ge \frac{\delta}{2}\, j_G^*(x,y).$$

Proof. Fix $x, y \in G$. By symmetry, we may suppose that $d(x) \le d(y)$. Denote $r = d(x)$, and choose a point $z \in \partial G$ such that $r = |x-z|$. By condition $H(\delta)$, $w \in \mathbb{R}^2 \setminus G$ exists such that $B^2(w,\delta r) \subset B^2(z,r) \cap (\mathbb{R}^2 \setminus G)$. Denote $G_1 = \mathbb{R}^2 \setminus \overline{B}^2(w,\delta r)$. By

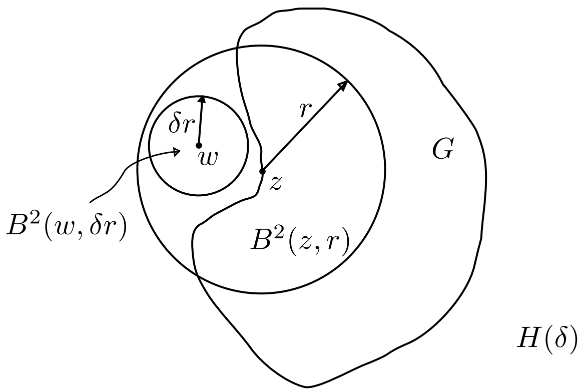

FIGURE 3. Condition $H(\delta)$.

the monotonicity of v_G with respect to the domain, we have

$$v_G(x,y) \geq v_{G_1}(x,y).$$

Geometrically, $v_{G_1}(x,y)$ can be found by considering the circle through x,y externally tangent to $B^2(w,\delta r)$. Suppose this circle is $B^2(\widetilde{c},\widetilde{R})$. In order to find a lower bound for v_{G_1}, we need an upper bound for $\widetilde{R}$. By elementary geometry, $\widetilde{R} \leq R$ where $B^2(c,R)$ corresponds to the case where $y = y_1 = x + [(x-w)/|x-w|]|x-y|$. Then, $|x-y_1| = |x-y|$. Using the power of the point w with respect to the circle $\partial B^2(c,R)$, we have

$$\delta r(\delta r + 2R) = |x-w||y_1-w| = |x-w|(|x-w|+|x-y|),$$

and hence,

$$2R = \frac{|x-w|}{\delta r}(|x-w|+|x-y|) - \delta r.$$

In the same manner as in the proof of Theorem 3.1, we utilize the law of the sine to obtain

$$R = \frac{|x-y_1|}{2\sin v_{G_1}(x,y_1)} = \frac{|x-y|}{2\sin v_{G_1}(x,y_1)}\,.$$

Observing that $|x-w| \le |x-z|+|z-w| \le d(x)+(1-\delta)\,d(x)$, we have

$$\begin{aligned}\sin v_{G_1}(x,y) &\ge \frac{|x-y|}{((2-\delta)/\delta)(|x-w|+|x-y|)-\delta r}\\ &\ge \frac{|x-y|}{((2-\delta)/\delta)((2-\delta)\,d(x)+|x-y|)-\delta d(x)}\\ &= \frac{\delta}{2-\delta}\cdot\frac{t}{t+(4-4\delta)/(2-\delta)} \ge \frac{\delta}{2-\delta}\cdot\frac{t}{t+2}\\ &= \frac{\delta}{2-\delta}\cdot\frac{e^{j_G(x,y)}-1}{e^{j_G(x,y)}+1} \ge \frac{\delta}{2}\,j_G^*(x,y),\end{aligned}$$

where $t=|x-y|/d(x)$. $\square$

4. Lipschitz conditions. One of the main reasons for studying metrics in geometric function theory is the distortion theory of mappings: the study of how far a map transforms two given points. In this section, we will study the triangular ratio metric and other aforementioned metrics from this point of view. We begin our discussion with the following result of Gehring and Osgood [**9**]. We assume that the reader is familiar with the basic facts of the theory of K-quasiconformal/K-quasiregular maps [**20, 24**]. In particular, we follow Väisälä's definition of K-quasiconformality [**20,** page 42].

Theorem 4.1. *Let $f: G\to G'$ be a K-quasiconformal homeomorphism between domains $G, G'\subset\mathbb{R}^n$. Then, there exists a constant $c=c(n,K)$ dependent only upon n and K such that, for all $x,y\in G$,*

$$k_{G'}(f(x)f(y)) \le \max\{k_G(x,y)^\alpha,\ k_G(x,y)\},\quad \alpha=K^{1/(1-n)}.$$

It is a natural question whether a similar result holds for the metrics considered in this paper. This and other questions have already been studied elsewhere [**4, 11**]. Before proceeding, we mention a few well-known cases where the above result can be refined.

Lemma 4.2. *Let $f: G\to G'=fG$ be a Möbius transformation where $G, G'\subset\mathbb{R}^n$ are domains. Then*:

(i) $j_G(x,y)/2 \le j_{G'}(f(x),f(y)) \le 2j_G(x,y)$,
(ii) $k_G(x,y)/2 \le k_{G'}(f(x),f(y)) \le 2k_G(x,y)$,

for all $x,y\in G$.

Proof. See [**9,** Proof of Theorem 4] and [**10,** Corollary 2.5]. $\square$

Lemma 4.3. *Let $f : G \to G'$ be a conformal map between domains $G, G' \subset \mathbb{R}^2$. Then,*

$$\frac{k_G(x,y)}{4} \le k_{G'}(f(x), f(y)) \le 4k_G(x,y),$$

for all $x, y \in G$.

Proof. See [**15,** Proposition 1.6]. $\square$

These results may be further refined, for instance, if $G = G' = \mathbb{B}^n$ as shown in [**15**] or if $G = G' = \mathbb{R}^n \setminus \{0\}$. Here, our goal is to study the extent to which these results have counterparts for the triangular ratio metric.

Väisälä [**21**] has proved that an L-bilipschitz map with respect to the quasihyperbolic metric is a quasiconformal map with the linear dilatation $4L^2$. Partly motivated by his work, we consider bilipschitz maps with respect to the triangular ratio metric, and our result gives a refined upper bound L^2 of the linear dilatation in the case of Euclidean spaces.

Theorem 4.4. *Let $G \subsetneq \mathbb{R}^n$ be a domain, and let $f : G \to fG \subset \mathbb{R}^n$ be a sense-preserving homeomorphism satisfying the L-bilipschitz condition with respect to the triangular ratio metric, i.e.,*

$$\frac{s_G(x,y)}{L} \le s_{fG}(f(x), f(y)) \le L s_G(x,y)$$

holds for all $x, y \in G$. Then, f is quasiconformal with the linear dilatation $H(f) \le L^2$.

Proof. If $x, y \in G$ satisfy $|x-y| < \min\{d(x), d(y)\}$ and $w \in \partial G$ with $d(x) = |x-w|$ is a point, then it is easy to see that

$$s_G(x,y) \ge \frac{|x-y|}{|x-w|+|w-y|} \ge \frac{|x-y|}{2\min\{d(x),d(y)\}+|x-y|}$$

and

$$s_G(x,y) \le \frac{|x-y|}{d(x)+d(y)} \le \frac{|x-y|}{2\min\{d(x),d(y)\} - |x-y|}\,,$$

from which we conclude that

$$\frac{2\min\{d(x),d(y)\}}{1/s_G(x,y)+1} \le |x-y| \le \frac{2\min\{d(x),d(y)\}}{1/s_G(x,y)-1}\,. \tag{4.1}$$

For an arbitrary point $z \in G$, let $x, y \in G$ with $|x-z| = |y-z| = r$, where r is small enough such that the following argument is meaningful, i.e., all of the terms are positive. Let

$$A(x,y,z) = \frac{\min\{d(f(x)),\, d(f(z))\}}{\min\{d(f(y)),\, d(f(z))\}}\,,$$

which tends to 1 as x, y tend to z. Then, by estimate (4.1), we obtain

$$\begin{aligned}
&\frac{|f(x)-f(z)|}{|f(y)-f(z)|}\\
&\quad\le A(x,y,z)\frac{1/s_{fG}(f(y),f(z))+1}{1/s_{fG}(f(x),f(z))-1}\\
&\quad\le A(x,y,z)\frac{L/s_G(y,z)+1}{1/(Ls_G(x,z))-1}\\
&\quad\le A(x,y,z)\frac{L/(|y-z|/(2\min\{d(y),d(z)\}+|y-z|))+1}{1/(L|x-z|/(2\min\{d(x),d(z)\}-|x-z|))-1}\\
&\quad= A(x,y,z)\frac{2L^2\min\{d(y),d(z)\}+(L^2+L)|y-z|}{2\min\{d(x),d(z)\}-(L+1)|x-z|}\,\frac{|x-z|}{|y-z|} \longrightarrow L^2,
\end{aligned}$$

when $r = |x-z| = |y-z| \to 0$. Hence,

$$H(f,z) = \limsup_{|x-z|=|y-z|=r\to 0^+} \frac{|f(x)-f(z)|}{|f(y)-f(z)|} \le L^2. \qquad \square$$

Corollary 4.5. *Let $G \subset \mathbb{R}^n$ be a domain, and let $f : G \to fG \subset \mathbb{R}^n$ be a sense-preserving homeomorphism satisfying the L-bilipschitz condition with respect to the distance ratio metric or quasihyperbolic metric. Then, f is quasiconformal with linear dilatation $H(f) \le L^2$.*

Proof. By Lemma 2.1, $j_G^*(x,y) \le s_G(x,y) \le (e^{j_G(x,y)}-1)/2$ for all $x, y \in G$. It follows that, for arbitrary $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all $x, y \in G$ satisfying $j_G(x,y) < \delta$, we have that

$$\frac{j_G(x,y)}{2(1+\varepsilon)} \le s_G(x,y) \le \frac{1+\varepsilon}{2}\, j_G(x,y).$$

For an L-bilipschitz mapping with respect to the j-metric, we choose $x, y \in G$ such that $j_G(x,y) < \delta/L$. Then,

$$\begin{aligned} s_{fG}(f(x), f(y)) &\le \frac{1+\varepsilon}{2}\, j_{fG}(f(x), f(y)) \le \frac{L(1+\varepsilon)}{2}\, j_G(x,y) \\ &\le L(1+\varepsilon)^2 s_G(x,y). \end{aligned}$$

Similarly, we also have

$$s_{fG}(f(x), f(y)) \ge \frac{j_{fG}(f(x), f(y))}{2(1+\varepsilon)} \ge \frac{j_G(x,y)}{2L(1+\varepsilon)} \ge \frac{s_G(x,y)}{L(1+\varepsilon)^2}\,.$$

Hence, an L-bilipschitz mapping with respect to the j-metric is in fact locally $L(1+\varepsilon)^2$-bilipschitz with respect to the s-metric, from which we obtain that the mapping is quasiconformal with linear dilatation $H(f) \le L^2(1+\varepsilon)^4$. Since ε is arbitrary, we conclude that the mapping is actually quasiconformal with linear dilatation $H(f) \le L^2$.

Because $0 < \lambda < 1$, $x \in G$ and $y \in B^n(x, \lambda d(x))$, we have by **[24,** Lemma 3.7], that

$$j_G(x,y) \le k_G(x,y) \le j_G(x,y)/(1-\lambda);$$

the same argument applies to L-bilipschitz mapping in the k-metric, i.e., an L-bilipschitz mapping in the k-metric is quasiconformal with linear dilatation $H(f) \le L^2$. $\square$

Corollary 4.6. *Let $G \subset \mathbb{R}^n$ be a domain, and let $f : G \to fG \subset \mathbb{R}^n$ be a sense-preserving isometry with respect to the triangular ratio metric distance ratio metric or quasihyperbolic metric. Then, f is a conformal mapping. In particular, for $n \ge 3$, the mapping f is the restriction of a Möbius map.*

Proof. The result follows from the fact that a 1-quasiconformal mapping is conformal and Liouville's theorem in higher dimensions. $\square$

Hästö **[12]** considered the isometries of the quasihyperbolic metric on plane domains and proved that, except for the trivial case of a half-plane where the quasihyperbolic metric coincides with the hyperbolic metric, the isometries are exactly similarity mappings. Note that an additional condition of C^3 smoothness of the boundary of the domain is needed.

Theorem 4.7. *Let $f:\mathbb{B}^n\to G$, $G\in\{\mathbb{B}^n,\mathbb{H}^n\}$ be a Möbius transformation. Then, for $x,y\in\mathbb{B}^n$, we have*

$$s_G(f(x),f(y))\le\frac{2s_{\mathbb{B}^n}(x,y)}{1+s^2_{\mathbb{B}^n}(x,y)}\,.$$

Proof. For $G=\mathbb{H}^n$, by (2.7), [**24,** (2.21)] and Lemma 2.6, we have, for all $x,y\in\mathbb{B}^n$,

$$\begin{aligned}s_{\mathbb{H}^n}(f(x),f(y))&=\operatorname{th}\left(\frac{\rho_{\mathbb{H}^n}(f(x),f(y))}{2}\right)=\operatorname{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{2}\\&=\frac{2\operatorname{th}(\rho_{\mathbb{B}^n}(x,y)/4)}{1+\operatorname{th}^2(\rho_{\mathbb{B}^n}(x,y)/4)}\le\frac{2s_{\mathbb{B}^n}(x,y)}{1+s^2_{\mathbb{B}^n}(x,y)}\,.\end{aligned}$$

Similarly, for $G=\mathbb{B}^n$, by Lemma 2.6 and [**24,** (2.20)], we have, for all $x,y\in\mathbb{B}^n$,

$$\begin{aligned}s_{\mathbb{B}^n}(f(x),f(y))&\le\operatorname{th}\left(\frac{\rho_{\mathbb{B}^n}(f(x),f(y))}{2}\right)=\operatorname{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{2}\\&=\frac{2\operatorname{th}(\rho_{\mathbb{B}^n}(x,y)/4)}{1+\operatorname{th}^2(\rho_{\mathbb{B}^n}(x,y)/4)}\le\frac{2s_{\mathbb{B}^n}(x,y)}{1+s^2_{\mathbb{B}^n}(x,y)}\,.\end{aligned}$$ □

Theorem 4.8.

(i) *Let $f:\mathbb{B}^n\to\mathbb{H}^n$ be a Möbius transformation. Then, for $x,y\in\mathbb{B}^n$, we have*:

$$p_{\mathbb{B}^n}(x,y)\le p_{\mathbb{H}^n}(f(x),f(y))\le\frac{2p_{\mathbb{B}^n}(x,y)}{1+p^2_{\mathbb{B}^n}(x,y)}\,.$$

(ii) *Let $f:\mathbb{B}^n\to\mathbb{B}^n$ be a Möbius transformation. Then, for $x,y\in\mathbb{B}^n$, we have*:

$$\frac{p_{\mathbb{B}^n}(x,y)}{1+\sqrt{1-p^2_{\mathbb{B}^n}(x,y)}}\le p_{\mathbb{B}^n}(f(x),f(y))\le\frac{2p_{\mathbb{B}^n}(x,y)}{1+p^2_{\mathbb{B}^n}(x,y)}\,.$$

(iii) *Let $f : \mathbb{H}^n \to \mathbb{B}^n$ be a Möbius transformation. Then, for $x, y \in \mathbb{H}^n$, we have:*

$$\frac{p_{\mathbb{B}^n}(x,y)}{1+\sqrt{1-p_{\mathbb{B}^n}^2(x,y)}} \le p_{\mathbb{H}^n}(f(x),f(y)) \le \frac{2p_{\mathbb{B}^n}(x,y)}{1+p_{\mathbb{B}^n}^2(x,y)}\,.$$

Proof.

(i) For the second inequality, by (2.7), Theorem 4.7 and Lemma 2.6, we have, for all $x, y \in \mathbb{B}^n$,

$$p_{\mathbb{H}^n}(f(x),f(y)) = s_{\mathbb{H}^n}(f(x),f(y)) \le \frac{2s_{\mathbb{B}^n}(x,y)}{1+s_{\mathbb{B}^n}^2(x,y)} \le \frac{2p_{\mathbb{B}^n}(x,y)}{1+p_{\mathbb{B}^n}^2(x,y)}\,.$$

For the first inequality, we have by Lemma 2.6,

$$p_{\mathbb{H}^n}(f(x),f(y)) = \mathrm{th}\left(\frac{\rho_{\mathbb{H}^n}(f(x),f(y))}{2}\right) = \mathrm{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{2} \ge p_{\mathbb{B}^n}(x,y).$$

(ii) By Lemma 2.6 and **[24,** (2.20)],

$$\begin{aligned} p_{\mathbb{B}^n}(f(x),f(y)) &\le \mathrm{th}\frac{\rho_{\mathbb{B}^n}(f(x),f(y))}{2} = \frac{2\mathrm{th}(\rho_{\mathbb{B}^n}(x,y)/4)}{1+\mathrm{th}^2(\rho_{\mathbb{B}^n}(x,y)/4)} \\ &\le \frac{2p_{\mathbb{B}^n}(x,y)}{1+p_{\mathbb{B}^n}^2(x,y)}\,. \end{aligned}$$

For the first inequality, we have again by Lemma 2.6,

$$\begin{aligned} p_{\mathbb{B}^n}(f(x),f(y)) &\ge \mathrm{th}\frac{\rho_{\mathbb{B}^n}(f(x),f(y))}{4} = \mathrm{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{4} \\ &\ge \frac{p_{\mathbb{B}^n}(x,y)}{1+\sqrt{1-p_{\mathbb{B}^n}^2(x,y)}}\,. \end{aligned}$$

(iii) By Lemma 2.6,

$$\begin{aligned} p_{\mathbb{B}^n}(f(x),f(y)) &\le \mathrm{th}\frac{\rho_{\mathbb{B}^n}(f(x),f(y))}{2} = \frac{2\mathrm{th}(\rho_{\mathbb{H}^n}(x,y)/4)}{1+\mathrm{th}^2(\rho_{\mathbb{H}^n}(x,y)/4)} \\ &\le \frac{2p_{\mathbb{H}^n}(x,y)}{1+p_{\mathbb{H}^n}^2(x,y)}\,. \end{aligned}$$

For the first inequality, we have again by Lemma 2.6,

$$p_{\mathbb{B}^n}(f(x),f(y)) \geq \operatorname{th}\frac{\rho_{\mathbb{B}^n}(f(x),f(y))}{4} = \operatorname{th}\frac{\rho_{\mathbb{H}^n}(x,y)}{4} \geq \frac{p_{\mathbb{H}^n}(x,y)}{1+\sqrt{1-p^2_{\mathbb{H}^n}(x,y)}}. \qquad \square$$

Theorem 4.9. *Let $f:\mathbb{B}^n\to G$, $G\in\{\mathbb{B}^n,\mathbb{H}^n\}$ be a K-quasiregular mapping. Then, for $x,y\in\mathbb{B}^n$, we have*

$$s_G(f(x),f(y)) \leq \lambda_n^{1-\alpha}\left(\frac{2s_{\mathbb{B}^n}(x,y)}{1+s^2_{\mathbb{B}^n}(x,y)}\right)^{\alpha}, \quad \alpha = K^{1/(1-n)},$$

where $\lambda_n\in[4,2e^{n-1})$, $\lambda_2=4$, is the Grötzsch ring constant dependent only upon n [**24,** Lemma 7.22].

Proof. For $G=\mathbb{B}^n$, by Lemma 2.6 and [**4,** Theorem 5.4], we have, for all $x,y\in\mathbb{B}^n$,

$$s_{\mathbb{B}^n}(f(x),f(y)) \leq \operatorname{th}\left(\frac{\rho_{\mathbb{B}^n}(f(x),f(y))}{2}\right) \leq \lambda_n^{1-\alpha}\left(\operatorname{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{2}\right)^{\alpha} = \lambda_n^{1-\alpha}\left(\frac{2\operatorname{th}(\rho_{\mathbb{B}^n}(x,y)/4)}{1+\operatorname{th}^2(\rho_{\mathbb{B}^n}(x,y)/4)}\right)^{\alpha} \leq \lambda_n^{1-\alpha}\left(\frac{2s_{\mathbb{B}^n}(x,y)}{1+s^2_{\mathbb{B}^n}(x,y)}\right)^{\alpha}.$$

Similarly, for $G=\mathbb{H}^n$ by (2.7), Lemma 2.6, and [**4,** 5.4], we have, for all $x,y\in\mathbb{B}^n$,

$$s_{\mathbb{H}^n}(f(x),f(y)) = \operatorname{th}\left(\frac{\rho_{\mathbb{H}^n}(f(x),f(y))}{2}\right) \leq \lambda_n^{1-\alpha}\left(\operatorname{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{2}\right)^{\alpha} = \lambda_n^{1-\alpha}\left(\frac{2\operatorname{th}(\rho_{\mathbb{B}^n}(x,y)/4)}{1+\operatorname{th}^2(\rho_{\mathbb{B}^n}(x,y)/4)}\right)^{\alpha} \leq \lambda_n^{1-\alpha}\left(\frac{2s_{\mathbb{B}^n}(x,y)}{1+s^2_{\mathbb{B}^n}(x,y)}\right)^{\alpha}. \square$$

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