ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 46, Number 6, 2016

ON SCHAUDER BASIS PROPERTIES OF MULTIPLY GENERATED GABOR SYSTEMS

MORTEN NIELSEN

ABSTRACT. Let $\mathcal{A}$ be a finite subset of $L^2(\mathbb{R})$ and $p, q \in \mathbb{N}$. We characterize the Schauder basis properties in $L^2(\mathbb{R})$ of the Gabor system

$$G(1, p/q, \mathcal{A}) = \{e^{2\pi imx} g(x - np/q) : m, n \in \mathbb{Z}, g \in \mathcal{A}\},$$

with a specific ordering on $\mathbb{Z} \times \mathbb{Z} \times \mathcal{A}$. The characterization is given in terms of a Muckenhoupt matrix A_2 condition on an associated Zibulski-Zeevi type matrix.

1. Introduction. For a fixed function $g \in L^2(\mathbb{R})$, the corresponding Gabor system is the collection of functions

$$G(a, b, g) = \{e^{2\pi i \max} g(x - bn) : m, n \in \mathbb{Z}\}.$$

Such systems were introduced by Gabor with the aim of creating sparse and efficient time-frequency localized expansions of signals using elements from $G(a, b, g)$. A major contribution to the theory of Gabor systems was made by Daubechies et al. **[3]** by studying the problem of obtaining expansions relative to $G(a, b, g)$ in a Hilbert space frame setup giving a much more systematic approach to such expansions. The frame approach in **[3]** paved the way for a very extensive study of the frame properties of Gabor systems, see e.g., **[2, 4, 6]** and the references therein.

In this paper, we consider multiple-generated Gabor systems of the form

$$G(1, p/q, \mathcal{A}) = \{e^{2\pi imx} g(x - np/q) : m, n \in \mathbb{Z}, g \in \mathcal{A}\},$$

with $\mathcal{A}$ a finite subset of $L^2(\mathbb{R})$ and $p, q \in \mathbb{N}$.

2010 AMS *Mathematics subject classification.* Primary 42C15, 46B15, Secondary 42C40.

Keywords and phrases. Multiple Gabor systems, Schauder basis, Muckenhoupt matrix condition, conditional convergence.

Received by the editors on February 27, 2015.

DOI:10.1216/RMJ-2016-46-6-2043

There are a number of interesting stability questions related to $G(1, p/q, \mathcal{A})$. One immediate question regards completeness. We let $\mathcal{G} = \overline{\text{Span}}\,(G(1, p/q, \mathcal{A}))$. The question is then whether $\mathcal{G} = L^2(\mathbb{R})$ or $\mathcal{G}$ is a proper subspace of $L^2(\mathbb{R})$. This question was addressed by Zibulski and Zeevi [**14**], where a complete characterization of completeness is given in terms of properties of a certain corresponding matrix generated by the so-called Zak transform. The Zak transform-based approach by Zibulski and Zeevi will play a central role in the present paper.

Frame properties of $G(1, p/q, \mathcal{A})$ were considered [**14**] and further studied in [**1, 5, 10**]. Multiple-generated Gabor Riesz bases for $L^2(\mathbb{R})$ and/or for $\mathcal{G}$ (so-called Riesz sequences) were also characterized by Bownik and Christensen [**1**], see also [**5**] for the single window case.

Expansions relative to a Riesz basis converge unconditionally. In the present paper, we move one step further to the borderline case where the convergence might be conditional and depend upon the precise ordering of the system $G(1, p/q, \mathcal{A})$.

We recall that an ordered family $B = \{x_n : n \in \mathbb{N}\}$ of vectors in a Hilbert space H is a Schauder basis for H if for every $x \in H$ there exists a unique sequence $\{\alpha_n := \alpha_n(x) : n \in \mathbb{N}\}$ of scalars such that

$$\lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n x_n = x \tag{1.1}$$

in the norm topology of H. Clearly, any Riesz basis for H is also a Schauder basis for H. For Riesz bases the convergence in equation (1.1) is unconditional. However, it is well known that conditional Schauder bases exist.

We give a complete characterization of when the system $G(1, p/q, \mathcal{A})$ with the proper ordering forms a Schauder basis for $\mathcal{G}$ and $L^2(\mathbb{R})$. The problem of characterizing Gabor Schauder bases in the case of one generator was first considered by Heil and Powell [**7**]. They obtained a complete characterization in terms of a so-called Muckenhoupt A_2 condition on a certain multivariate weight obtained by the Zak transform. Our result will reproduce the result obtained [**7**] in the case of a singleton $\mathcal{A}$ and $p = q = 1$. Our characterization will be given in terms of a certain matrix Muckenhoupt A_2 condition first introduced by the author [**9**].

2. Zak transform analysis of Gabor systems. Let us introduce some notation that will be used throughout the paper. For $p \in \mathbb{N}$, we define the domain as

$$T_p := [0,1) \times [0,1/p). \tag{2.1}$$

We call a measurable (periodic) matrix function $W : T_p \to \mathbb{C}^{N\times N}$ a *matrix weight* if $W(x)$ is a non-negative definite Hermitian matrix for almost every x, and W is in L^1, i.e., the matrix norm $\|W\|$ belongs to $L^1(T_p)$.

We define the matrix weighted L^2, denoted $L^2(T_p, W)$ to be the set of vector functions $\mathbf{f} : T_p \to \mathbb{C}^N$ such that

$$\begin{aligned}\|\mathbf{f}\|^2_{L^2(T_p,W)} &:= \int_0^1 \int_0^{1/p} |W^{1/2}(x,u)\mathbf{f}(x,u)|^2 \, dx \, du \\ &= \int_0^1 \int_0^{1/p} \langle W(x,u)\mathbf{f}(x,u), \mathbf{f}(x,u)\rangle_{\mathbb{C}^N} \, dx \, du < \infty,\end{aligned}$$

where the Lebesgue measure is used. We can turn $L^2(T_p, W)$ into a Hilbert space by factorizing over $N = \{\mathbf{f} : \|\mathbf{f}\|_{L^2(T_p,W)} = 0\}$. Whenever $W(x)$ is strictly positive definite, N is exactly the set of vector functions $\mathbf{f}$ defined on T_p that vanish almost everywhere.

The main tool we will use to analyze Gabor systems is the Zak transform. The Zak transform is defined for $f \in L^2(\mathbb{R})$ by

$$Zf(x,u) = \sum_{k\in\mathbb{Z}} f(x-k)e^{2\pi iku}, \quad x,u \in \mathbb{T}.$$

The Zak transform is a unitary transform of $f \in L^2(\mathbb{R})$ onto $L^2(\mathbb{T}^2)$.

We now turn to the multiple-generated Gabor setup. Suppose we have L generators $\mathcal{A} = \{g^\ell\}_{l=0}^{L-1} \subset L^2(\mathbb{R})$. For fixed $p, q \in \mathbb{N}$, the associated Gabor system is given by

$$\mathcal{G} := G(1, p/q, \mathcal{A}) = \{g^\ell_{m,n}\}_{m,n\in\mathbb{Z}},$$

with $g^\ell_{m,n} := e^{2\pi imx} g^\ell(x - np/q)$. A straightforward calculation shows, see e.g., [**5,** Lemma 3.2],

$$Zg^\ell_{m,(nq+r)}(x,u) = e^{2\pi imx}e^{-2\pi inpu} Zg^\ell\left(x-r\frac{p}{q}, u\right), \quad 0 \le r < q. \tag{2.2}$$

For notational convenience, we set

$$E_{m,n}(x,u) := e^{2\pi imx}e^{-2\pi inpu}. \tag{2.3}$$

Let us consider a finite expansion

$$f = \sum_{\ell=0}^{L-1}\sum_{r=0}^{q-1}\sum_{m,n\in\mathbb{Z}} c^{\ell}_{m,nq+r}g^{\ell}_{m,nq+r}. \tag{2.4}$$

Put

$$\tau_r^l(x,u) = \sum_{m,n\in\mathbb{Z}} c^{\ell}_{m,nq+r}E_{m,n}(x,u). \tag{2.5}$$

Then, by equation (2.2),

$$Zf(x,u) = \sum_{\ell=0}^{L-1}\sum_{r=0}^{q-1}\tau_r^{\ell}(x,u)Zg^{\ell}\left(x - r\frac{p}{q}, u\right). \tag{2.6}$$

We now follow the approach of Zibulski and Zeevi **[14]** and introduce a convenient matrix notation. We let $G^\ell := G^\ell(x,u)$ be the $q \times p$-matrix given by

$$G^{\ell}_{r,k} = Zg^{\ell}\left(x - r\frac{p}{q}, u + \frac{k}{p}\right), \quad 0 \le r < q,\ 0 \le k < p. \tag{2.7}$$

We form the $Lq \times p$-matrix as

$$G = \begin{bmatrix} G^0 \\ G^1 \\ \vdots \\ G^{L-1} \end{bmatrix}, \tag{2.8}$$

and put

$$W = GG^* \ge 0. \tag{2.9}$$

We mention that in order to study completeness in $L^2(\mathbb{R})$ of the system $G(1, p/q, \mathcal{A})$ and to study its frame properties one often turns to the $p\times p$-matrix G^*G, see **[14, 1]**; for example, $\mathcal{G} = L^2(\mathbb{R})$ if and only if G^*G has full rank almost everywhere. However, as we will see below, basis properties of $G(1, p/q, \mathcal{A})$ are more closely related to properties of the matrix in equation (2.9).

Notice that W given by equation (2.9) is an $Lq \times Lq$-matrix, with entry $(sq+r, tq+\ell)$ given by

$$\sum_{k=0}^{p-1} Zg^s\Big(x - r\frac{p}{q}, u + \frac{k}{p}\Big)\overline{Zg^t\Big(x - \ell\frac{p}{q}, u + \frac{k}{p}\Big)}.$$

Also notice that each entry in W is in $L^1(T_p)$. This follows from the Cauchy-Schwarz inequality using that $Zg^s \in L^2(\mathbb{T}^2)$, $s = 0, \dots, L-1$.

We now form the vector $\boldsymbol{\tau}(u,x) \in \mathbb{C}^{Lq}$ by letting $(\tau(u,x))_{sq+r} = \tau_r^s(x,u)$, $0 \le s \le L-1$, $0 \le r < q$. We now obtain the following result.

Theorem 2.1. *Let $p, q \in \mathbb{N}$. Suppose that*

$$\mathcal{A} = \{g^0, \dots, g^{L-1}\} \subset L^2(\mathbb{R}),$$
$$\mathcal{G} = \overline{\operatorname{Span}}\{G(1, p/q, \mathcal{A})\},$$

and W is the non-negative definite matrix given by equation (2.9)*. Then the map $\mathcal{Z} : L^2(T_p, W) \to \mathcal{G}$ given by*

(2.10)
$$\mathcal{Z}(\mathbf{f}) = Z^{-1}\Bigg(\sum_{\ell=0}^{L-1}\sum_{r=0}^{q-1}(\mathbf{f})_{\ell q+r} Zg^\ell\Big(x - r\frac{p}{q}, u\Big)\Bigg), \quad \mathbf{f} = [f_0, \dots, f_{Lq-1}]^T,$$

is an isometric isomorphism.

Proof. Let $\{\mathbf{e}_j\}_{j=0}^{Lq-1}$ denote the standard basis for $\mathbb{C}^{Lq}$. It follows from equation (2.2) and equation (2.10) that

$$\mathcal{Z}(\mathbf{e}_{\ell q+r} E_{m,n}) = g^\ell_{m,nq+r}. \tag{2.11}$$

Now, take a vector-function $\boldsymbol{\tau}(x,u) \in L^2(T_p, W)$ of the form

$$\boldsymbol{\tau}(x,u) := \sum_{\ell=0}^{L-1}\sum_{r=0}^{q-1} \tau_r^\ell(x,u)\mathbf{e}_{\ell q+r},$$

where each

$$\tau_r^\ell(x,u) = \sum_{m,n\in\mathbb{Z}} c^\ell_{m,nq+r} E_{m,n}(x,u)$$

is a trigonometric polynomial. We notice that, by equation (2.11) and linearity,

$$f := \mathcal{Z}(\boldsymbol{\tau}) = \sum_{\ell=0}^{L-1}\sum_{r=0}^{q-1}\sum_{m,n\in\mathbb{Z}} c^{\ell}_{m,nq+r} g^{\ell}_{m,nq+r}.$$

Hence, using $\langle f, f\rangle = \langle Zf, Zf\rangle$ and equation (2.6), we obtain

$$\begin{aligned}
\langle f, f\rangle &= \int_0^1\int_0^1 \sum_{\ell=0}^{L-1}\sum_{r=0}^{q-1} \tau_r^{\ell}(x,u) Zg^{\ell}\left(x - r\frac{p}{q}, u\right) \\
&\qquad \times \overline{\sum_{t=0}^{L-1}\sum_{s=0}^{q-1} \tau_s^{t}(x,u) Zg^{t}\left(x - s\frac{p}{q}, u\right)}\, dx\, du \\
&= \int_0^1\int_0^{1/p} \sum_{r,\ell}\sum_{s,t} \tau_r^{\ell}(x,u) \\
&\qquad \times \left[\sum_{k=0}^{p-1} Zg^{\ell}\left(x - r\frac{p}{q}, u + \frac{k}{p}\right)\overline{Zg^{t}\left(x - s\frac{p}{q}, u + \frac{k}{p}\right)}\right]\overline{\tau_s^{t}(x,u)}\, dx\, du \\
&= \|\boldsymbol{\tau}\|^2_{L^2(T_p, W)}.
\end{aligned}$$

The vectors with trigonometric polynomial entries are dense in $L^2(T_p, W)$, and the images under $\mathcal{Z}$ of such vectors are clearly dense in $\mathcal{G}$. Hence, we may conclude that $\mathcal{Z}$ extends to an isometric isometry from $L^2(T_p, W)$ onto $\mathcal{G}$. $\square$

3. Bi-orthogonal systems and Schauder bases. Let us recall some elementary facts about Schauder bases and bi-orthogonal sequences in a Hilbert space H. Suppose that $B = \{x_n : n \in \mathbb{N}\}$ is a Schauder basis for H, i.e., that for every $x \in H$ there exists a unique sequence $\{\alpha_n := \alpha_n(x) : n \in \mathbb{N}\}$ of scalars such that

$$\lim_{N\to\infty}\sum_{n=1}^{N} \alpha_n x_n = x$$

in the norm topology of H. The unique choice of scalars, and the fact that we are in a Hilbert space, implies that $x \to \alpha_n(x)$ is a continuous linear functional for every $n \in \mathbb{N}$. Furthermore, for every $n \in \mathbb{N}$, there exists a unique vector y_n such that $\alpha_n(x) = \langle x, y_n\rangle$. It follows that

$$\langle x_m, y_n\rangle = \delta_{m,n}, \quad m, n \in \mathbb{N}. \tag{3.1}$$

A pair of sequences $(\{u_n\}_{n\in\mathbb{N}}, \{v_n\}_{n\in\mathbb{N}})$ in H is a *bi-orthogonal system* if $\langle u_m, v_n\rangle = \delta_{m,n}$, $m, n \in \mathbb{N}$. We say that $\{v_n\}_{n\in\mathbb{N}}$ is a *dual sequence* to $\{u_n\}_{n\in\mathbb{N}}$, and vice versa.

A dual sequence is not necessarily uniquely defined. In fact, it is unique if and only if the original sequence is complete in H, i.e., if the span of the original sequence is dense in H.

Suppose that $B = \{x_n : n \in \mathbb{N}\}$ is complete and has a unique dual sequence $\{y_n\}$. Then, B is a Schauder basis for H if and only if the partial sum operators

$$S_N(x) = \sum_{n=1}^{N} \langle x, y_n\rangle x_n$$

are uniformly bounded on H.

Finally, we call $B = \{x_n : n \in \mathbb{N}\}$ a *basis sequence* if it is a Schauder basis for its closed linear span.

3.1. Bi-orthogonal Gabor systems. We obtain the next result for multiple-generated Gabor systems. We let $\{\mathbf{e}_j\}_{j=0}^{Lq-1}$ denote the standard basis for $\mathbb{C}^{Lq}$.

Proposition 3.1. *Let $p, q \in \mathbb{N}$. Suppose $\mathcal{A} = \{g^0, \ldots, g^{L-1}\} \subset L^2(\mathbb{R})$, and define $\mathcal{G} = \overline{\mathrm{Span}}\{G(1, p/q, \mathcal{A})\}$. Let W be the non-negative definite matrix given by equation* (2.9). *Then $G(1, p/q, \mathcal{A})$ has a uniquely defined bi-orthogonal system if and only if $W^{-1} \in L^1(T_p)$. In the case where $W^{-1} \in L^1(T_p)$, the dual element to $g^\ell_{m,nq+r}$, $m, n \in \mathbb{Z}$, $0 \le r < q$, is given by*

$$\widetilde{g^\ell_{m,nq+r}} := p\mathcal{Z}(W^{-1}E_{m,n}\mathbf{e}_{\ell q+r}).$$

Proof. Let us first suppose that $W^{-1} \in L^1(T_p)$. We notice that

$$\|W^{-1}E_{m,n}\mathbf{e}_{\ell q+r}\|^2_{L^2(T_p,W)} = \int_0^1 \int_0^{1/p} \langle WW^{-1}E_{m,n}\mathbf{e}_{\ell q+r}, W^{-1}E_{m,n}\mathbf{e}_{\ell q+r}\rangle_{\mathbb{C}^{Lq}}\, dx\, du$$

$$= \int_0^1 \int_0^{1/p} (W^{-1})_{\ell q+r,\ell q+r}(x,u)\,dx\,du < \infty,$$

so $\widetilde{g^\ell_{m,nq+r}} = \mathcal{Z}(W^{-1}E_{m,n}\mathbf{e}_{\ell q+r})$ is well defined. We have, for $n, n', m, m' \in \mathbb{Z}$, $0 \le r$, $r' < q$ and $0 \le \ell$, $\ell < L$,

$$\begin{aligned}
&\langle \widetilde{g^{\ell'}_{m',n'q+r'}}, g^\ell_{m,nq+r}\rangle_{L^2(\mathbb{R})}\\
&\qquad = p\langle \mathcal{Z}(W^{-1}E_{m',n'}\mathbf{e}_{\ell' q+r'}), \mathcal{Z}(\mathbf{e}_{\ell q+r}E_{m,n})\rangle_{L^2(\mathbb{R})}\\
&\qquad = p\langle W^{-1}E_{m',n'}\mathbf{e}_{\ell' q+r'}, \mathbf{e}_{\ell q+r}E_{m,n}\rangle_{L^2(T_p,W)}\\
&\qquad = p\int_0^1 \int_0^{1/p} E_{m-m',n'-n}\langle \mathbf{e}_{\ell' q+r'}, \mathbf{e}_{\ell q+r}\rangle_{\mathbb{C}^{Lq}}\,dx\,du\\
&\qquad = \delta_{m,m'}\delta_{n,n'}\delta_{\ell,\ell'}\delta_{r,r'},
\end{aligned}$$

so $\{\widetilde{g^\ell_{m,n}}\}$ is indeed a dual sequence to $G(1,p/q,\mathcal{A})$.

We turn to the converse statement. Suppose that $\{\widetilde{g^\ell_{m,n}}\} \subset \mathcal{G} \subseteq L^2(\mathbb{R})$ is a dual sequence to $G(1,p/q,\mathcal{A})$. The map $\mathcal{Z}$ is onto $\mathcal{G}$, so we can write $\widetilde{g^\ell_{m,n}} = \mathcal{Z}(\mathbf{f}^\ell_{m,n})$ for some $\mathbf{f}^\ell_{m,n} \in L^2(T_p,W)$. Then

$$\begin{aligned}
\delta_{m,m'}\delta_{n,n'}\delta_{\ell,\ell'}\delta_{r,r'} &= \langle g^\ell_{m,nq+r}, \widetilde{g^{\ell'}_{m',n'q+r'}}\rangle_{L^2(\mathbb{R})}\\
&= \langle \mathcal{Z}(\mathbf{e}_{\ell q+r}E_{m,n}), \mathcal{Z}(\mathbf{f}^{\ell'}_{m',n'})\rangle_{L^2(\mathbb{R})}\\
&= \int_0^1 \int_0^{1/p} (\mathbf{f}^{\ell'}_{m',n'})^H W \mathbf{e}_{\ell q+r}E_{m,n}(x,u)\,dx\,du.
\end{aligned}$$

We have $(\mathbf{f}^{\ell'}_{m',n'})^H W\mathbf{e}_{\ell q+r} \in L^1(T_p)$ since $\mathbf{f}^{\ell'}_{m',n'} \in L^2(T_p,W)$ and $W \in L^1(T_p)$. Also, $\{pE_{m,n}\}$ forms an orthonormal trigonometric basis for $L^2(T_p)$, and since the Fourier transform is injective on $L^1(T_p)$, we get that for almost all $(x,u) \in T_p$,

$$(\mathbf{f}^{\ell'}_{m',n'}(x,u))^H W(x,u) = pE_{-m,-n}(x,u)\mathbf{e}^T_{\ell q+r}.$$

Hence, W has full rank almost everywhere, and we may solve for $\mathbf{f}_{m',n'}$ to obtain

$$\mathbf{f}_{m',n'} = pW^{-1}(E_{m,n}\mathbf{e}_{\ell q+r}). \qquad \square$$

3.2. Rectangular partial sums. As before, let us consider $\mathcal{A} = \{g^0,\dots,g^{L-1}\} \subset L^2(\mathbb{R})$, and let $\mathcal{G} = \overline{\operatorname{Span}}\{G(1,p/q,\mathcal{A})\}$. Let us suppose that $W^{-1} \in L^1(T_p)$ so that $G(1,p/q,\mathcal{A})$ has a dual system in $\mathcal{G}$.

For any $f = \mathcal{Z}(\boldsymbol{\tau}) \in \mathcal{G}$, and $N_1, N_2 \in \mathbb{N}$, we consider the rectangular partial sum operator $T_{N_1,N_2} : \mathcal{G} \to \mathcal{G}$ given by

$$T_{N_1,N_2} f = \sum_{\ell=0}^{L-1} \sum_{r=0}^{q-1} \sum_{|m|\le N_1, |n|\le N_2} \langle f, \widetilde{g^{\ell}_{m,nq+r}} \rangle g^{\ell}_{m,nq+r}.$$

We would like to study the boundedness properties of $\{T_{N_1,N_2}\}_{N_1,N_2}$ on $\mathcal{G}$. We mention that it is necessary for T_{N_1,N_2} to be uniformly bounded on $\mathcal{G}$ if $G(1, p/q, \mathcal{A})$ forms a Schauder basis for $\mathcal{G}$ with an ordering "compatible" with rectangular partial sums. However, proving uniform boundedness of $\{T_{N_1,N_2}\}_{N_1,N_2}$ is not quite enough to conclude that $G(1, p/q, \mathcal{A})$ forms a Schauder basis for $\mathcal{G}$. We study this in detail in subsection 3.3.

From Proposition 3.1 we obtain that $\widetilde{g^{\ell}_{m,nq+r}} := p\mathcal{Z}(W^{-1} E_{m,n} \mathbf{e}_{\ell q+r})$. Therefore,

$$\begin{aligned}
&S_{N_1,N_2}\boldsymbol{\tau} \\
&\quad := \sum_{\ell=0}^{L-1} \sum_{r=0}^{q-1} \Bigg(\sum_{|m|\le N_1, |n|\le N_2} \langle \mathcal{Z}(\boldsymbol{\tau}), \mathcal{Z}(pE_{m,n} W^{-1} \mathbf{e}_{\ell q+r}) \rangle_{L^2(\mathbb{R})} E_{m,n} \Bigg) \mathbf{e}_{\ell q+r} \\
&\quad = \sum_{\ell=0}^{L-1} \sum_{r=0}^{q-1} \Bigg(\sum_{|m|\le N_1, |n|\le N_2} \langle \boldsymbol{\tau}, pE_{m,n} W^{-1} \mathbf{e}_{\ell q+r} \rangle_{L^2(T_p, W)} E_{m,n} \Bigg) \mathbf{e}_{\ell q+r} \\
&\quad = \sum_{\ell=0}^{L-1} \sum_{r=0}^{q-1} \Bigg(\sum_{|m|\le N_1, |n|\le N_2} \langle \boldsymbol{\tau}, pE_{m,n} \mathbf{e}_{\ell q+r} \rangle_{L^2(T_p)} E_{m,n} \Bigg) \mathbf{e}_{\ell q+r}.
\end{aligned}$$

Also notice that $\mathcal{Z}(T_{N_1,N_2}\boldsymbol{\tau}) = pS_{N_1,N_2} f$ using equation (2.2). And, since $\|f\|_{L^2(\mathbb{R})} = \|\boldsymbol{\tau}\|_{L^2(T_p,W)}$, we deduce that

$$p\|S_{N_1,N_2}\|_{\mathcal{G}\to\mathcal{G}} = \|T_{N_1,N_2}\|_{L^2(T_p,W)\to L^2(T_p,W)}.$$

We can thus study the boundedness of $\{S_{N_1,N_2}\}_{N_1,N_2}$ by studying properties of the trigonometric system in $L^2(T_p, W)$. The connection between convergence of the Fourier series in a weighted L^2-space and the so-called Muckenhoupt condition on the weight was first made precise in the seminal paper by Hunt, Muckenhoupt and Wheeden [**8**]. In this paper, we need a Muckenhoupt condition in the matrix setting. Muckenhoupt matrix weights and their connection to convergence of

Fourier series was studied by Treil and Volberg [**12, 13**]. The following class of product Muckenhoupt weights was introduced in [**9**].

Definition 3.2. Let W be a $N\times N$ matrix weight on T_p, i.e., a $(1,1/p)$-periodic measurable function defined on T_p whose values are positive semi-definite $N\times N$ matrices. We say that W satisfies the Muckenhoupt product A_2-matrix-condition, provided that

$$\sup_{I\times J}\left\|\left(\frac{1}{|I\times J|}\int_I\int_J W(x,u)\,dx\,du\right)^{1/2}\times\left(\frac{1}{|I\times J|}\int_I\int_J W^{-1}(x,u)\,dx\,du\right)^{1/2}\right\|<\infty, \tag{3.2}$$

where the sup is over all rectangles $I\times J\subset T_p$. The collection of all such weights is denoted by $\mathbb{A}_2(T_p)$.

We note that $W\in\mathbb{A}_2$ implies $W,W^{-1}\in L^1(T_p)$. It is not difficult to prove, see [**9,** Lemma 3.4], that $W\in\mathbb{A}_2$ implies that $W(x,\cdot)$ and $W(\cdot,u)$ are uniform in the corresponding univariate matrix A_2 class for almost every x, respectively almost every u. So, for the u variable, we have

$$\operatorname*{ess\,sup}_{u\in[0,1/p)}\sup_I\left\|\left(\frac{1}{|I|}\int_I W(x,u)\,dx\,du\right)^{1/2}\times\left(\frac{1}{|I|}\int_I W^{-1}(x,u)\,dx\right)^{1/2}\right\|<\infty, \tag{3.3}$$

and similarly for $W(\cdot,u)$. This fact will be used in the proof of Theorem 3.1.

We now call on the following product version of the Muckenhoupt, Hunt and Wheeden theorem proved by the author in [**9**].

Theorem 3.3. *Let $W:T_p\to\mathbb{C}^{Lq\times Lq}$ be a matrix weight with $W,W^{-1}\in L^1(T_p)$. Let $\{\mathbf{e}_j\}_{j=0}^{Lq-1}$ denote the standard basis for $\mathbb{C}^N$. Then, the rectangular partial sum operators*

$$S_{N_1,N_2}\mathbf{f}(x,u) := \sum_{\ell=0}^{L-1}\sum_{r=0}^{q-1}\Bigg(\sum_{\substack{m,n\in\mathbb{Z}\\ |m|\le N_1,|n|\le N_2}} \langle \mathbf{f}, pE_{m,n}\mathbf{e}_{\ell q+r}\rangle_{L^2(T_p)} E_{m,n}(x,u)\Bigg)\mathbf{e}_{\ell q+r},$$

are uniformly bounded on $L_2(T_p;W)$ *if and only if* $W\in\mathbb{A}_2$.

Remark 3.4. Theorem 3.3 is stated for weights on the torus $\mathbb{T}^2$ [**9**], but the proof in [**9**] translates verbatim to the case of domain T_p.

We can now deduce the next result.

Corollary 3.5. *Let* $p,q\in\mathbb{N}$. *Suppose that* $\mathcal{A}=\{g^0,\dots,g^{L-1}\}\subset L^2(\mathbb{R})$, *and define* $\mathcal{G}=\overline{\mathrm{Span}}\{G(1,p/q,\mathcal{A})\}$. *Let* W *be the non-negative definite matrix given by equation* (2.7). *Then the partial sum operators* $\{S_{N_1,N_2}\}_{\tilde{N}_1,\tilde{N}_2\in\mathbb{N}}$ *are uniformly bounded on* $\mathcal{G}$ *if and only if* $W\in\mathbb{A}_2(T_p)$.

3.3. Schauder bases. We now turn to the question of turning the system $G(1,p/q,\mathcal{A})$ into a Schauder basis. Guided by Corollary 3.5, we need to find an enumeration of $G(1,p/q,\mathcal{A})$ that respects the rectangular partial sums.

We follow Heil and Powell [**7**] and consider the following class of enumerations.

Definition 3.6. Let Λ be the set containing all enumerations $\{(k_j,n_j)\}_{j=1}^{\infty}$ of $\mathbb{Z}^2$ defined in the following recursive manner.

(1) The initial terms $(k_1,n_1)\dots(k_{J_1},n_{J_1})$ are either

$$(0,0),(1,0),(-1,0),\dots(A_1,0),(-A_1,0)$$

or

$$(0,0),(0,1),(0,-1),\dots,(0,B_1),(0,-B_1),$$

for some positive integers A_1 or B_1.

(2) If $\{(k_j,n_j)\}_{j=1}^{J_k}$ has been constructed to be of the form

$$\{-A_k,\dots,A_k\}\times\{-B_k,\dots,B_k\}$$

for some non-negative integers A_k, B_k, then terms are added either to the top and bottom or the left and right sides to obtain either the rectangle

$$\{-A_k, \dots, A_k\} \times \{-(B_k+1), \dots, B_k+1\}$$

or

$$\{-(A_k+1), \dots, A_k+1\} \times \{-B_k, \dots, B_k\}.$$

For example, terms would first be added to the left side ordered as

$$(-(A_k+1), 0), (-(A_k+1), 1), (-(A_k+1), -1), \dots,\\ (-(A_k+1), B_k), (-(A_k+1), -B_k),$$

and likewise for the right side. The top and bottom proceed analogously.

Given $\sigma \in \Lambda$, we lift σ to an enumeration $\widetilde{\sigma}$ of $\{0, 1, \dots, L-1\} \times \mathbb{Z}^2$, defined as follows

$$(0, \sigma(1)), (1, \sigma(1)), \dots, (Lq-1, \sigma(1)), (0, \sigma(2)), \dots,\\ (Lq-1, \sigma(2)), (0, \sigma(3)), \dots . \tag{3.4}$$

Let us now assume that $\mathcal{A} = \{g^0, g^1, \dots, g^{L-1}\} \subset L_2(\mathbb{R})$ such that the system $\mathcal{G}(1, p/q, \mathcal{A})$ has a unique dual system $\{\widetilde{g^\ell_{m,n}}\}$ in $\mathcal{G} = \overline{\mathrm{Span}}\{G(1, p/q, \mathcal{A})\}$.

For $j \in \mathbb{N}$, we write $\widetilde{\sigma}(j) = (\ell_j, m_j, n_j)$. We let

$$G(\widetilde{\sigma}(j)) := g^{\ell_j}_{m_j, n_j},$$
$$F(\widetilde{\sigma}(j)) = \widetilde{g^{\ell_j}_{m_j, n_j}},$$

and introduce the partial sum operators

$$\mathcal{T}^\sigma_J f := \sum_{j=1}^{J} \langle f, F(\tilde{\sigma}(j))\rangle G(\widetilde{\sigma}(j)), \quad f \in \mathcal{G}.$$

We also need to consider the associated partial sum operator in $L_2(\mathbb{T}^2; W)$. Let $\{\mathbf{e}_j\}_{j=0}^{Lq-1}$ be the standard basis for $\mathbb{C}^{Lq}$. Put $e(\ell, m, n) := E_{m,n}\mathbf{e}_\ell$, and $\widetilde{e}(\ell, m, n) := \mathcal{Z}(pE_{m,n}W^{-1}\mathbf{e}_\ell)$. Then

$$\mathcal{S}_J^\sigma \boldsymbol{\tau} := \sum_{j=1}^{J} \langle \boldsymbol{\tau}, \widetilde{e}(\widetilde{\sigma}(j)) \rangle_{L_2(T_p,W)} e(\widetilde{\sigma}(j)), \quad \tau \in L_2(T_p, W),$$

satisfies $\mathcal{Z}(\mathcal{S}_J^\sigma \tau) = \mathcal{T}_J^\sigma f$ for $\mathcal{Z}(\boldsymbol{\tau}) = f \in \mathcal{G}$.

It is now immediate from our general discussion of Schauder bases that the following conditions are equivalent:

(i) the system $G(1, p/q, \mathcal{A})$ is a Schauder basis for $\mathcal{G}$ with the ordering induced by $\sigma \in \Lambda$;
(ii) the partial sum operators T_J^σ are uniformly bounded on $\mathcal{G}$.

This leads to the next result, Theorem 3.7. The first part of the proof of Theorem 3.7 follows from the approach outlined by the author in [**9,** Corollary 3.4]. We include here the details for the benefit of the reader.

Theorem 3.7. *Let $p, q \in \mathbb{N}$. Suppose that $\mathcal{A} = \{g^0, \dots, g^{L-1}\} \subset L^2(\mathbb{R})$, and define $\mathcal{G} = \overline{\mathrm{Span}}\{G(1, p/q, \mathcal{A})\}$. Let W be the non-negative definite matrix given by equation* (2.7)*. Then the following statements are equivalent*:

(a) $\sup_{\sigma\in\Lambda} \sup_J \|\mathcal{T}_J^\sigma\| < \infty$.
(b) $W \in \mathbb{A}_2(T_p)$.

Moreover, at the critical density $Lq = p$, $W \in \mathbb{A}_2(T_p)$ implies that $G(1, p/q, \mathcal{A})$ forms a Schauder basis for $L^2(\mathbb{R})$.

Remark 3.8. For $L = p = q = 1$, the condition in Theorem 3.7 reduces to the scalar condition $|Zg^0|^2 \in A_2(\mathbb{T}^2)$, which is exactly that derived by Powell and Heil [**7**].

We will need the following facts for the proof of Theorem 3.7. We let D_N denote the univariate Dirichlet kernel, given by

$$D_0(t) = 1, \qquad D_N(t) = \frac{\sin 2\pi(N + 1/2)t}{\sin \pi t}, \quad N \geq 1, \tag{3.5}$$

and, for $f \in L_2(\mathbb{T})$, $N \geq 0$,

$$S_N(f) := \sum_{k=-N}^{N} \widehat{f}(k) e^{2\pi i k \cdot} = f * D_N := \int_{\mathbb{T}} f(t) D_N(\cdot - t)\, dt. \tag{3.6}$$

We lift S_N to the vector setting by letting

$$S_N(\boldsymbol{\tau}) := \sum_{j=0}^{Lq-1} S_N(\langle \boldsymbol{\tau}, \mathbf{e}_j\rangle)\mathbf{e}_j.$$

It is known that

$$\operatorname*{ess\,sup}_{u\in[0,1/p)} \sup_N \|S_N\|_{L^2(\mathbb{T},W(\cdot,u))\to L^2(\mathbb{T},W(\cdot,u))} < \infty \tag{3.7}$$

for weights W satisfying the univariate A_2 condition equation (3.3), see [**9,** Corollary 3.2]. This is very closely related to the fact that the Hilbert transform is bounded for such weights, see [**12, 13**]. We refer to [**9**] for further details. The same result of course holds for $W(x,\cdot)$.

Proof of Theorem 3.7. It suffices to consider the operator $\mathcal{S}_J^\sigma \tau$ on $L^2(T_p, W)$. Given a rectangle

$$R = \{-N_1, \dots, N_1\} \times \{-N_2, \dots, N_2\}, \quad N_1, N_2 \in \mathbb{N}_0,$$

we can use Definition 3.6 to construct an enumeration $\sigma \in \Lambda$ such that $\sigma(\{1,\dots,J\}) = R$ for some $J \in \mathbb{N}$. Then $\mathcal{S}_{N\cdot J}^\sigma = S_{N_1,N_2}$, and therefore, $\sup_{N_1,N_2\geq 0} \|S_{N_1,N_2}\| < \infty$. Hence, $W \in \mathbb{A}_2$ by Corollary 3.5. Conversely, we fix $f \in \mathcal{G}$ and pick $\sigma \in \Lambda$. For any J, we let N_J be the largest integer $N_j \leq J$ for which $\mathcal{T}_{N_j}^\sigma f = T_{L,K} f$, for some integers L, K. Now, by Corollary 3.5,

$$\begin{aligned}\|\mathcal{T}_J^\sigma f\|_{L_2(\mathbb{R})} &\leq \|T_{L,K} f\|_{L_2(\mathbb{R})} + \|(\mathcal{T}_J^\sigma - T_{L,K})f\|_{L_2(\mathbb{R})}\\ &\leq C\|f\|_{L_2(\mathbb{R})} + \|(\mathcal{T}_J^\sigma - T_{L,K})f\|_{L_2(\mathbb{R})}.\end{aligned}$$

Hence, it suffices to bound the norm of the term

$$(T_J^\sigma - T_{L,K})f = \sum_{j=N_J}^{J} \langle f, F(\widetilde{\sigma}(j))\rangle G(\widetilde{\sigma}(j)). \tag{3.8}$$

According to Definition 3.6, the sum (3.8) contains terms that have been added to the top and bottom or left and right side of an rectangle. The cases are treated in a similar fashion. For definiteness, assume that equation (3.8) adds terms to the top of the rectangle.

We study the equivalent problem in $L^2(T_p, W)$. Choose τ with $\mathcal{Z}(\tau) = f$, so that $\mathcal{Z}(\mathcal{S}_J^\sigma \tau) = \mathcal{T}_J^\sigma f$. Note that the ordering $\widetilde{\sigma}$ given by equation (3.4) ensures that the sum $(\mathcal{S}_J^\sigma - S_{L,K})\tau$ can be rewritten as

$$
\begin{aligned}
&(\mathcal{S}_J^\sigma - S_{L,K})\boldsymbol{\tau} = \\
&\sum_{\ell=0}^{L-1}\sum_{r=0}^{q-1}\sum_{m=-M}^{M} \langle \boldsymbol{\tau}, p e^{2\pi i m x} e^{-2\pi i (K+1) p u}\mathbf{e}_{\ell q + r}\rangle_{L^2(T_p)} \\
&\qquad \times e^{2\pi i m x} e^{-2\pi i (K+1) p u}\mathbf{e}_{\ell q + r} + R,
\end{aligned}
\tag{3.9}
$$

where the remainder R is a sum of at most $2Lq-1$ terms of the type $\langle \boldsymbol{\tau}, pE_{m,n}\mathbf{e}_\ell\rangle E_{m,n}\mathbf{e}_\ell$. We observe that, in general,

$$
\|\langle \boldsymbol{\tau}, pE_{m,n}\mathbf{e}_\ell\rangle E_{m,n}\mathbf{e}_\ell\|_{L_2(\mathbb{T}^2;W)} \le p\|W\|_{L_1(T_p)}\|W^{-1}\|_{L_1(T_p)}\|\boldsymbol{\tau}\|_{L_2(T_p,W)},
$$

which follows from Hölder's inequality. We can therefore use the triangle inequality uniformly to estimate the remainder R in equation (3.9) in terms of $\|\boldsymbol{\tau}\|_{L_2(\mathbb{T}^2;W)}$. Next, we note that

$$
\begin{aligned}
&\Bigg\| \sum_{\ell=0}^{L-1}\sum_{r=0}^{q-1}\sum_{m=-M}^{M} \langle \boldsymbol{\tau}, p e^{2\pi i m x} e^{-2\pi i (K+1) p u}\mathbf{e}_{\ell q + r}\rangle_{L^2(T_p)} \\
&\qquad\qquad \times e^{2\pi i m x} e^{-2\pi i (K+1) p u}\mathbf{e}_{\ell q + r}\Bigg\|_{L_2(T_p,W)} \\
&\qquad = \Bigg\| \sum_{\ell=0}^{L-1}\sum_{r=0}^{q-1}\sum_{m=-M}^{M} p\langle \boldsymbol{\tau} e^{2\pi i (K+1) p u}, \\
&\qquad\qquad e^{2\pi i m x}\mathbf{e}_{\ell q + r}\rangle_{L^2(T_p)} e^{2\pi i m x}\mathbf{e}_{\ell q + r}\Bigg\|_{L_2(T_p,W)}.
\end{aligned}
$$

For notational convenience, we define the vector function as

$$
\mathbf{f}(x) := p\int_0^{1/p} \boldsymbol{\tau}(x,u) e^{2\pi i (K+1) p u}\, du = S_0(\boldsymbol{\tau}(x,\cdot) e^{2\pi i (K+1) p \cdot}),
$$

where S_0 is the 0-order partial sum operator. We recall that $W(\cdot,u)$ and $W(x,\cdot)$ are uniformly matrix A_2-weights for almost every u and almost every x, respectively. We now use the boundedness estimate (3.7) to obtain that

$$
\Bigg\| \sum_{\ell=0}^{L-1}\sum_{r=0}^{q-1}\sum_{m=-M}^{M} p\langle \boldsymbol{\tau} e^{2\pi i (K+1) p u}, e^{2\pi i m x}\mathbf{e}_{\ell q + r}\rangle_{L^2(T_p)} e^{2\pi i m x}\mathbf{e}_{\ell q + r}\Bigg\|_{L_2(T_p,W)}
$$

$$
\begin{aligned}
&= \int_0^{1/p}\int_0^1 |W^{1/2}(x,u)D_M * \mathbf{f}|^2 dx\, du \\
&\le C\int_0^{1/p}\int_0^1 |W^{1/2}(x,u)\mathbf{f}|^2 dx\, du \\
&= C\int_0^{1/p}\int_0^1 |W^{1/2}(x,u)S_0(\boldsymbol{\tau}(x,\cdot)e^{2\pi i(K+1)p\cdot})|^2 du\, dx \\
&\le C'\int_0^{1/p}\int_0^1 |W^{1/2}(x,u)\boldsymbol{\tau}e^{2\pi i(K+1)pu}|^2 du\, dx \\
&= C'\|\boldsymbol{\tau}\|^2_{L_2(T_p,W)},
\end{aligned}
$$

where we also used that S_0 is bounded uniformly on $L_2(\mathbb{T};W(x,\cdot))$ for almost every x. Combining estimates, we conclude that

$$
\begin{aligned}
\|(T_J^\sigma - T_{L,K})f\|_{L_2(\mathbb{R})} &= \|(S_J^\sigma - S_{L,K})\boldsymbol{\tau}\|_{L_2(T_p,W)} \\
&\le C'\|\boldsymbol{\tau}\|_{L_2(T_p,W)} \\
&= C'\|f\|_{L_2(\mathbb{R})},
\end{aligned}
$$

with C' independent of J.

For the last part, we notice that, whenever the $Lq \times Lq$ matrix $W = GG^*$ is positive definite almost everywhere, then $p = Lq = \operatorname{rank}(GG^*) \le \operatorname{rank}(G) \le p$ almost everywhere since G is of size $Lq \times p$. Hence, $\operatorname{rank}(G) = p$ almost everywhere. The fact that $\mathcal{G} = L^2(\mathbb{R})$ now follows from [**14,** Theorem 2]. $\square$

Remark 3.9. Suppose that $G(1,p/q,\mathcal{A})$ ordered by $\sigma \in \Lambda$ forms a Schauder basis for $\mathcal{G} = \overline{\operatorname{Span}}\{G(1,p/q,\mathcal{A})\}$ with $\mathcal{A} = \{g^0,\dots,g^{L-1}\}$. Then clearly, by Theorem 3.7, the corresponding weight W satisfies $W \in A_2(T_p)$. It is easy to directly check that, for any subset $\mathcal{A}' \subset \mathcal{A}$, $G(1,p/q,\mathcal{A}')$ forms a Schauder basis for $\mathcal{G}' = \overline{\operatorname{Span}}\{G(1,p/q,\mathcal{A}')\}$. Let W' be the weight corresponding to $G(1,p/q,\mathcal{A}')$, where we notice that W' is a submatrix of W. We deduce from Theorem 3.7 that the submatrix W' must also belong to $A_2(T_p)$.

4. Example. We conclude this paper by giving an example of a conditional multiple-generated Gabor Schauder basis for $L^2(\mathbb{R})$. Let us consider the case $L = p$, $p \ge 2$, and $q = 1$. Take univariate polynomials

$P_0(x), \ldots P_{L-1}(x)$ and exponents $a_0, \ldots, a_{L-1} \in \mathbb{R}$ satisfying

$$-1 < \deg(P_j)a_j < 1, \quad j = 0, 1, \ldots, L-1.$$

Then it is well known that $|P_i|^{a_i}$ satisfies the scalar A_2-condition, i.e.,

$$\sup_I \left(\frac{1}{|I|} \int_I |P_j|^{a_j} \, dx \cdot \frac{1}{|I|} \int_I |P_j|^{-a_j} \, dx \right) < +\infty, \tag{4.1}$$

where the sup is over all intervals $I \subset [0,1)$, see **[11,** Chapter 5]. We now set

$$g^\ell(x) := \chi_{[0,1)}(x+\ell)|P_\ell(x+\ell)|^{a_\ell/2}.$$

It is easy to verify that $g^\ell \in L^2(\mathbb{R})$. Notice that

$$Zg^\ell(x,u) = |P_\ell(x)|^{a_\ell/2} e^{2\pi i \ell u}.$$

We now form the matrix $W = GG^*$ defined by equation (2.9). Notice that entry r, s of W is given by

$$\begin{aligned}
&\sum_{k=0}^{L-1} Zg^r\left(x, u + \frac{k}{L}\right) \overline{Zg^s\left(x, u + \frac{k}{L}\right)} \\
&\qquad = \sum_{k=0}^{L-1} |P_r(x)|^{a_r/2} |P_s(x)|^{a_s/2} e^{2\pi i r(u+k/L)} e^{-2\pi i s(u+k/L)} \\
&\qquad = |P_r(x)|^{a_r/2} |P_s(x)|^{a_s/2} e^{2\pi i (r-s)u} \sum_{k=0}^{L-1} e^{2\pi i (r-s)k/L} \\
&\qquad = L\delta_{r,s} |P_r(x)|^{a_r}.
\end{aligned}$$

It is now straightforward to use equation (4.1) to verify the diagonal matrix $W \in \mathbb{A}_2$. Hence, for $\mathcal{A} = \{g^0, \ldots, g^{L-1}\}$, the system $G(1, L, \mathcal{A})$ forms a Schauder basis for $L^2(\mathbb{R})$ according to Theorem 3.7 since $Lq = L = p$.

Moreover, we note that, by choosing appropriate polynomials P_j, we can easily obtain a matrix G containing unbounded row vectors on $[0,1) \times [0, 1/p)$ and/or row vectors not bounded away from 0 in norm on $[0,1) \times [0, 1/p)$. It thus follows from **[1,** Theorem 2.2] that the corresponding system $G(1, L, \mathcal{A})$ cannot form an unconditional Riesz basis for $L^2(\mathbb{R})$. We thus obtain an example of a conditional multiple-generated Gabor Schauder basis for $L^2(\mathbb{R})$.

REFERENCES

1. M. Bownik and O. Christensen. *Characterization and perturbation of Gabor frame sequences with rational parameters*, J. Approx. Th. **147** (2007), 67–80.

2. O. Christensen, *Frames and bases*, in *Applied and numerical harmonic analysis*, *An introductory course*, Birkhäuser Boston, Inc., Boston, 2008.

3. I. Daubechies, A. Grossmann and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys. **27** (1986), 1271–1283.

4. H.G. Feichtinger and T. Strohmer, eds., *Advances in Gabor analysis*, in *Applied and numerical harmonic analysis*, Birkhäuser Boston, Inc., Boston, 2003.

5. J.-P. Gabardo and D. Han, *Balian-Low phenomenon for subspace Gabor frames*, J. Math. Phys. **45** (2004), 3362–3378.

6. C. Heil, *A basis theory primer*, in *Applied and numerical harmonic analysis*, expanded edition, Birkhäuser/Springer, New York, 2011.

7. C. Heil and A.M. Powell, *Gabor Schauder bases and the Balian-Low theorem*, J. Math. Phys. **47** (2006), `article id no. 113506`.

8. R. Hunt, B. Muckenhoupt and R. Wheeden. *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. **176** (1973), 227–251.

9. M. Nielsen, *On stability of finitely generated shift-invariant systems*, J. Fourier Anal. Appl. **16** (2010), 901–920.

10. A. Ron and Z. Shen, *Weyl-Heisenberg frames and Riesz bases in* $L_2(\mathbf{R}^{\mathbf{d}})$, Duke Math. J. **89** (1997), 237–282.

11. E.M. Stein, *Harmonic analysis*: *Real-variable methods, orthogonality, and oscillatory integrals*, Princeton Math. **43**, Princeton University Press, Princeton, 1993.

12. S. Treil and A. Volberg, *Continuous frame decomposition and a vector Hunt-Muckenhoupt-Wheeden theorem*, Ark. Mat. **35** (1997), 363–386.

13. ______, *Wavelets and the angle between past and future*, J. Funct. Anal. **143** (1997), 269–308.

14. M. Zibulski and Y.Y. Zeevi, *Analysis of multiwindow Gabor-type schemes by frame methods*, Appl. Comp. Harmonic Anal. **4** (1997), 188–221.

Department of Mathematical Sciences, Aalborg University, Fr. Bajers Vej 7G, DK-9220 Aalborg East, Denmark

Email address: **mnielsen@math.aau.dk**