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POSITIVE SOLUTIONS FOR THE NONHOMOGENEOUS p-LAPLACIAN EQUATION IN $\mathbb{R}^N$

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ABSTRACT. In this paper, we study a class of nonhomogeneous sublinear-superlinear p-Laplacian equations in $\mathbb{R}^N$. By applying a minimization method on the Nehari manifold $\mathcal{N}^\alpha$, the existence of positive solutions and the continuity in the perturbation term are obtained.

1. Introduction and main results. In this paper, we are interested in the existence of positive solutions for the following nonhomogeneous sublinear-superlinear p-Laplacian problem:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + |u|^{m-2}u = |u|^{q-2}u + f(x) & x \in \mathbb{R}^N, \\ u(x) \in \mathcal{D}^{1,p}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $1 < p < N$, $1 < q < p \leq m < p^* = pN/(N-p)$. Problem (1.1) may be considered as a perturbation of the homogeneous problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + |u|^{m-2}u = |u|^{q-2}u & x \in \mathbb{R}^N, \\ u(x) \in \mathcal{D}^{1,p}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N). \end{cases} \tag{1.2}$$

Recently, Lyberopoulos **[13]** studied the existence of the ground state solution for the p-Laplacian equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u + H(x)|u|^{s-2}u = h(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N, \tag{1.3}$$

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where the parameters p, q, s satisfy one of the following assumptions:

(A_1) $1 < q < \min\{p, s\}$ or $q > \max\{p, s\}$;
(A_2) $s < q < p$;
(A_3) $p < q < s < p^*$,

and the nonnegative functions $V(x)$, $h(x)$ and $H(x)$ verify

(A_4) there exists a $\theta \in (0, p)$ such that $|x|^\theta V(x) \to \alpha > 0$ as $|x| \to \infty$;
(A_5) $(h(x))^{p^*-p}(V(x))^{q-p^*} \to 0$, $(H(x))^{p^*-p}(V(x))^{p^*-s} \to 0$ as $|x| \to \infty$.

Similarly, Su and Wang **[17]** investigated the existence of entire solutions of nonlinear elliptic equations of the form

$$\begin{cases} -\mathrm{div}(A(|x|)|\nabla u|^{p-2}\nabla u) + V(|x|)|u|^{p-2}u = Q(|x|)f(u) & x \in \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{1.4}$$

where $f(u) = o(|u|^\mu)$, $\mu > p$, as $u \to 0$.

It is worth noting that, as $|x| \to \infty$, the functions satisfy $V(x)$, $h(x)$, $H(x) \to 0$ in (1.3) and $Q(|x|) \to 0$ in (1.4). Similar studies may be found in **[3, 9, 12, 18, 20, 21]** and the references therein.

In striking contrast to the rich variety of the aforementioned studies, however, very little seems to be known for problem (1.1). A general method exists for solving the analogue of problem (1.1) in a bounded domain, see **[1, 4, 8]**. While in $\mathbb{R}^N$, problem (1.1) is not compact, that is, the minimizing sequence may be bounded, but not pre-compact, in the Sobolev space $W^{1,p}(\mathbb{R}^N)$. In order to overcome this difficulty, the authors in **[13]** used assumptions (A_4)–(A_5) to obtain the compact embedding $E_p(\mathbb{R}^N, V) \hookrightarrow L^q(\mathbb{R}^N, h)$ ($L^s(\mathbb{R}^N, H)$) and then proved the existence of solutions for (1.4), where the weighted Sobolev space $E \equiv E_p(\mathbb{R}^N, V)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_E = \left(\int_{\mathbb{R}^N} (|\nabla u|^p + V|u|^p)\, dx \right)^{1/p}.$$

The other method for dealing with this problem is to work in weighted Sobolev spaces of radial functions and then establish a compact embedding theorem, see **[17, 18]**. In this paper, we are motivated by **[3, 13, 17, 18]** and study the existence of positive solutions for (1.1). We shall use the Nehari manifold and the fibering map methods proposed by Drabek and Pohozaev **[6, 14]** (also see **[2]**) to study problem (1.1).

In order to state our main results, we introduce some Lebesgue spaces and norms. Let $L^s(\mathbb{R}^N)$, $s \geq 1$, be the usual Lebesgue spaces with the norm

$$\|u\|_s = \left(\int_{\mathbb{R}^N} |u|^s dx \right)^{1/s}$$

and

$$X = \mathcal{D}^{1,p}(\mathbb{R}^N) = \left\{ u \in L^{p^*}(\mathbb{R}^N) \mid \frac{\partial u}{\partial x_i} \in L^p(\mathbb{R}^N), \ i = 1, 2, \dots, N \right\}$$

endowed with the norm $\|u\|_X = \|\nabla u\|_p$.

The following Gagliardo-Nirenberg-Sobolev inequality is well known. There is a constant $S > 0$, dependent only upon p and N, such that

$$S\left(\int_{\mathbb{R}^N} |u|^{p^*} dx \right)^{p/p^*} \leq \int_{\mathbb{R}^N} |\nabla u|^p dx \quad \text{for all } u \in C_0^\infty(\mathbb{R}^N). \tag{1.5}$$

Since $C_0^\infty(\mathbb{R}^N)$ is a dense subset of X, the embedding inequality (1.5) holds on X.

For problem (1.1), we introduce the Banach space $E \equiv \mathcal{D}^{1,p}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ with the norm

$$\|u\|_E = \|\nabla u\|_p + \|u\|_q. \tag{1.6}$$

By (1.5) and the interpolation inequality, there exists an $S_r > 0$ such that, for $r \in [q, p^*]$,

$$\|u\|_r \leq S_r \|u\|_E \quad \text{for all } u \in E. \tag{1.7}$$

Definition 1.1. A function $u \in E$ is said to be a *weak* solution of (1.1) if, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, the following holds:

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \nabla \varphi + |u|^{m-2} u \varphi)\, dx = \int_{\mathbb{R}^N} |u|^{q-2} u \varphi\, dx + \int_{\mathbb{R}^N} f(x) \varphi\, dx. \tag{1.8}$$

Let $J(u) : E \to \mathbb{R}$ be the energy functional associated with problem (1.1) defined by

$$J(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{m}\|u\|_m^m - \frac{1}{q}\|u\|_q^q - \int_{\mathbb{R}^N} f(x)u\,dx. \tag{1.9}$$

It is easy to see that, for all $\varphi \in E$, the functional $J \in C^1(E, \mathbb{R})$ and its Gateaux derivative are given by

$$\begin{aligned} J'(u)\varphi = &\int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u\nabla\varphi + |u|^{m-2}u\varphi)\,dx \\ &- \int_{\mathbb{R}^N} |u|^{q-2}u\varphi\,dx - \int_{\mathbb{R}^N} f(x)\varphi\,dx. \end{aligned} \tag{1.10}$$

Clearly, the solutions of (1.1) correspond to critical points of J in E.

Our main result in this paper is as follows.

Theorem 1.2. *Let $1 < p < N$ and $1 < q < p \le m < p^* = pN/(N-p)$. In addition, suppose that the function f is nontrivial and nonnegative, and $f \in L^{q'}(\mathbb{R}^N) \cap L^\gamma(\mathbb{R}^N)$, where*

$$q' = \frac{q}{q-1}, \qquad \gamma = \frac{p^*}{p^*-1}.$$

Then, problem (1.1) admits a least positive positive solution $u \in E$ which converges to 0 in E as $\|f\|_{q'} \to 0$.

This paper is organized as follows. In Section 2, we set up the variational framework and derive some lemmas. We give the proof of Theorem 1.1 in Section 3.

2. Preliminaries. In this section, we make some assumptions regarding Theorem 1.1 and establish some lemmas. In order to obtain solutions of problem (1.1), we look for critical points of the functional J. Since J is not bounded on E, we introduce the following open subset of E. Let $\alpha > p-1$. Denote

$$E^\alpha = \left\{ u \in E \mid \|\nabla u\|_p^p + \|u\|_m^m > \frac{\alpha}{p-1}\|u\|_q^q \right\} \tag{2.1}$$

and the Nehari manifold as

(2.2)
$$\mathcal{N}^\alpha = \Big\{u \in E^\alpha \mid J'(u)u = \|\nabla u\|_p^p + \|u\|_m^m - \|u\|_q^q - \int_{\mathbb{R}^N} fu\,dx = 0\Big\}.$$

For $u \in E \setminus \{0\}$, we consider the fibering maps $\phi_u(t) : [0, \infty) \to \mathbb{R}$, defined by

(2.3)
$$\begin{aligned}
\phi_u(t) &= J(tu) = \frac{t^p}{p}\|\nabla u\|_p^p + \frac{t^m}{m}\|u\|_m^m - \frac{t^q}{q}\|u\|_q^q - t\int_{\mathbb{R}^N} fu\,dx,\\
\phi_u'(t) &= t^{p-1}\|\nabla u\|_p^p + t^{m-1}\|u\|_m^m - t^{q-1}\|u\|_q^q - \int_{\mathbb{R}^N} fu\,dx,\\
\phi_u''(t) &= (p-1)t^{p-2}\|\nabla u\|_p^p + (m-1)t^{m-2}\|u\|_m^m - (q-1)t^{q-2}\|u\|_q^q.
\end{aligned}$$

In order to proceed, we first establish the following result.

Lemma 2.1. *The Nehari manifold $\mathcal{N}^\alpha$ defined by* (2.2) *is not an empty set.*

Proof. We first prove $E^\alpha \neq \emptyset$. Since $f(x) \geq 0$ and $f(x) \not\equiv 0$ in $\mathbb{R}^N$, there exist $x_0 \in \mathbb{R}^N$ and $r > 0$ such that $f(x) > 0$ for $x \in B_r(x_0) \equiv \{x \in \mathbb{R}^N |x - x_0| < r\}$. Then, we take $\nu(x) \in C_0^2(\mathbb{R}^N)$ with $\operatorname{supp}\nu(x) \subset B_r(x_0)$ such that

$$\int_{\mathbb{R}^N} f(x)\nu(\sigma(x - x_0))\,dx > 0 \quad \text{for any } \sigma \geq 1.$$

Set $u(x) = \nu(\sigma(x - x_0))$. Then, we claim that $u \in E^\alpha$ if σ is large enough. In fact, the inequality

$$\int_{\mathbb{R}^N} |\nabla u(x)|^p dx + \int_{\mathbb{R}^N} |u(x)|^m dx > \frac{\alpha}{p-1}\int_{\mathbb{R}^N} |u(x)|^q dx$$

is equivalent to

$$\sigma^p\int_{\mathbb{R}^N} |\nabla \nu(y)|^p dy + \int_{\mathbb{R}^N} |\nu(y)|^m dy > \frac{\alpha}{p-1}\int_{\mathbb{R}^N} |\nu(y)|^q dy.$$

Clearly, it is true if σ is large enough. Therefore, $E^\alpha \neq \emptyset$.

In the following, we prove $\mathcal{N}^\alpha \neq \emptyset$. Denote $\phi_u(t) = J(tu)$. Let $t_0 > 0$ be the unique root of the equation

$$(p-1)(t_0^{p-q}\|\nabla u\|_p^p + t_0^{m-q}\|u\|_m^m) = \alpha\|u\|_q^q. \tag{2.4}$$

Then,

$$\begin{aligned}\phi_u'(t_0) &= t_0^{p-1}\|\nabla u\|_p^p + t_0^{m-1}\|u\|_m^m - t_0^{q-1}\|u\|_q^q - \int_{\mathbb{R}^N} fu\,dx \\ &= \frac{\alpha-p+1}{p-1}\,t_0^{q-1}\|u\|_q^q - \int_{\mathbb{R}^N} fu\,dx.\end{aligned} \tag{2.5}$$

Note that

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla u(x)|^p dx = \sigma^{p-N}\int_{\mathbb{R}^N} |\nabla \nu(y)|^p dy, \\ &\int_{\mathbb{R}^N} |u(x)|^s dx = \sigma^{-N}\int_{\mathbb{R}^N} |\nu(y)|^s dy, \quad s = m, q.\end{aligned} \tag{2.6}$$

We have from (2.4) and (2.6) that $t_0 \in (0,1]$ for large σ, and so

$$\begin{aligned} t_0 &\le \left(\frac{\alpha\|u\|_q^q}{(p-1)(\|\nabla u\|_p^p + \|u\|_m^m)}\right)^{1/(m-q)} \\ &= \left(\frac{\alpha\|\nu\|_q^q}{(p-1)(\sigma^p\|\nabla \nu\|_p^p + \|\nu\|_m^m)}\right)^{1/(m-q)} \\ &\le C_1\sigma^{-p/(m-q)},\end{aligned} \tag{2.7}$$

where C_1 is independent of σ. On the other hand, there exists a $\beta_0 > 0$ independent of σ such that

$$\int_{\mathbb{R}^N} f(x)u(x)\,dx = \sigma^{-N}\int_{\mathbb{R}^N} f(x_0+y/\sigma)\nu(y)\,dy \ge \beta_0\sigma^{-N} \quad \text{for } \sigma \text{ large.} \tag{2.8}$$

Then, it follows from (2.5), (2.7) and (2.8) that

$$\phi_u'(t_0) \le \sigma^{-N}\left(\frac{\alpha-p+1}{p-1}\,C_1^{q-1}\sigma^{-[p(q-1)]/(m-q)} - \beta_0\right) < 0 \quad \text{for } \sigma \text{ large.} \tag{2.9}$$

In addition, we note that $\phi_u'(t_0) < 0$ and $\lim_{t\to\infty}\phi_u'(t) = \infty$. Thus, there exists a minimum $t_1 > t_0$ of $\phi_u(t)$ such that

$$0 = \phi_u'(t_1) = t_1^{p-1}\|\nabla u\|_p^p + t_1^{m-1}\|u\|_m^m - t_0^{q-1}\|u\|_q^q - \int_{\mathbb{R}^N} fu\,dx. \tag{2.10}$$

Since

$$t_1 > t_0 \Longrightarrow (p-1)(t_1^{p-q}\|\nabla u\|_p^p + t_1^{m-q}\|u\|_m^m) > \alpha\|u\|_q^q, \tag{2.11}$$

we obtain $v = t_1 u \in \mathcal{N}^\alpha$. This completes the proof. □

Lemma 2.2. *Problem* (1.2) *admits only the trivial solution in E.*

Proof. Let u be a solution of problem (1.2). By the Pohozaev identity for the p-Laplacian equation **[7, 10, 15]**, we have, for any $\beta \in \mathbb{R}$,

$$\left(\frac{N-p}{p} - \beta\right)\|\nabla u\|_p^p + \left(\frac{N}{m} - \beta\right)\|u\|_m^m + \left(\beta - \frac{N}{q}\right)\|u\|_q^q = 0. \tag{2.12}$$

In particular, letting $\beta = N/q$ gives $u = 0$, and thus, the conclusion holds. □

Lemma 2.3. *The functional J is bounded below on $\overline{\mathcal{N}^\alpha}$, where*

$$\overline{\mathcal{N}^\alpha} = \left\{u \in E \mid J'(u)u = 0,\ \|\nabla u\|_p^p + \|u\|_m^m \ge \frac{\alpha}{p-1}\|u\|_q^q\right\}. \tag{2.13}$$

Proof. Suppose that there exists a sequence $\{u_n\} \subset \overline{\mathcal{N}^\alpha}$ such that $J(u_n) \to -\infty$. Since

$$J'(u_n)u_n = \|\nabla u_n\|_p^p + \|u_n\|_m^m - \|u_n\|_q^q - \int_{\mathbb{R}^N} f u_n \, dx = 0 \tag{2.14}$$

and

$$T_n \equiv \|\nabla u_n\|_p^p + \|u_n\|_m^m \ge \frac{\alpha}{p-1}\|u_n\|_q^q,$$

we have

$$\begin{aligned} J(u_n) &= \frac{1-p}{p}\|\nabla u_n\|_p^p + \frac{1-m}{m}\|u_n\|_m^m + \frac{q-1}{q}\|u_n\|_q^q \\ &\ge \frac{1-p}{p}\|\nabla u_n\|_p^p + \frac{1-m}{m}\|u_n\|_m^m. \end{aligned} \tag{2.15}$$

This shows that $T_n \to \infty$ as $n \to \infty$. Furthermore, from (2.14), we obtain

$$1 = \frac{\|u_n\|_q^q}{T_n} + \frac{\int_{\mathbb{R}^N} f u_n dx}{T_n} \le \frac{p-1}{\alpha} + \frac{\int_{\mathbb{R}^N} f u_n dx}{T_n} \tag{2.16}$$

If $\|\nabla u_n\|_p \to \infty$ as $n \to \infty$, we have

$$T_n^{-1} \int_{\mathbb{R}^N} |fu_n|\, dx \le T_n^{-1} \|u_n\|_{p^*} \|f\|_\gamma \le S^{-1/p} \|\nabla u_n\|_p^{1-p} \|f\|_\gamma \longrightarrow 0,$$

where S is given in (1.6) and $\gamma = p^*/(p^* - 1)$. If $\|u_n\|_m \to \infty$ as $n \to \infty$, we have

$$T_n^{-1} \int_{\mathbb{R}^N} |fu_n|\, dx \le T_n^{-1} \|u_n\|_m \|f\|_{m'} \le \|u_n\|_m^{1-m} \|f\|_{m'} \longrightarrow 0,$$

with $m' = m/(m-1)$. Here, we use the fact that $f \in L^{q'}(\mathbb{R}^N) \cap L^\gamma(\mathbb{R}^N)$ implies $f \in L^{m'}(\mathbb{R}^N)$.

Letting $n \to \infty$ in (2.16), we obtain $\alpha \le p - 1$. This is a contradiction. Thus, J is bounded below on $\overline{\mathcal{N}^\alpha}$. This concludes the proof. $\square$

Lemma 2.4. *Assume $\{u_n\} \subset E$ satisfies $J'(u_n)u_n = 0$ for any $n \in \mathbb{N}$ and $\{J(u_n)\}$ is bounded. Then $\{u_n\}$ is bounded in E.*

Proof. Since $J'(u_n)u_n = 0$, we see that

$$-J(u_n) = \left(\frac{1}{q} - \frac{1}{p}\right) \|\nabla u_n\|_p^p + \left(\frac{1}{q} - \frac{1}{m}\right) \|u_n\|_m^m + \frac{q-1}{q} \int_{\mathbb{R}^N} f u_n\, dx.$$

By Hölder's and Young's inequalities with small $\varepsilon > 0$, we have

$$\int_{\mathbb{R}^N} |fu_n|\, dx \le \varepsilon \|u_n\|_m^m + C_\varepsilon \|f\|_{m'}^{m'},$$

and then,

$$-J(u_n) \ge \left(\frac{1}{q} - \frac{1}{p}\right) \|\nabla u_n\|_p^p + \left(\frac{1}{q} - \frac{1}{m} - \varepsilon\right) \|u_n\|_m^m - C_\varepsilon \|f\|_{m'}^{m'}.$$

The fact that $J(u_n)$ is bounded gives that the sequences $\{\|\nabla u_n\|_p\}$ and $\{\|u_n\|_m\}$ are bounded. Furthermore, it follows from (2.14) that $\{\|u_n\|_q\}$ is also bounded. Thus, $\{u_n\}$ is in E. Hence, the proof is finished. $\square$

Lemma 2.5. *Let $\alpha = p - 1 + \epsilon$ with small $\epsilon > 0$. Then*

$$d = \inf_{\overline{\mathcal{N}^\alpha}} J(u) = \inf_{\mathcal{N}^\alpha} J(u).$$

Proof. Assume that there exists a minimizing sequence $\{u_n\} \subset \overline{\mathcal{N}^\alpha}$ with $J(u_n) \to d, J'(u_n)u_n = 0$, and

$$\|\nabla u_n\|_p^p + \|u_n\|_m^m = \frac{\alpha}{p-1}\|u_n\|_q^q. \tag{2.17}$$

Clearly, from Lemma 2.4, there is a $b > 0$ such that $\|u_n\|_q^q \le b$ for all $n \in \mathbb{N}$. Then, we obtain from (2.14) and (2.17) that

$$\begin{aligned} J(u_n) &= \frac{1-p}{p}\|\nabla u_n\|_p^p + \frac{1-m}{m}\|u_n\|_m^m + \frac{q-1}{q}\|u_n\|_q^q \\ &= \left(1 - \frac{1}{q} - \frac{\alpha}{p}\right)\|u_n\|_q^q + \left(\frac{1}{m} - \frac{1}{p}\right)\|u_n\|_m^m \\ &\ge \left(1 - \frac{1}{q} - \frac{\alpha}{p}\right)\|u_n\|_q^q + \left(\frac{1}{m} - \frac{1}{p}\right)\frac{\alpha}{p-1}\|u_n\|_q^q \\ &= -\eta_1\|u_n\|_q^q \ge -b\eta_1. \end{aligned} \tag{2.18}$$

Here, and in the sequel,

$$\begin{aligned} \eta_0 &= \frac{\eta_1}{\eta_2}, \\ \eta_1 &= \frac{1}{q} + \frac{\alpha(m-1)}{m(p-1)} - 1 > 0, \\ \eta_2 &= (p-1)\left(\frac{1}{p} - \frac{q-1}{q\alpha}\right) > 0. \end{aligned}$$

We now take $u_0 \in E$ such that

$$b\eta_0 \le \|\nabla u_0\|_p^p + \|u_0\|_m^m < \|u_0\|_q^q \quad \text{and} \quad \int_{\mathbb{R}^N} f u_0 \, dx > 0. \tag{2.19}$$

This is possible if we choose $u_0(x) = k|x|^{-\tau}$ for $|x| \ge 1$ and $u_0(x) = k$ for $|x| < 1$, where k is large and $\tau = \rho + N/q$ with small $\rho > 0$. Furthermore, we let $\gamma(t) = J(tu_0), t \ge 0$. Then,

$$\begin{aligned} \gamma'(0) &= -\int_{\mathbb{R}^N} f u_0 \, dx < 0, \\ \gamma'(1) &= \|\nabla u_0\|_p^p + \|u_0\|_m^m - \|u_0\|_q^q - \int_{\mathbb{R}^N} f u_0 \, dx \\ &< -\int_{\mathbb{R}^N} f u_0 \, dx < 0, \end{aligned}$$

and $\gamma'(t) \to \infty$ as $t \to \infty$. Therefore, there exists a $t_0 > 1$ such that $\gamma'(t_0) = 0$. This implies that

$$\|t_0 \nabla u_0\|_p^p + \|t_0 u_0\|_m^m = \|t_0 u_0\|_q^q + t_0 \int_{\mathbb{R}^N} f u_0 \, dx > \frac{\alpha}{p-1} \|t_0 u_0\|_q^q, \tag{2.20}$$

where $\alpha = p - 1 + \varepsilon$ with small $\varepsilon > 0$. Also, (2.20) shows that the function $v = t_0 u_0 \in E^\alpha$. Then, it follows from (2.18) and (2.20) that

$$\begin{aligned}
J(v) &= \frac{1}{p}\|t_0 \nabla u_0\|_p^p + \frac{1}{m}\|t_0 u_0\|_m^m - \frac{1}{q}\|t_0 u_0\|_q^q - t_0 \int_{\mathbb{R}^N} f u_0 \, dx \\
&= \frac{1-p}{p}\|t_0 \nabla u_0\|_p^p + \frac{1-m}{m}\|t_0 u_0\|_m^m + \frac{q-1}{q}\|t_0 u_0\|_q^q \\
&< \frac{1-p}{p}\|t_0 \nabla u_0\|_p^p + \frac{1-m}{m}\|t_0 u_0\|_m^m \\
&\quad + \frac{(p-1)(q-1)}{\alpha q}(\|t_0 \nabla u_0\|_p^p + \|t_0 u_0\|_m^m) \\
&< -\eta_2(\|t_0 \nabla u_0\|_p^p + \|t_0 u_0\|_m^m) \\
&< -\eta_2(\|\nabla u_0\|_p^p + \|u_0\|_m^m) < -\eta_2 \eta_0 b \\
&= -b\eta_1 \le J(u_n) \longrightarrow d.
\end{aligned}$$

Therefore, we have

$$d = \inf_{u \in \overline{\mathcal{N}^\alpha}} J(u) \le J(v) < -b\eta_1 \le d. \tag{2.21}$$

This is a contradiction. Thus, $u_n \in \mathcal{N}^\alpha$ for all $n \in \mathbb{N}$. Now the proof is complete. $\square$

Lemma 2.6. *Under the assumptions of Theorem* 1.1, *problem* (1.1) *admits a solution* $u \in \overline{\mathcal{N}^\alpha}$ *with* $J(u) = d$ *and*

$$\|\nabla u\|_p^p + \|u\|_m^m \ge \frac{\alpha}{p-1}\|u\|_q^q. \tag{2.22}$$

Proof. By analogy with the proof of Wu **[19]**, we can show that a minimizing sequence $\{u_n\} \subset \overline{\mathcal{N}^\alpha}$ exists such that

$$J(u_n) = d + o(1) \quad \text{and} \quad J'(u_n) = o(1) \text{ in } E^*. \tag{2.23}$$

By Lemma 2.5, we assume $u_n \in \mathcal{N}^\alpha$, and thus, $J(u_n) \to d$ and $J'(u_n)u_n = 0$. Furthermore, it follows from Lemma 2.4 that $\{u_n\}$ is bounded in E. Therefore, there exists a $u \in E$ such that $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^r_{\text{loc}}(\mathbb{R}^N)$, $1 < r < p^*$ and $u_n \to u$ almost everywhere in $\mathbb{R}^N$, up to a subsequence.

Since $u_n \in \mathcal{N}^\alpha$, then $J'(u_n)u_n = 0$, and

$$\int_{\mathbb{R}^N} fu_n dx = \|\nabla u_n\|_p^p + \|u_n\|_m^m - \|u_n\|_q^q > \frac{\alpha - p + 1}{p - 1}\|u_n\|_q^q. \tag{2.24}$$

By the weak lower semi-continuity of the norm, we obtain

$$\int_{\mathbb{R}^N} fu\, dx \geq \frac{\alpha - p + 1}{p - 1} \varliminf_{n \to \infty} \|u_n\|_q^q \geq \frac{\alpha - p + 1}{p - 1}\|u\|_q^q. \tag{2.25}$$

Thus, it follows from (2.24) that

$$\|\nabla u\|_p^p + \|u\|_m^m - \|u\|_q^q = \int_{\mathbb{R}^N} fu\, dx \geq \frac{\alpha - p + 1}{p - 1}\|u\|_q^q.$$

This is (2.22).

Next, we prove $J(u) = d$. Obviously, it is sufficient to show that $u_n \to u$ in E. We note that $\|u\|_E \leq \varliminf_{n\to\infty} \|u_n\|_E$, and the following claims become evident.

Claim 1. Under the assumptions of Theorem 1.1, the case $\|u\|_E < \varliminf_{n\to\infty} \|u_n\|_E$ is impossible.

First, we prove that an unbounded sequence $\{y_n\} \subset \mathbb{R}^N$ exists such that

$$v_n(x + y_n) \equiv u_n(x + y_n) - u(x + y_n) \rightharpoonup U(x) \neq 0$$

in E as $n \to \infty$. Suppose that, for any $\{y_n\} \subset \mathbb{R}^N$, $v_n(x + y_n) \rightharpoonup 0$ in E. Then, for any $r > 0$,

$$\sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |v_n(x)|^q dx \longrightarrow 0 \quad \text{as } n \to \infty, \tag{2.26}$$

where $B_r(y) = \{x \in \mathbb{R}^N \mid |x - y| < r\}$. By **[11,** Lemma I.1], it is seen that $v_n \to 0$ in $L^s(\mathbb{R}^N)$ for all $s \in [q, p^*)$.

On the other hand, assumptions $J'(u_n) \to 0$ in E^* and $v_n(x) = u_n(x) - u(x) \rightharpoonup 0$ in E yield

$$\begin{aligned} J'(u_n)v_n = \int_{\mathbb{R}^N} &[(|\nabla u_n|^{p-2}\nabla u_n \nabla v_n + |u_n|^{m-2}u_n v_n)\, dx \\ &- (|u_n|^{q-2}u_n + f(x))v_n]\, dx \longrightarrow 0, \end{aligned} \tag{2.27}$$

and

$$A_n = \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla v_n + |u|^{m-2} u v_n)\, dx \longrightarrow 0.$$

Since

$$\int_{\mathbb{R}^N} |u_n|^{q-1}|v_n|\, dx \le \|u_n\|_q^{q-1}\|v_n\|_q \le C\|v_n\|_q \longrightarrow 0,$$

$$\int_{\mathbb{R}^N} |f v_n|\, dx \le \|f\|_{q'}\|v_n\|_q \longrightarrow 0,$$

we have from (2.27) that

$$B_n = \int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}\nabla u_n \nabla v_n + |u_n|^{m-2} u_n v_n)\, dx \longrightarrow 0.$$

Note that

$$\begin{aligned} B_n - A_n = &\int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla v_n\, dx \\ &+ \int_{\mathbb{R}^N} (|u_n|^{m-2}u_n - |u|^{m-2}u) v_n\, dx \\ &\ge c_0(\|\nabla(u_n - u)\|_p^p + \|u_n - u\|_m^m) \end{aligned} \tag{2.28}$$

with some constant $c_0 > 0$. Then $B_n - A_n \to 0$ implies that $\|\nabla(u_n - u)\|_p \to 0$ and $\|u_n\|_E \to \|u\|_E$. This is a contradiction. Hence, there exists a $\{y_n\} \subset \mathbb{R}^N$ such that $v_n(x + y_n) \rightharpoonup U(x) \neq 0$ in E.

In the following, we show that the sequence $\{y_n\}$ is unbounded. Suppose that $\{y_n\}$ is bounded. Without loss of generality, we assume that $y_n \to y$ in $\mathbb{R}^N$. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$. By $y_n \to y$ and $v_n(x) \rightharpoonup 0$ in E, it follows that

$$\int_{\mathbb{R}^N} \varphi(x - y_n) v_n(x)\, dx \longrightarrow 0.$$

Since $v_n(x + y_n) \rightharpoonup U(x)$ in E, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \varphi(x - y_n) v_n(x)\, dx &= \int_{\mathbb{R}^N} \varphi(y) v_n(y + y_n)\, dy \\ &\longrightarrow \int_{\mathbb{R}^N} \varphi(y) U(y)\, dy = 0 \end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Hence, $U(x) = 0$ almost everywhere in $\mathbb{R}^N$. This is a contradiction. Thus, $\{y_n\}$ is unbounded in $\mathbb{R}^N$.

In the following, we show that $U(x)$ is a solution of (1.2). For this, we prove $u_n(x+y_n) \rightharpoonup U(x)$ in E. Since $u(x+y_n)$ is bounded in E, there exists a $w \in E$ such that $u(x+y_n) \rightharpoonup w(x)$ in E and

$$\int_{\mathbb{R}^N} u(x+y_n)\varphi(x)\,dx \longrightarrow \int_{\mathbb{R}^N} w(x)\varphi(x)\,dx \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N).$$

However, it follows from [**3,** Lemma 3.5] that

$$\int_{\mathbb{R}^N} u(x+y_n)\varphi(x)\,dx = \int_{\mathbb{R}^N} u(y)\varphi(y-y_n)\,dy \longrightarrow 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Hence, we obtain

$$\int_{\mathbb{R}^N} w(x)\varphi(x)\,dx = 0 \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N)$$

and $w(x) = 0$ almost everywhere in $\mathbb{R}^N$. We have reached the conclusion that $v_n(x+y_n) = u_n(x+y_n) - u(x+y_n) \rightharpoonup U(x)$ in E.

On the other hand, the fact that $J'(u_n) \to 0$ in E^* by (2.23) ensures that $J'(u_n)\varphi(x-y_n) \to 0$, where

(2.29)

$$\begin{aligned}
J'(u_n)\varphi(x-y_n) &= \int_{\mathbb{R}^N} |\nabla u_n(x)|^{p-2}\nabla u_n(x)\nabla\varphi(x-y_n)\,dx \\
&\quad + \int_{\mathbb{R}^N} |u_n(x)|^{m-2}u_n(x)\varphi(x-y_n)\,dx \\
&\quad - \int_{\mathbb{R}^N} |u_n(x)|^{q-2}u_n(x))\varphi(x-y_n)\,dx \\
&\quad - \int_{\mathbb{R}^N} f(x)\varphi(x-y_n)\,dx \\
&= \int_{\mathbb{R}^N} |\nabla u_n(y+y_n)|^{p-2}\nabla u_n(y+y_n)\nabla\varphi(y)\,dy \\
&\quad + \int_{\mathbb{R}^N} |u_n(y+y_n)|^{m-2}u_n(y+y_n)\varphi(y)\,dy \\
&\quad - \int_{\mathbb{R}^N} |u_n(y+y_n)|^{q-2}u_n(y+y_n)\varphi(y)\,dy \\
&\quad - \int_{\mathbb{R}^N} f(x)\varphi(x-y_n)\,dx.
\end{aligned}$$

Similarly, we have from **[3,** Lemma 3.5] that

$$\int_{\mathbb{R}^N} f(x)\varphi(x-y_n)\,dx \longrightarrow 0, \tag{2.30}$$

and the limit $u_n(x+y_n) \rightharpoonup U(x)$ in E yields

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^N} &(|\nabla u_n(y+y_n)|^{p-2}\nabla u_n(y+y_n)\nabla\varphi(y) \\ &+ |u_n(y+y_n)|^{m-2}u_n(y+y_n)\varphi(y))\,dy \\ &= \int_{\mathbb{R}^N}(|\nabla U(y)|^{p-2}\nabla U(y)\nabla\varphi(y) + |U(y)|^{m-2}U(y)\varphi(y))\,dy.\end{aligned} \tag{2.31}$$

Moreover, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n(y+y_n)|^{q-2}u_n(y+y_n)\varphi(y)dy = \int_{\mathbb{R}^N}|U(y)|^{q-2}U(y)\varphi(y)dy. \tag{2.32}$$

In fact, since $u_n(x+y_n) \to U(x)$ in $L^q(\operatorname{supp}\varphi)$, there exists a subsequence, still denoted by u_n, $h \in L^q(\mathbb{R}^N)$, such that

$$|u_n(x+y_n)|^{q-2}u_n(x+y_n)\varphi(x) \longrightarrow |U(x)|U(x)\varphi(x)$$

almost everywhere in $\mathbb{R}^N$, and

$$|u_n(x+y_n)|^{q-1}|\varphi(x)| \le |h(x)|^{q-1}|\varphi(x)| \in L^1(\mathbb{R}^N). \tag{2.33}$$

By the Lebesgue dominated convergence theorem and (2.29)–(2.33), it follows that

$$\begin{aligned}\int_{\mathbb{R}^N}&(|\nabla U(y)|^{p-2}\nabla U(y)\nabla\varphi(y) + |U(y)|^{m-2}U(y)\varphi(y))\,dy \\ &= \int_{\mathbb{R}^N}|U(y)|^{q-2}U(y)\varphi(y)\,dy.\end{aligned} \tag{2.34}$$

This shows that $U(x)$ is a weak solution of (1.2) in E. By Lemma 2.2, $U(x) = 0$ almost everywhere in $\mathbb{R}^N$. This is a contradiction. Thus, the first case $\|u\|_E < \underline{\lim}_{n\to\infty}\|u_n\|_E$ does not hold, and the only possible case is $\|u\|_E = \underline{\lim}_{n\to\infty}\|u_n\|_E$.

Claim 2. If $\|u\|_E = \underline{\lim}_{n\to\infty}\|u_n\|_E$, then we have $u_n \to u$ in E and $J(u_n) \to J(u) = d$. Up to a subsequence, we let $\|u\|_E = \lim_{n\to\infty}\|u_n\|_E$. Since

$$\overline{\lim_{n\to\infty}}\,\|u_n\|_q = \overline{\lim_{n\to\infty}}\,(\|\nabla u_n\|_p + \|u_n\|_q - \|\nabla u_n\|_p) \tag{2.35}$$

$$
\begin{aligned}
&\leq \overline{\lim_{n\to\infty}} \|u_n\|_E - \underline{\lim}_{n\to\infty} \|\nabla u_n\|_p \\
&= \lim_{n\to\infty} \|u_n\|_E - \underline{\lim}_{n\to\infty} \|\nabla u_n\|_p \\
&= \|\nabla u\|_p - \underline{\lim}_{n\to\infty} \|\nabla u_n\|_p + \|u\|_q \leq \|u\|_q,
\end{aligned}
$$

we have

$$
\|u\|_q \leq \underline{\lim}_{n\to\infty} \|u_n\|_q \leq \overline{\lim_{n\to\infty}} \|u_n\|_q \leq \|u\|_q. \tag{2.36}
$$

This shows $\|u_n\|_q \to \|u\|_q$. By the Brezis-Lieb lemma **[5]**, $u_n \to u$ in $L^q(\mathbb{R}^N)$. On the other hand, since $\|u_n\|_E \to \|u\|_E$, we obtain $\|\nabla u_n\|_p \to \|\nabla u\|_p$. Again, by the Brezis-Lieb lemma, $\|\nabla(u_n - u)\|_p \to 0$. By the Sobolev inequality, this implies $\|u_n - u\|_{p^*} \leq C\|\nabla(u_n - u)\|_p \to 0$.

Since $1 < q < p \leq m < p^*$, there exists a $t \in (0,1)$ such that $m = tq + (1-t)p^*$ and

$$
\|u_n - u\|_m^m \leq \|u_n - u\|_q^{tq} \|u_n - u\|_{p^*}^{(1-t)p^*} \longrightarrow 0. \tag{2.37}
$$

Similarly, we derive that

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} f u_n \, dx = \int_{\mathbb{R}^N} f u \, dx. \tag{2.38}
$$

Hence, $u_n \to u$ in E and $J(u_n) \to J(u) = d$ as $n \to \infty$.

We now prove that u is a critical point for J in E, that is, $J'(u)v = 0$ for all $v \in E$, and thus, $J'(u) = 0$ in E^*.

For every $v \in E$, we choose $\varepsilon > 0$ such that $u + sv \neq 0$ for all $s \in (-\varepsilon, \varepsilon)$. Define a function $\varphi : (-\varepsilon, \varepsilon) \times (0, \infty) \to \mathbb{R}$ by

$$
\begin{aligned}
\varphi(s,t) &= J'(t(u+sv))t(u+sv) \\
&= t^p \|\nabla(u+sv)\|_p^p + t^m \|u+sv\|_m^m \\
&\quad - t^q \|u+sv\|_q^q - t \int_{\mathbb{R}^N} f(u+sv) \, dx.
\end{aligned} \tag{2.39}
$$

Then,

$$
\varphi(0,1) = J'(u)v = \|\nabla u\|_p^p + \|u\|_m^m - \|u\|_q^q - \int_{\mathbb{R}^N} f u \, dx = 0, \tag{2.40}
$$

and

$$\begin{aligned}\frac{\partial \varphi}{\partial t}(0,1) &= p\|\nabla u\|_p^p + m\|u\|_m^m - q\|u\|_q^q - \int_{\mathbb{R}^N} fu\,dx \\ &= (p-1)\|\nabla u\|_p^p + (m-1)\|u\|_m^m + (1-q)\|u\|_q^q \\ &\geq (p-1)\|\nabla u\|_p^p + (m-1)\|u\|_m^m \\ &\quad - \frac{(q-1)(p-1)}{\alpha}\left(\|\nabla u\|_p^p + \|u\|_m^m\right) \\ &= \frac{(p-1)(\alpha-q+1)}{\alpha}\|\nabla u\|_p^p \\ &\quad + \frac{\alpha(m-1)-(p-1)(q-1)}{\alpha}\|u\|_m^m > 0.\end{aligned} \tag{2.41}$$

Thus, by the implicit function theorem, there exists a C^1 function $t : (-\varepsilon_0, \varepsilon_0)\ (\subseteq (-\varepsilon, \varepsilon)) \to \mathbb{R}$ such that $t(0) = 1$ and $\varphi(s, t(s)) = 0$ for all $s \in (-\varepsilon_0, \varepsilon_0)$. This also shows that $t(s) \neq 0$, at least for ε_0 very small. Therefore, $t(s)(u + sv) \in \mathcal{N}$. Denote $t = t(s)$ and

$$\begin{aligned}\phi(s) = J(t(u+sv)) &= \frac{1}{p}\|\nabla t(u+sv)\|_p^p \\ &\quad + \frac{1}{m}\|t(u+sv)\|_m^m - \frac{1}{q}\|t(u+sv)\|_q^q \\ &\quad - t\int_{\mathbb{R}^N} f(u+sv)\,dx.\end{aligned}$$

We see that the function $\phi(s)$ is differentiable and has a minimum point at $s = 0$. Thus,

$$0 = \phi'(0) = t'(0)\left(\|\nabla u\|_p^p + \|u\|_m^m - \|u\|_q^q - \int_{\mathbb{R}^N} fu\,dx\right) + J'(u)v. \tag{2.42}$$

It follows from (2.40) that $J'(u)v = 0$ for every $v \in E$, and thus, $J'(u) = 0$ in E^*, that is, u is a critical point of J and u is a weak solution of (1.1) in E. This completes the proof. □

3. Proof of Theorem 1.1. The existence of solution u of problem (1.1) follows from Lemma 2.6. We now prove that this solution is positive. Consider the function

$$\psi(t) = \frac{1}{p}\|t\nabla u\|_p^p + \frac{1}{m}\|tu\|_m^m - \frac{1}{q}\|tu\|_q^q - t\int_{\mathbb{R}^N} f|u|\,dx, \quad t \geq 0. \tag{3.1}$$

Then,

$$\psi'(0) = -\int_{\mathbb{R}^N} f|u|\,dx < 0 \quad \text{and} \quad \lim_{t\to+\infty} \psi'(t) \longrightarrow +\infty.$$

Thus, there exists a $t_0 > 0$ such that $\psi'(t_0) = 0$ and $\psi(t_0) = \inf_{t\geq 0}\psi(t)$.

Since $\psi'(0) < 0$, there exists a $t_1 > 0$ such that $\psi'(t) < 0$ in $(0, t_1)$, that is, $\psi(t)$ is non-increasing in $(0, t_1)$. Similarly, the fact that $\psi'(t) \to +\infty$ as $t \to +\infty$ implies that there exists a $T_1 > t_1$ such that $\psi'(t) > 0$, that is, $\psi(t)$ is increasing in (T_1, ∞). Therefore, $t_0 \in (t_1, T_1)$. Moreover, the fact that u is a solution of (1.1) gives that

$$\begin{aligned}\psi'(1) &= \|\nabla u\|_p^p + \|u\|_m^m - \|u\|_q^q - \int_{\mathbb{R}^N} f(x)|u|\,dx \\ &= \int_{\mathbb{R}^N} f(x)u\,dx - \int_{\mathbb{R}^N} f(x)|u|\,dx \leq 0.\end{aligned} \tag{3.2}$$

We claim that $\psi'(1) = 0$. Otherwise, if $\psi'(1) < 0$, we have $t_0 > 1$. It follows from (2.22) that

$$\|t_0\nabla u\|_p^p + \|t_0 u\|_m^m > \frac{\alpha}{p-1}\|t_0 u\|_q^q,$$

that is, $v = t_0|u| \in \mathcal{N}^\alpha$. Note that

$$d \leq J(v) = \psi(t_0) < \psi(1) = J(u) \leq J(|u|) \leq J(u) = d.$$

This is a contradiction. Thus, $\psi'(1) = 0$ and $\int_{\mathbb{R}^N} f(x)(u - |u|)\,dx = 0$. Furthermore, the assumption $f \geq 0$ implies that $u = |u|$ almost everywhere in $\mathbb{R}^N$. Therefore, u is a nonnegative weak solution of (1.1). By the maximum principle **[16]**, u is a positive solution of (1.1).

Finally, we prove continuity of the solutions. Let $f = f_n \to 0 \in L^{q'}(\mathbb{R}^N)$ in (1.1) as $n \to \infty$, and let u_n be the solution of (1.1) given by Lemma 2.6. Since u_n satisfies (1.1) and $u_n \in \mathcal{N}^\alpha$, we see that

$$\|u_n\|_q^q + \int_{\mathbb{R}^N} f_n u_n\,dx = \|\nabla u_n\|_p^p + \|u_n\|_m^m \geq \frac{\alpha}{p-1}\|u_n\|_q^q \tag{3.3}$$

and

$$\frac{\alpha - p + 1}{p-1}\|u_n\|_q^q \leq \int_{\mathbb{R}^N} f_n u_n\,dx \leq \|f_n\|_{q'}\|u\|_q. \tag{3.4}$$

Therefore,

$$\|u_n\|_q \le \left(\frac{p-1}{\alpha-p+1}\right)^{1/(q-1)} \|f_n\|_{q'}^{1/(q-1)}. \tag{3.5}$$

This shows that $\|u_n\|_q \to 0$ as $f_n \to 0$ in $L^{q'}(\mathbb{R}^N)$. Furthermore, it follows from (3.3) that $\|\nabla u_n\|_p^p \to 0$ and $u_n \to 0$ in E. This completes the proof. □

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