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EXISTENCE OF PERIODIC SOLUTIONS FOR $2n$TH-ORDER NONLINEAR p-LAPLACIAN DIFFERENCE EQUATIONS

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ABSTRACT. By using the critical point theory, the existence of periodic solutions for $2n$th-order nonlinear p-Laplacian difference equations is obtained. The main approaches used in our paper are variational techniques and the Saddle Point theorem. The problem is to solve the existence of periodic solutions for $2n$th-order p-Laplacian difference equations. The results obtained successfully generalize and complement the existing ones.

1. Introduction. In this paper, we consider the following $2n$th-order p-Laplacian difference equation

(1.1)
$$\Delta^n \left(r_{k-n}\varphi_p\left(\Delta^n u_{k-1}\right)\right) + (-1)^n q_k \Delta\varphi_p\left(\Delta u_{k-1}\right) = (-1)^n f(k,u_k), \quad n \in \mathbf{Z}(1), \quad k \in \mathbf{Z},$$

where Δ is the forward difference operator $\Delta u_k = u_{k+1} - u_k$, $\Delta^n u_k = \Delta(\Delta^{n-1}u_k)$, $\varphi_p(s)$ is the p-Laplacian operator $\varphi_p(s) = |s|^{p-2}s$, $1 < p < \infty$, $\{r_k\}$ and $\{q_k\}$ are real sequences, $f \in C(\mathbf{Z}\times\mathbf{R},\mathbf{R})$, T is a given positive integer, $r_{k+T} = r_k > 0$, $q_{k+T} = q_k \geq 0$, $f(k+T,v) = f(k,v)$.

We may think of equation (1.1) as a discrete analogue of the following $2n$th-order differential equation:

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$$\frac{d^n}{dt^n}\left[r(t)\varphi_p\left(\frac{d^n u(t)}{dt^n}\right)\right] + (-1)^n q(t)(\varphi_p(u'(t)))' = (-1)^n f(t, u(t)), \quad t \in \mathbf{R}. \tag{1.2}$$

Existence of periodic solutions of higher-order differential equations has been the subject of many investigations [**8, 15, 16, 17, 32, 36, 37**]. By using various methods and techniques, such as fixed point theory, the Kaplan-Yorke method, critical point theory, coincidence degree theory, bifurcation theory and dynamical system theory, etc., a series of existence results for periodic solutions have been obtained in the literature. Recently, the difference equations have widely occurred as the mathematical models describing real life situations in probability theory, matrix theory, electrical circuit analysis, combinatorial analysis, queuing theory, number theory, psychology and sociology, etc. For the general background of difference equations, one can refer to the monographs [**1, 2, 3, 28**]. Since the last decade, there has been much progress on the qualitative properties of difference equations, which included results on stability and attractivity [**18, 28, 31, 46**] and results on oscillation and other topics [**1, 2, 3, 23, 24, 25, 31, 42, 43, 44, 45**]. Only a few papers discuss the periodic solutions of higher-order difference equations. Therefore, it is worthwhile to explore this topic.

The widely used tools for the existence of periodic solutions of difference equations are the various fixed point theorems in cones [**1, 2, 3, 28**]. It is well known that critical point theory is a very powerful tool that deals with the problems of differential equations [**8, 11, 14, 21, 22, 32, 38, 41**]. Only since 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions of difference equations. By using the critical point theory, Guo and Yu [**23, 24, 25**] and Shi et al. [**39**] established sufficient conditions on the existence of periodic solutions of second-order nonlinear difference equations. Compared to first-order or second-order difference equations, the study of higher-order equations has received considerably less attention (see, for example, [**4, 5, 8, 12, 13, 19, 28, 33, 35**] and the references contained therein). Ahlbrandt and Peterson [**4**] in 1994 studied the $2n$th-order difference equation of the form

$$\sum_{i=0}^{n} \Delta^i \left(r_i(k-i)\Delta^i u(k-i)\right) = 0 \tag{1.3}$$

in the context of the discrete calculus of variations, and Peil and Peterson [**35**] studied the asymptotic behavior of solutions of (1.3) with $r_i(k) \equiv 0$ for $1 \le i \le n-1$. In 1998, Anderson [**5**] considered (1.3) for $k \in \mathbf{Z}(a)$ and obtained a formulation of generalized zeros and (n, n)-disconjugacy for (1.3). Migda [**33**] in 2004 studied an mth-order linear difference equation.

If $q_k \equiv 0$, $p = 2$ and $n = 2$, and replacing $f(k, u_k)$ with $-f(k, u_k)$, (1.1) reduces to the following equation:

$$\Delta^2 \left(r_{k-2}\Delta^2 u_{k-2}\right) + f(k, u_k) = 0, \quad k \in \mathbf{Z}. \tag{1.4}$$

In 2005, Cai, Yu and Guo [**10**] obtained some criteria for the existence of periodic solutions of the fourth-order difference equation for (1.4).

Recently, Cai and Yu [**9**] obtained some criteria for the existence of periodic solutions of the following $2n$th-order difference equation

$$\Delta^n \left(r_{k-n}\Delta^n u_{k-n}\right) + f(k, u_k) = 0, \quad n \in \mathbf{Z}(3),\ k \in \mathbf{Z}, \tag{1.5}$$

for the case where f grows superlinearly at both 0 and ∞.

A great deal of work has also been done on the study of the existence of solutions to discrete boundary value problems with the p-Laplacian operator. Because of potential applications in many fields, we refer the reader to the monograph by Agarwal, et al., and some recent contributions [**6, 7, 8, 26, 27, 29, 30, 40**]. However, to the best of our knowledge, the results on periodic solutions of higher-order nonlinear difference equations with p-Laplacian are very scarce in the literature.

The main intention of this paper is to give some sufficient conditions for the existence and multiplicity of periodic and subharmonic solutions for $2n$th-order nonlinear p-Laplacian difference equations. The proof is based on the Saddle Point theorem in combination with the variational technique. In particular, our results generalize and complement the results in the literature [**9, 10**]. In fact, one can see the following Remark 1.4 for details. The motivation for the present work stems from the recent papers [**13, 20**].

Let

$$\underline{r} = \min_{k\in \mathbf{Z}(1,T)} \{r_k\}, \qquad \overline{r} = \max_{k\in \mathbf{Z}(1,T)} \{r_k\},$$
$$\underline{q} = \min_{k\in \mathbf{Z}(1,T)} \{q_k\}, \qquad \overline{q} = \max_{k\in \mathbf{Z}(1,T)} \{q_k\}.$$

Now we state the main results of this paper.

Theorem 1.1. *Assume that the following hypotheses are satisfied:*

(F_1) *there exists a functional* $F(k,v) \in C^1(\boldsymbol{Z}\times \boldsymbol{R}, \boldsymbol{R})$,

$$\frac{\partial F(k,v)}{\partial v} = f(k,v)$$

such that

$$F(k+T,v) = F(k,v);$$

(F_2) *there exists a constant* $M_0 > 0$ *for all* $(k,v) \in \boldsymbol{Z}\times\boldsymbol{R}$ *such that* $|f(k,v)| \le M_0$;

(F_3) $F(k,v) \to +\infty$ *uniformly for* $k \in \boldsymbol{Z}$ *as* $|v| \to +\infty$.

Then, for any given positive integer $m > 0$, *equation* (1.1) *has at least one* mT*-periodic solution.*

Remark 1.2. Assumption (F_2) implies that there exists a constant $M_1 > 0$ such that

$$(F_2') \qquad |F(k,v)| \le M_1 + M_0|v|, \quad \text{for all } (k,v) \in \mathbf{Z}\times\mathbf{R}.$$

Theorem 1.3. *Assume that* (F_1) *holds; further,*

(F_4) *there exist constants* $R_1 > 0$ *and* α, $1 < \alpha < 2$, *such that, for* $k \in \boldsymbol{Z}$ *and* $|v| \ge R_1$,

$$0 < f(k,v)v \le \frac{\alpha}{2} pF(k,v);$$

(F_5) *there exist constants* $a_1 > 0$, $a_2 > 0$ *and* γ, $1 < \gamma \le \alpha$ *such that*

$$F(k,v) \ge a_1|v|^{(\gamma/2)p} - a_2, \quad \textit{for all } (k,v) \in \boldsymbol{Z}\times\boldsymbol{R}.$$

Then, for any given positive integer $m > 0$, *equation* (1.1) *has at least one* mT*-periodic solution.*

Remark 1.4. Assumption (F_4) implies that, for each $k \in \mathbf{Z}$, there exist constants $a_3 > 0$ and $a_4 > 0$ such that

(F_4') $F(k,v) \le a_3|v|^{(\alpha/2)p} + a_4, \quad \text{for all } (k,v) \in \mathbf{Z} \times \mathbf{R}.$

Remark 1.5. The results of Theorems 1.1 and 1.3 ensure that equation (1.1) has at least one mT-periodic solution. However, in some cases, we are interested in the existence of nontrivial periodic solutions for equation (1.1).

In this case, we have

Theorem 1.6. *Assume that* (F_1) *holds*; *further,*

(F_6) $F(k,0) = 0$, $f(k,v) = 0$ *if and only if* $v = 0$, *for all* $k \in \boldsymbol{Z}$;

(F_7) *there exists a constant* α, $1 < \alpha < 2$, *such that, for* $k \in \boldsymbol{Z}$,

$$0 < f(k,v)v \le \frac{\alpha}{2} pF(k,v), \quad \textit{for all } v \ne 0;$$

(F_8) *there exist constants* $a_5 > 0$ *and* γ, $1 < \gamma \le \alpha$, *such that*

$$F(k,v) \ge a_5|v|^{(\gamma/2)p}, \quad \textit{for all } (k,v) \in \boldsymbol{Z} \times \boldsymbol{R}.$$

Then, for any given positive integer $m > 0$, *equation* (1.1) *has at least one nontrivial* mT*-periodic solution.*

Theorem 1.7. *Assume that* (F_1)–(F_3) *and* (F_6) *hold*; *further,*

(F_9) *there exist constants* $a_6 > 0$ *and* θ, $0 < \theta < 2$, *such that*

$$F(k,v) \ge a_6|v|^{(\theta/2)p}, \quad \textit{for all } (k,v) \in \boldsymbol{Z} \times \boldsymbol{R}.$$

Then, for any given positive integer $m > 0$, *equation* (1.1) *has at least one nontrivial* mT*-periodic solution.*

If $q_k \equiv 0$, $p = 2$ and $n = 2$, replacing $f(k,u_k)$ with $(-1)^{n+1}f(k,u_k)$, equation (1.1) reduces to equation (1.5). Then, we have the following results.

Theorem 1.8. *Assume that* (F_1) *holds*; *further,*

(F_{10}) $F(k,0) = 0$, *for all* $k \in \boldsymbol{Z}$;

(F_{11}) *there exists a constant* α, $1 < \alpha < 2$, *such that, for* $k \in \boldsymbol{Z}$,

$$\alpha F(k, v) \leq v f(k, v) < 0, \quad \textit{for all } |v| \neq 0;$$

(F_{12}) *there exist constants* $a_7 > 0$ *and* γ, $1 < \gamma \leq \alpha$, *such that*

$$F(k, v) \leq -a_7 |v|^{\gamma}, \quad \textit{for all } (k, v) \in \boldsymbol{Z} \times \boldsymbol{R}.$$

Then, for any given positive integer $m > 0$, *equation* (1.5) *has at least one nontrivial* mT*-periodic solution.*

Theorem 1.9. *Assume that* (F_1), (F_2) *and* (F_6) *hold*; *further,*

(F_{13}) $F(k, v) \to -\infty$ *uniformly for* $k \in \boldsymbol{Z}$ *as* $v \to +\infty$;

(F_{14}) *there exist constants* $a_8 > 0$ *and* θ, $0 < \theta < 2$, *such that*

$$F(k, v) \leq -a_8 |v|^{\theta}, \quad \textit{for all } (k, v) \in \boldsymbol{Z} \times \boldsymbol{R}.$$

Then, for any given positive integer $m > 0$, *equation* (1.5) *has at least one nontrivial* mT*-periodic solution.*

Remark 1.10. When $\beta > 2$, Cai, Yu and Guo [**10,** Theorem 1.1] have obtained some criteria for the existence of periodic solutions of (1.4) and Cai and Yu [**9,** Theorem 1.1] have obtained some criteria for the existence of periodic solutions of (1.5). When $\beta < 2$, we can still find the periodic solutions of equations (1.4) and (1.5). Hence, Theorems 1.6–1.9 generalize and complement the existing ones.

The rest of the paper is organized as follows. First, in Section 2, we shall establish the variational framework associated with (1.1) and transfer the problem of the existence of periodic solutions of (1.1) into that of the existence of critical points of the corresponding functional. Some related fundamental results will also be recalled. Then, in Section 3, we shall complete the proof of the results by using the critical point method. Finally, in Section 4, we shall give two examples to illustrate the main results.

Regarding the basis for variational methods, we refer the reader to [**32, 34, 38**].

2. Variational structure and some lemmas. In order to apply the critical point theory, we shall establish the corresponding variational framework for (1.1) and give some lemmas which will be of fundamental importance in proving our main results. We start with some basic notation.

Let S be the set of sequences

$$u = (\dots, u_{-k}, \dots, u_{-1}, u_0, u_1, \dots, u_k, \dots) = \{u_k\}_{k=-\infty}^{+\infty},$$

that is,

$$S = \{\{u_k\} \mid u_k \in \mathbf{R},\ k \in \mathbf{Z}\}.$$

For any $u, v \in S$, $a, b \in \mathbf{R}$, $au + bv$ is defined by

$$au + bv = \{au_k + bv_k\}_{k=-\infty}^{+\infty}.$$

Then S is a vector space.

For any given positive integers m and T, E_{mT} is defined as a subspace of S by

$$E_{mT} = \{u \in S \mid u_{k+mT} = u_k, \quad \text{for all } k \in \mathbf{Z}\}.$$

Clearly, E_{mT} is isomorphic to $\mathbf{R}^{mT}$. E_{mT} can be equipped with the inner product

$$\langle u, v\rangle = \sum_{j=1}^{mT} u_j v_j, \quad \text{for all } u, v \in E_{mT}, \tag{2.1}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \left(\sum_{j=1}^{mT} u_j^2\right)^{1/2}, \quad \text{for all } u \in E_{mT}. \tag{2.2}$$

It is obvious that E_{mT} with the inner product (2.1) is a finite-dimensional Hilbert space and linearly homeomorphic to $\mathbf{R}^{mT}$.

On the other hand, we define the norm $\|\cdot\|_s$ on E_{mT} as follows:

$$\|u\|_s = \left(\sum_{j=1}^{mT} |u_j|^s\right)^{1/s}, \tag{2.3}$$

for all $u \in E_{mT}$ and $s > 1$.

Since $\|u\|_s$ and $\|u\|_2$ are equivalent, there exist constants $c_1,\ c_2$ such that $c_2 \geq c_1 > 0$, and

$$c_1\|u\|_2 \leq \|u\|_s \leq c_2\|u\|_2, \quad \text{for all } u \in E_{mT}. \tag{2.4}$$

Clearly, $\|u\| = \|u\|_2$. For all $u \in E_{mT}$, define the functional J on E_{mT} as follows:
(2.5)
$$\begin{aligned} J(u) &= -\frac{1}{p}\sum_{k=1}^{mT} r_{k-1}\left|\Delta^n u_{k-1}\right|^p - \frac{1}{p}\sum_{k=1}^{mT} q_k\left|\Delta u_k\right|^p + \sum_{k=1}^{mT} F(k,u_k) \\ &:= -H(u) + \sum_{k=1}^{mT} F(k,u_k), \end{aligned}$$

where

$$H(u) = \frac{1}{p}\sum_{k=1}^{mT} r_{k-1}\left|\Delta^n u_{k-1}\right|^p + \frac{1}{p}\sum_{k=1}^{mT} q_k\left|\Delta u_k\right|^p, \qquad \frac{\partial F(k,v)}{\partial v} = f(k,v).$$

It is evident that $J \in C^1(E_{mT}, \mathbf{R})$ and, for any $u = \{u_k\}_{k\in\mathbf{Z}} \in E_{mT}$, by using $u_0 = u_{mT}$ and $u_1 = u_{mT+1}$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_k} = -(-1)^n \Delta^n\left(r_{k-n}\varphi_p\left(\Delta^n u_{k-1}\right)\right) - q_k\Delta\varphi_p\left(\Delta u_{k-1}\right) + f(k,u_k).$$

Thus, u is a critical point of J on E_{mT} if and only if

$$\begin{aligned} \Delta^n\left(r_{k-n}\varphi_p\left(\Delta^n u_{k-1}\right)\right) + (-1)^n q_k\Delta\varphi_p\left(\Delta u_{k-1}\right) \\ = (-1)^n f(k,u_k), \quad \text{for all } k \in \mathbf{Z}(1,mT). \end{aligned}$$

Due to the periodicity of $u = \{u_k\}_{k\in\mathbf{Z}} \in E_{mT}$ and $f(k,v)$ in the first variable k, we reduce the existence of periodic solutions of (1.1) to the existence of critical points of J on E_{mT}, that is, the functional J is just the variational framework of (1.1).

Let

$$P = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

be an $mT \times mT$ matrix. By matrix theory, we see that the eigenvalues of P are

$$\lambda_j = 2\left(1 - \cos\frac{2j}{mT}\pi\right), \quad j = 0, 1, 2, \ldots, mT - 1. \tag{2.6}$$

Thus, $\lambda_0 = 0$, $\lambda_1 > 0$, $\lambda_2 > 0, \ldots, \lambda_{mT-1} > 0$. Therefore,

$$\begin{cases} \lambda_{\min} = \min\{\lambda_1, \lambda_2, \ldots, \lambda_{mT-1}\} = 2\left(1 - \cos\frac{2}{mT}\pi\right) \\ \lambda_{\max} = \max\{\lambda_1, \lambda_2, \ldots, \lambda_{mT-1}\} \end{cases} \tag{2.7}$$

$$= \begin{cases} 4 & \text{when } mT \text{ is even,} \\ 2\left(1 + \cos\frac{1}{mT}\pi\right) & \text{when } mT \text{ is odd.} \end{cases}$$

Let

$$W = \ker P = \{u \in E_{mT} \mid Pu = 0 \in \mathbf{R}^{mT}\}.$$

Then

$$W = \{u \in E_{mT} \mid u = \{c\},\ c \in \mathbf{R}\}.$$

Let V be the direct orthogonal complement of E_{mT} to W, i.e., $E_{mT} = V \oplus W$. For convenience, we identify $u \in E_{mT}$ with $u = (u_1, u_2, \cdots, u_{mT})^*$.

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (abbreviated PS condition) if any sequence $\{u^{(i)}\} \subset E$ for which $\{J(u^{(i)})\}$ is bounded and $J'(u^{(i)}) \to 0$ $(i \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ, and let ∂B_ρ denote its boundary.

Lemma 2.1 (Saddle Point theorem [**32, 38**]). *Let E be a real Banach space, $E = E_1 \oplus E_2$, where $E_1 \neq \{0\}$ and is finite-dimensional. Suppose that $J \in C^1(E, \boldsymbol{R})$ satisfies the* PS *condition and*

(J_1) *there exist constants $\sigma,\ \rho > 0$ such that $J|_{\partial B_\rho \cap E_1} \leq \sigma$;*

(J_2) *there exists $e \in B_\rho \cap E_1$ and a constant $\omega \geq \sigma$ such that $J_{e+E_2} \geq \omega$.*

Then J possesses a critical value $c \geq \omega$, where

$$c = \inf_{h\in\Gamma} \max_{u\in B_\rho\cap E_1} J(h(u)), \qquad \Gamma = \{h \in C(\overline{B}_\rho \cap E_1, E) \mid h|_{\partial B_\rho \cap E_1} = \mathrm{id}\}$$

and id *denotes the identity operator.*

Lemma 2.2. *Assume that (F_1)–(F_3) are satisfied. Then J satisfies the* PS *condition.*

Proof. Let $\{u^{(i)}\} \subset E_{mT}$ be such that $\{J(u^{(i)})\}$ is bounded and $J'(u^{(i)}) \to 0$ as $i \to \infty$. Then there exists a positive constant M_2 such that $|J(u^{(i)})| \leq M_2$.

Let $u^{(i)} = v^{(i)} + w^{(i)} \in V + W$. For i large enough, since

$$-\|u\|_2 \leq \langle J'(u^{(i)}), u\rangle = -\langle H'(u^{(i)}), u\rangle + \sum_{k=1}^{mT} f(k, u_k^{(i)}) u_k,$$

and combining with (F_2) and (F_3), we have

$$\begin{aligned}
\langle H'(u^{(i)}), v^{(i)}\rangle &\leq \sum_{k=1}^{mT} f(k, u_k^{(i)}) v_k^{(i)} + \|v^{(i)}\|_2 \\
&\leq M_0 \sum_{k=1}^{mT} |v_k^{(i)}| + \|v^{(i)}\|_2 \\
&\leq \left(M_0\sqrt{mT} + 1\right) \|v^{(i)}\|_2.
\end{aligned}$$

On the other hand, we know that

$$\langle H'(u^{(i)}), v^{(i)}\rangle = \sum_{k=1}^{mT} r_k |\Delta^n v_k^{(i)}|^p + \sum_{k=1}^{mT} q_k |\Delta v_k^{(i)}|^p = pH(v^{(i)}).$$

Since

$$
\begin{aligned}
H(v^{(i)}) &\geq \frac{\underline{r}}{p} c_1^p [(x^{(i)})^* P(x^{(i)})]^{p/2} + \frac{\underline{q}}{p} c_1^p [(v^{(i)})^* P(v^{(i)})]^{p/2} \\
&\geq \frac{\underline{r}}{p} c_1^p \lambda_{\min}^{p/2} \|x^{(i)}\|_2^p + \frac{\underline{q}}{p} c_1^p \lambda_{\min}^{p/2} \|v^{(i)}\|_2^p,
\end{aligned}
$$

$$
\begin{aligned}
H(v^{(i)}) &\leq \frac{\overline{r}}{p} c_2^p [(x^{(i)})^* P(x^{(i)})]^{p/2} + \frac{\overline{q}}{p} c_2^p [(v^{(i)})^* P(v^{(i)})]^{p/2} \\
&\leq \frac{\overline{r}}{p} c_2^p \lambda_{\max}^{p/2} \|x^{(i)}\|_2^p + \frac{\overline{q}}{p} c_2^p \lambda_{\max}^{p/2} \|v^{(i)}\|_2^p,
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_{\min}^{(n-1)p/2} \|v^{(i)}\|_2^p \leq \|x^{(i)}\|_2^p &= \sum_{k=1}^{mT} (\Delta^{n-2} v_{k+1}^{(i)} - \Delta^{n-2} v_k^{(i)})^p \\
&\leq \lambda_{\max}^{p/2} \sum_{k=1}^{mT} (\Delta^{n-2} v_k^{(i)})^p \leq \lambda_{\max}^{(n-1)p/2} \|v^{(i)}\|_2^p,
\end{aligned}
$$

where $x^{(i)} = (\Delta^{n-1} v_1^{(i)}, \Delta^{n-1} v_2^{(i)}, \ldots, \Delta^{n-1} v_{mT}^{(i)})^*$, we get

$$
\begin{aligned}
&\frac{\underline{r}}{p} c_1^p \lambda_{\min}^{np/2} \|v^{(i)}\|_2^p + \frac{\underline{q}}{p} c_1^p \lambda_{\min}^{p/2} \|v^{(i)}\|_2^p \leq H(v^{(i)}) \\
&\qquad \leq \frac{\overline{r}}{p} c_2^p \lambda_{\max}^{np/2} \|v^{(i)}\|_2^p + \frac{\overline{q}}{p} c_2^p \lambda_{\max}^{p/2} \|v^{(i)}\|_2^p.
\end{aligned} \tag{2.8}
$$

Thus, we have

$$
\underline{r} c_1^p \lambda_{\min}^{(np)/2} \|v^{(i)}\|_2^p + \frac{\underline{q}}{p} c_1^p \lambda_{\min}^{p/2} \|v^{(i)}\|_2^p \leq (M_0 \sqrt{mT} + 1) \|v^{(i)}\|_2.
$$

The above inequality implies that $\{v^{(i)}\}$ is bounded.

Next, we shall prove that $\{w^{(i)}\}$ is bounded. Since

$$
\begin{aligned}
M_2 &\geq J(u^{(i)}) \\
&= -H(u^{(i)}) + \sum_{k=1}^{mT} F(k, u_k^{(i)}) \\
&= -H(v^{(i)}) + \sum_{k=1}^{mT} [F(k, u_k^{(i)}) - F(k, w_k^{(i)})] + \sum_{k=1}^{mT} F(k, w_k^{(i)}),
\end{aligned}
$$

combining with (2.8), we get

$$\begin{aligned}\sum_{k=1}^{mT} F(k,w_k^{(i)}) &\le M_2 + H(v^{(i)}) + \sum_{k=1}^{mT} |F(k,u_k^{(i)}) - F(k,w_k^{(i)})| \\ &\le M_2 + \left(\frac{\overline{r}}{p} c_2^p \lambda_{\max}^{(np)/2} + \frac{\overline{q}}{p} c_2^p \lambda_{\max}^{p/2}\right) \|v^{(i)}\|_2^p \\ &\quad + \sum_{k=1}^{mT} |f(k, w_k^{(i)} + \theta v_k^{(i)}) v_k^{(i)}| \\ &\le M_2 + \left(\frac{\overline{r}}{p} c_2^p \lambda_{\max}^{(np)/2} + \frac{\overline{q}}{p} c_2^p \lambda_{\max}^{p/2}\right) \|v^{(i)}\|_2^p + M_0\sqrt{mT}\|v^{(i)}\|_2,\end{aligned}$$

where $\theta \in (0,1)$. It is not difficult to see that

$$\left\{ \sum_{k=1}^{mT} F(k, w_k^{(i)}) \right\}$$

is bounded.

By (F_3), $\{w^{(i)}\}$ is bounded. Otherwise, assume that $\|w^{(i)}\|_2 \to +\infty$ as $i \to \infty$. Since there exist $z^{(i)} \in \mathbf{R}$, $i \in \mathbf{N}$, such that $w^{(i)} = (z^{(i)}, z^{(i)}, \dots, z^{(i)})^* \in E_{mT}$, then

$$\|w^{(i)}\|_2 = \left(\sum_{k=1}^{mT} |w_k^{(i)}|^2\right)^{1/2} = \left(\sum_{k=1}^{mT} |z^{(i)}|^2\right)^{1/2} = \sqrt{mT}|z^{(i)}| \to +\infty$$

as $i \to \infty$. Since $F(k, w_k^{(i)}) = F(k, z^{(i)})$, then $F(k, w_k^{(i)}) \to +\infty$ as $i \to \infty$. This contradicts the fact that $\{\sum_{k=1}^{mT} F(k, w_k^{(i)})\}$ is bounded. Thus the PS condition is verified. □

Lemma 2.3. *Assume that (F_1), (F_4) and (F_5) are satisfied. Then J satisfies the* PS *condition.*

Proof. Let $\{u^{(i)}\} \subset E_{mT}$ be such that $\{J(u^{(i)})\}$ is bounded and $J'(u^{(i)}) \to 0$ as $i \to \infty$. Then there exists a positive constant M_3 such that $|J(u^{(i)})| \le M_3$.

For i large enough, we have

$$|\langle J'(u^{(i)}), u^{(i)}\rangle| \le \|u^{(i)}\|_2.$$

So

$$M_3 + \frac{1}{p}\|u^{(i)}\|_2 \geq J(u^{(i)}) - \frac{1}{p}\langle J'(u^{(i)}), u^{(i)}\rangle$$
$$= \sum_{k=1}^{mT} F(k, u_k^{(i)}) - \frac{1}{p}\sum_{k=1}^{mT} f(k, u_k^{(i)})u_k^{(i)}.$$

Take

$$I_1 = \{k \in \mathbf{Z}(1, mT) \mid |u_k^{(i)}| \geq R_1\},$$
$$I_2 = \{k \in \mathbf{Z}(1, mT) \mid |u_k^{(i)}| < R_1\}.$$

By (F_4), we have

$$\begin{aligned} M_3 + \frac{1}{p}\|u^{(i)}\|_2 &\geq \sum_{k=1}^{mT} F(k, u_k^{(i)}) - \frac{1}{p}\sum_{k\in I_1} f(k, u_k^{(i)})u_k^{(i)} - \frac{1}{p}\sum_{k\in I_2} f(k, u_k^{(i)})u_k^{(i)} \\ &\geq \sum_{k=1}^{mT} F(k, u_k^{(i)}) - \frac{\alpha}{2}\sum_{k\in I_1} F(k, u_k^{(i)}) - \frac{1}{p}\sum_{k\in I_2} f(k, u_k^{(i)})u_k^{(i)} \\ &= \left(1 - \frac{\alpha}{2}\right)\sum_{k=1}^{mT} F(k, u_k^{(i)}) \\ &\quad + \frac{1}{p}\sum_{k\in I_2}\left[\frac{\alpha}{2}pF(k, u_k^{(i)}) - f(k, u_k^{(i)})u_k^{(i)}\right]. \end{aligned}$$

The continuity of $(\alpha/2)pF(k, v) - f(k, v)v$ with respect to the second variable implies that there exists a constant $M_4 > 0$ such that

$$\frac{\alpha}{2}pF(k, v) - f(k, v)v \geq -M_6,$$

for $k \in \mathbf{Z}(1, mT)$ and $|v| \leq R_1$. Therefore,

$$M_3 + \frac{1}{p}\|u^{(i)}\|_2 \geq \left(1 - \frac{\alpha}{2}\right)\sum_{k=1}^{mT} F(k, u_k^{(i)}) - \frac{1}{p}mTM_4.$$

By (F_5), we get

$$M_3 + \frac{1}{p}\|u^{(i)}\|_2 \geq \left(1 - \frac{\alpha}{2}\right)a_1\sum_{k=1}^{mT} |u_k^{(i)}|^{(\gamma/2)p} - \left(1 - \frac{\alpha}{2}\right)a_2 mT - \frac{1}{p}mTM_4$$

$$\geq \left(1-\frac{\alpha}{2}\right)a_1\sum_{k=1}^{mT}|u_k^{(i)}|^{(\gamma/2)p}-M_5,$$

where $M_5=(1-(\alpha/2))a_2mT+(1/p)mTM_4$.

Combining with (2.4), we have

$$M_3+\frac{1}{p}\|u^{(i)}\|_2\geq \left(1-\frac{\alpha}{2}\right)a_1c_1^{(\gamma/2)p}\|u^{(i)}\|_2^{(\gamma/2)p}-M_5.$$

Thus,

$$\left(1-\frac{\alpha}{2}\right)a_1c_1^{(\gamma/2)p}\|u^{(i)}\|_2^{(\gamma/2)p}-\frac{1}{p}\|u^{(i)}\|_2\leq M_3+M_5.$$

This implies that $\{\|u^{(i)}\|_2\}$ is bounded on the finite-dimensional space E_{mT}. As a consequence, it has a convergent subsequence. $\square$

3. Proof of the main results. In this section, we shall prove our main results by using the critical point method.

Proof of Theorem 1.1. By Lemma 2.2, we know that J satisfies the PS condition. In order to prove Theorem 1.1 by using the Saddle theorem, we shall prove conditions (J_1) and (J_2).

From (2.8) and (F_2'), for any $v\in V$,

$$\begin{aligned}J(v)&=-H(v)+\sum_{k=1}^{mT}F(k,v_k)\\&\leq-\left(\frac{r}{p}c_1^p\lambda_{\min}^{(np)/2}+\frac{q}{p}c_1^p\lambda_{\min}^{p/2}\right)\|v\|_2^p+mTM_1+M_0\sum_{k=1}^{mT}|v_k|\\&\leq-\left(\frac{r}{p}c_1^p\lambda_{\min}^{(np)/2}+\frac{q}{p}c_1^p\lambda_{\min}^{p/2}\right)\|v\|_2^p+mTM_1+M_0\sqrt{mT}\|v\|_2\longrightarrow-\infty\end{aligned}$$

as $\|v\|_2\to+\infty$. Therefore, it is easy to see that condition (J_1) is satisfied.

In the following, we shall verify condition (J_2). For any $w\in W$, $w=(w_1,w_2,\ldots,w_{mT})^*$, there exists a $z\in\mathbf{R}$ such that $w_k=z$, for all $k\in\mathbf{Z}(1,mT)$. By (F_3), we know that there exists a constant $R_0>0$ such that $F(k,z)>0$ for $k\in\mathbf{Z}$ and $|z|>R_0$. Let

$M_6 = \min_{k\in\mathbf{Z},\,|z|\le R_0} F(k,z)$, $M_7 = \min\{0, M_6\}$. Then

$$F(k,z) \ge M_7, \quad \text{for all } (k,z) \in \mathbf{Z}\times\mathbf{R}.$$

So, we have

$$J(w) = \sum_{k=1}^{mT} F(k,w_k) = \sum_{k=1}^{mT} F(k,z) \ge mTM_7, \quad \text{for all } w \in W.$$

The conditions of (J_1) and (J_2) are satisfied. $\square$

Proof of Theorem 1.2. By Lemma 2.3, J satisfies the PS condition. To apply the Saddle Point theorem, it suffices to prove that J satisfies conditions (J_1) and (J_2).

For any $w \in W$, since $H(w) = 0$, we have

$$J(w) = \sum_{k=1}^{mT} F(k,w_k).$$

By (F_5),

$$J(w) \ge a_1 \sum_{k=1}^{mT} |w_k|^{(\gamma/2)p} - a_2 mT \ge -a_2 mT.$$

Combining with (F_4'), (2,4) and (2.8), for any $v \in V$, we get, as before,

$$\begin{aligned} J(v) &\le -\left(\frac{r}{p} c_1^p \lambda_{\min}^{(np)/2} + \frac{q}{p} c_1^p \lambda_{\min}^{p/2}\right)\|v\|_2^p + a_3 \sum_{k=1}^{mT} |v_k|^{(\alpha/2)p} + a_4 mT \\ &\le -\left(\frac{r}{p} c_1^p \lambda_{\min}^{(np)/2} + \frac{q}{p} c_1^p \lambda_{\min}^{p/2}\right)\|v\|_2^p + a_3 c_2^{(\alpha/2)p}\|v\|_2^{(\alpha/2)p} + a_4 mT. \end{aligned}$$

Let $\mu = -a_2 mT$. Since $1 < \alpha < 2$, there exists a constant $\rho > 0$ large enough such that

$$J(v) \le \mu - 1 < \mu, \quad \text{for all } v \in V,\ \|v\|_2 = \rho.$$

Thus, by Lemma 2.1, equation (1.1) has at least one mT-periodic solution. $\square$

Proof of Theorem 1.3. Similarly to the proof of Lemma 2.3, we can prove that J satisfies the PS condition. We shall prove this theorem by means of the Saddle Point theorem. Firstly, we verify the condition (J_1).

In fact, (F_4) clearly implies (F_4'). For any $v \in V$, by (F_4') and equation (2.4), we again have $J(v) \to -\infty$ as $\|v\|_2 \to +\infty$.

Next, we show that J satisfies the condition (J_2) for any given $v_0 \in V$ and $w \in W$. Let $u = v_0 + w$. So

$$\begin{aligned}
J(u) &= -H(u) + \sum_{k=1}^{mT} F(k, u_k) \\
&= -H(v_0) + \sum_{k=1}^{mT} F(k, (v_0)_k + w_k) \\
&\geq -\left(\frac{\overline{r}}{p} c_2^p \lambda_{\max}^{(np)/2} + \frac{\overline{q}}{p} c_2^p \lambda_{\max}^{p/2}\right) \|v_0\|_2^p + a_5 \sum_{k=1}^{mT} |(v_0)_k + w_k)|^{(\gamma/2)p} \\
&\geq -\left(\frac{r}{p} c_2^p \lambda_{\max}^{(np)/2} + \frac{\overline{q}}{p} c_2^p \lambda_{\max}^{p/2}\right) \|v_0\|_2^p + a_5 \sum_{k=1}^{mT} |(v_0)_k + w_k|^{(\gamma/2)p} \\
&\geq -\left(\frac{\overline{r}}{p} c_2^p \lambda_{\max}^{(np)/2} + \frac{\overline{q}}{p} c_2^p \lambda_{\max}^{p/2}\right) \|v_0\|_2^p + a_5 c_1^{(\gamma/2)p} \left[\sum_{k=1}^{mT} |(v_0)_k + w_k|^2\right]^{(\gamma/4)p} \\
&= -\left(\frac{\overline{r}}{p} c_2^p \lambda_{\max}^{(np)/2} + \frac{\overline{q}}{p} c_2^p \lambda_{\max}^{p/2}\right) \|v_0\|_2^p + a_5 c_1^{(\gamma/2)p} \left[\|v_0\|_2^2 + \|w\|_2^2\right]^{(\gamma/4)p} \\
&\geq -\left(\frac{\overline{r}}{p} c_2^p \lambda_{\max}^{(np)/2} + \frac{\overline{q}}{p} c_2^p \lambda_{\max}^{p/2}\right) \|v_0\|_2^p + a_5 c_1^{(\gamma/2)p} \|v_0\|_2^{(\gamma/2)p} \\
&\quad + a_5 c_1^{(\gamma/2)p} \|w\|_2^{(\gamma/2)p}.
\end{aligned}$$

Since $1 < \gamma < 2$, there exists a constant $\delta > 0$ small enough such that

$$J(v_0 + w) \geq \delta^{(\gamma/2)p} \left[a_5 c_1^{(\gamma/2)p} - \left(\frac{\overline{r}}{p} c_2^p \lambda_{\max}^{(np)/2} + \frac{\overline{q}}{p} c_2^p \lambda_{\max}^{p/2}\right) \delta^{p-(\gamma/2)p}\right] > 0,$$

for $v_0 \in V$, $\|v_0\|_2 = \delta$ and for any $w \in W$.

Take

$$\nu = \delta^{(\gamma/2)p} \left[a_5 c_1^{(\gamma/2)p} - \left(\frac{\overline{r}}{p} c_2^p \lambda_{\max}^{(np)/2} + \frac{\overline{q}}{p} c_2^p \lambda_{\max}^{p/2}\right) \delta^{p-(\gamma/2)p}\right].$$

Then, for $v_0 \in V$ and for any $w \in W$, we get $\|v_0\|_2 = \delta$ and $J(v_0 + w) \geq \nu > 0$.

By the Saddle Point theorem, a critical point $\overline{u} \in E_{mT}$ exists which corresponds to an mT-periodic solution of (1.1).

In the following, we shall prove that $\overline{u}$ is nontrivial, i.e., $\overline{u} \notin W$. Otherwise, $\overline{u} \in W$. Since $J'(\overline{u}) = 0$, then

$$\Delta^n \left(r_{k-n}\varphi_p\left(\Delta^n \overline{u}_{k-1}\right)\right) + q_k \Delta\varphi_p\left(\Delta\overline{u}_{k-1}\right) = (-1)^n f(k, \overline{u}_k).$$

On the other hand, $\overline{u} \in W$ implies that there is a point $z \in \mathbf{R}$ such that $\overline{u}_k = z$, for all $k \in \mathbf{Z}(1, mT)$, that is,

$$\overline{u}_1 = \overline{u}_2 = \cdots = \overline{u}_k = \cdots = z.$$

Thus, $f(k, \overline{u}_k) = f(k, z) = 0$, for all $k \in \mathbf{Z}(1, mT)$. From (F_6), we know that $z = 0$. Therefore, by (F_6), we have

$$J(\overline{u}) = \sum_{k=1}^{mT} F(k, \overline{u}_k) = \sum_{k=1}^{mT} F(k, 0) = 0.$$

This contradicts $J(\overline{u}) \geq \nu > 0$. The proof of Theorem 1.6 is finished. $\square$

Remark 3.1. The techniques of the proof of Theorem 1.7 are exactly the same as those carried out in the proof of Theorem 1.6. We do not repeat them here.

Remark 3.2. Due to Theorems 1.6 and 1.7, the conclusion of Theorems 1.8 and 1.9 is obviously true.

4. Examples.

As an application of the main theorems, we give two examples to illustrate our results.

Example 4.1. For all $n \in \mathbf{Z}(1)$ and $k \in \mathbf{Z}$, assume that

(4.1)
$$\Delta^n \left(r_{k-n}\varphi_p\left(\Delta^n u_{k-1}\right)\right) + (-1)^n q_k \Delta\varphi_p\left(\Delta u_{k-1}\right) = (-1)^n \alpha p \psi(k) u_k^{\alpha p - 1},$$

where $\{r_k\}$ and $\{q_k\}$ are real sequences, ψ is continuously differentiable and $\psi(k) > 0$, T is a given positive integer, $r_{k+T} = r_k > 0$, $q_{k+T} =$

$q_k \geq 0$, $\psi(k+T) = \psi(k)$, $1 < p < \infty$, $1 < \alpha < 2$. We have

$$f(k,v) = \alpha p \psi(k) v^{\alpha p - 1} \quad \text{and} \quad F(k,v) = \psi(k) v^{\alpha p}.$$

Then

$$\frac{\partial F(k,v)}{\partial v} = \alpha p \psi(k) v^{\alpha p - 1}.$$

It is easy to verify that all assumptions of Theorem 1.6 are satisfied. Consequently, for any given positive integer $m > 0$, equation (4.1) has at least one nontrivial mT-periodic solution.

Example 4.2. For all $n \in \mathbf{Z}(1)$ and $k \in \mathbf{Z}$, assume that

$$\begin{aligned} \Delta^n \left(r_{k-n}\varphi_p\left(\Delta^n u_{k-1}\right)\right) + (-1)^n q_k \Delta\varphi_p\left(\Delta u_{k-1}\right) \\ = (-1)^n \theta p u_k \left(3 + \cos^2 \frac{k\pi}{T}\right) u_k^{\theta p - 1}, \end{aligned} \tag{4.2}$$

where $\{r_k\}$ and $\{q_k\}$ are real sequences, $r_{k+T} = r_k > 0$, $q_{k+T} = q_k \geq 0$, $1 < p < \infty$, $0 < \theta < 2$. We have

$$f(k,v) = \theta p \left(3 + \cos^2 \frac{k\pi}{T}\right) v^{\theta p - 1}$$

and

$$F(k,v) = \left(3 + \cos^2 \frac{k\pi}{T}\right) v^{\theta p}.$$

Then

$$\frac{\partial F(k,v)}{\partial v} = \theta p \left(3 + \cos^2 \frac{k\pi}{T}\right) v^{\theta p - 1}.$$

It is easy to verify that all of the assumptions of Theorem 1.7 are satisfied. Consequently, for any given positive integer $m > 0$, equation (4.2) has at least one nontrivial mT-periodic solution.

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