ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 46, Number 5, 2016

A CERTAIN CLASS OF APPROXIMATIONS FOR THE q-DIGAMMA FUNCTION

AHMED SALEM

ABSTRACT. In this paper, we derive a class of approximations of the q-digamma function $\psi_q(x)$. The infinite family

$$I_a(x;q) = \log[x+a]_q + \frac{q^x \log q}{1-q^x} - \left(\frac{1}{2} - a\right) H(q-1) \log q,$$

$a \in [0,1]$; $q > 0$, can be used as approximating functions for $\psi_q(x)$, where $[x]_q = (1-q^x)/(1-q)$ and $H(\cdot)$ is the Heaviside step function. We show that, for all $a \in [0,1]$, I_a is asymptotically equivalent to $\psi_q(x)$ for $q > 0$ and is a good pointwise approximation.

1. Introduction. The q-analogue of the digamma function $\psi_q(x)$ appeared in the work of Krattenthaler and Srivastava **[2]** where they studied the summations for basic hypergeometric series. Some of its properties were presented and proved in their work. In **[2]**, they proved that $\psi_q(x)$ tends to the digamma function $\psi(x)$ when letting $q \to 1$. Also, Salem **[6]** derived some properties and expansions associated with the q-digamma function. Some inequalities involving the q-digamma function have been introduced in **[1, 3, 7, 8, 9, 10]**. The q-digamma function $\psi_q(x)$ is defined as the logarithmic derivative of the q-gamma function,

$$\psi_q(x) = \frac{d}{dx}(\log \Gamma_q(x)) = \frac{\Gamma_q'(x)}{\Gamma_q(x)}, \tag{1.1}$$

where $\Gamma_q(x)$ is the q-gamma function defined as

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+x}}, \quad 0 < q < 1, \tag{1.2}$$

2010 AMS *Mathematics subject classification.* Primary 30E10, 33D05.
Keywords and phrases. q-Digamma function, approximations.
Received by the editors on January 17, 2013, and in revised form on November 6, 2014.
DOI:10.1216/RMJ-2016-46-5-1665

and

$$\Gamma_q(x) = (q-1)^{1-x} q^{x(x-1)/2} \prod_{n=0}^{\infty} \frac{1-q^{-(n+1)}}{1-q^{-(n+x)}}, \quad q > 1. \tag{1.3}$$

From (1.2), for $0 < q < 1$ and for all real variables $x > 0$, we obtain

$$\psi_q(x) = -\log(1-q) + \log q \sum_{k=0}^{\infty} \frac{q^{xk}}{1-q^k}, \tag{1.4}$$

and, from equation (1.3), for $q > 1$ and $x > 0$, we also get

$$\psi_q(x) = -\log(q-1) + \log q \left[x - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{q^{-xk}}{1-q^{-k}} \right]. \tag{1.5}$$

From the previous definitions, for a positive x and $q \geq 1$, we obtain

$$\Gamma_q(x) = q^{[(x-1)(x-2)]/2} \Gamma_{q^{-1}}(x), \tag{1.6}$$

$$\psi_q(x) = \frac{2x-3}{2} \log q + \psi_{q^{-1}}(x). \tag{1.7}$$

Muqattash and Yahdi **[5]** derived an infinite family of approximations for $\psi(x)$ on $\mathbb{R}_+$, denoted as $\{I_a, a \in [0,1]\}$, where

$$I_a(x) = \log(x+a) - \frac{1}{x}. \tag{1.8}$$

They proved that the functions I_a are shown to approximate locally and asymptotically independently of $a \in [0,1]$ with a perfect match $\psi(x) = I_a(x)$ for a certain a whenever x is fixed. Also, they found local and global bounding error functions and introduced new inequalities for the digamma function.

For any real numbers $a \in [0,1]$ and $q > 0$, suppose that $I_a(x;q)$ is the function defined for all real positive x by

$$I_a(x;q) = \log[x+a]_q + \frac{q^x \log q}{1-q^x} + \left(\frac{1}{2} - a\right) H(q-1) \log q, \tag{1.9}$$

where $[x]_q = (1-q^x)/(1-q)$ is the so-called basic number and $H(\cdot)$ is the Heaviside step function.

The main goal of this paper is to derive a class of approximations of the q-digamma function and, as a consequence, new inequalities for the q-digamma function. The infinite family $I_a(x;q) : a \in [0,1]$ can be used as approximating functions for $\psi_q(x)$. We show that, for all $a \in [0,1]$, I_a is asymptotically equivalent to $\psi_q(x)$ for $q > 0$ and is a good pointwise approximation.

2. Useful lemmas. We devote this section to establishing some preliminary facts and results needed in the proofs of the main results.

Lemma 2.1. *For all $x, q \in \mathbb{R}_+$, we have*

$$\begin{aligned}\log[x]_q + \frac{1}{2}H(q-1)\log q &\le \psi_q(x+1) \\ &\le \log[x+1]_q - \frac{1}{2}H(q-1)\log q.\end{aligned} \tag{2.1}$$

Proof. Suppose that the function

$$f_\alpha(x;q) = \psi_q(x+1) - \log[x+\alpha]_q, \quad 0 < q < 1,\ \alpha = 0, 1. \tag{2.2}$$

From equation (1.4) and the Taylor series of logarithm functions, the function $f_\alpha(x;q)$ can be rewritten as

$$f_\alpha(x;q) = \sum_{k=1}^{\infty} \frac{q^{xk}}{k(1-q^k)} g_\alpha(y), \quad y = q^k,$$

where $g_\alpha(y) = y\log y + y^\alpha(1-y)$ can be represented as

$$g_0(y) = y\sum_{n=2}^{\infty} \frac{\log^n(1/y)}{n!} > 0$$

$$g_1(y) = -y^2\sum_{n=2}^{\infty} \frac{\log^n(1/y)}{n!}(n-1) < 0.$$

Thus, the functions $f_0(x;q) > 0$ and $f_1(x;q) < 0$ for all $x > 0$. Therefore,

$$\log[x]_q \le \psi_q(x+1) \le \log[x+1]_q, \quad 0 < q < 1. \tag{2.3}$$

Now, let $q \ge 1$. Then equation (1.7) and the identity $[x]_{q^{-1}} = q^{x-1}[x]_q$

can be exploited to obtain

$$
\begin{aligned}
f_\alpha(x;q) &= \psi_{q^{-1}}(x+1) + \frac{2x-1}{2}\log q \\
&\qquad - \log[x+\alpha]_q - (x+\alpha-1)\log q \\
&= f_\alpha(x;q^{-1}) + \left(\frac{1}{2}-\alpha\right)\log q, \quad q \geq 1.
\end{aligned} \tag{2.4}
$$

In view of equations (2.2), (2.3) and (2.4), we obtain the desired result. ☐

Lemma 2.2. *For every $x, q \in \mathbb{R}_+$, the q-digamma function $\psi_q(x)$ is strictly increasing on $(0,\infty)$, and there exists a unique real number $x^* \in (1,2)$ such that $\psi_q(x^*) = 0$.*

Proof. For $q > 0$, Alzer and Grinshpan [**1**] stated that $\psi_q'(x)$ is strictly completely monotonic on $(0,\infty)$. This means that $\psi_q'(x) > 0$, which reveals that $\psi_q(x)$ is strictly increasing on $(0,\infty)$.

When $0 < q < 1$, (1.4) and the Taylor expansion of $\log(1-q)$ gives

$$
\psi_q(1) = -\log(1-q) + \log q \sum_{k=1}^{\infty} \frac{q^k}{1-q^k} = \sum_{k=1}^{\infty} \frac{q^k(1-q^k+k\log q)}{k(1-q^k)}.
$$

It is easy to show that

$$
1 - q^k + k\log q = -q^k \sum_{n=2}^{\infty} \frac{\log^n(q^{-k})}{n(n-2)!} < 0, \quad k \in \mathbb{N},
$$

which leads to the conclusion that $\psi_q(1) < 0$ for $0 < q < 1$. When $q > 1$, equation (1.7) gives $\psi_q(1) = -(1/2)\log q + \psi_{q^{-1}}(1) < 0$. This leads to the conclusion that $\psi_q(1) < 0$ for all $q > 0$.

Similarly, we can deduce that $\psi_q(2) > 0$ for all $q > 0$. In light of this proof we conclude that there exists a unique real number $x^* \in (1,2)$ such that $\psi_q(x^*) = 0$ for all $q > 0$. ☐

Lemma 2.3. *For each real number x, $q \in \mathbb{R}_+$. Then, for any fixed $a \in [0,1]$, the function $x \mapsto I_a(x;q)$ is positive and strictly increasing on $[2,\infty)$, and, whenever x is fixed in $[2,\infty)$, the function $a \mapsto I_a(x;q)$ is positive and strictly increasing on $[0,1]$.*

Proof. Differentiating equation (1.9) with respect to x yields

$$\frac{\partial}{\partial x} I_a(x;q) = \frac{-q^{x+a}\log q}{1-q^{x+a}} + \frac{q^x \log^2 q}{(1-q^x)^2} > 0, \quad q > 0,$$

which yields that the function $x \mapsto I_a(x;q)$ is strictly increasing on $[2,\infty)$.

Differentiating equation (1.9) with respect to a yields

$$\frac{\partial}{\partial a} I_a(x;q) = \frac{-q^{x+a}\log q}{1-q^{x+a}} > 0, \quad 0 < q < 1,$$

and

$$\frac{\partial}{\partial a} I_a(x;q) = \frac{-q^{x+a}\log q}{1-q^{x+a}} - \log q = \frac{-\log q}{1-q^{x+a}} > 0, \quad q \geq 1,$$

which yield that the function $a \mapsto I_a(x;q)$ is strictly increasing on $[0,1]$. Therefore, the minimum value of $I_a(x;q)$ can be computed for $0 < q < 1$ as

$$\begin{aligned} I_0(2;q) &= \log(1-q^2) - \log(1-q) + \frac{q^2\log q}{1-q^2} \\ &= \sum_{k=1}^{\infty} \frac{q^k(1-q^k+kq^k\log q)}{k}. \end{aligned}$$

A short calculation gives

$$1 - q^k + kq^k \log q = \sum_{n=2}^{\infty} \frac{\log^n(q^{-k})}{n!} > 0,$$

which reveals that $I_0(2;q) > 0$ for $0 < q < 1$. When $q \geq 1$, we have

$$I_0(2;q) = \log(1+q) + \frac{q^2\log q}{1-q^2} + \frac{1}{2}\log q.$$

Differentiation yields $(d/dq)I_0(2;q) = g(q)/(2(1-q^2)^2)$, where

$$g(q) = 4q^2\log q + (1-q^2)(1+2q-q^2).$$

Again, differentiation gives

$$g'(q) = 2 - 6q^2 + 4q^3 + 8q\log q = 2(2q+1)(q-1)^2 + 8q\log q > 0,$$

which concludes that $g(q)$ is increasing on $[1,\infty)$ and since $g(1)=0$, then $g(q)\geq 0$ for all $q\geq 1$. Therefore, the function $I_0(2;q)$ is increasing on $[1,\infty)$. Since $\lim_{q\to 1} I_0(2;q)=\log 2-(1/2)>0$, then $I_0(2;q)>0$ for all $q\geq 1$. This completes the proof. $\square$

Lemma 2.4. *Suppose $q\in\mathbb{R}_+$. Then we have*

$$\lim_{x\to\infty}\frac{I_0(x;q)}{I_1(x;q)}=1. \tag{2.5}$$

Proof. When $0<q<1$, equation (1.9) gives

$$\begin{aligned}\lim_{x\to\infty}\frac{I_0(x;q)}{I_1(x;q)} &= \lim_{x\to\infty}\frac{\log[x]_q+(q^x\log q)/(1-q^x)}{\log[x+1]_q+(q^x\log q)/(1-q^x)}\\ &= \lim_{x\to\infty}\frac{\log(1-q^x)-\log(1-q)+(q^x\log q)/(1-q^x)}{\log(1-q^{x+1})-\log(1-q)+(q^x\log q)/(1-q^x}\\ &= \frac{-\log(1-q)}{-\log(1-q)}=1.\end{aligned}$$

When $q\geq 1$, we obtain

$$\begin{aligned}\lim_{x\to\infty}\frac{I_0(x;q)}{I_1(x;q)} &= \lim_{x\to\infty}\frac{\log[x]_q+(q^x\log q)/(1-q^x)+1/2\log q}{\log[x+1]_q+(q^x\log q)/(1-q^x)-1/2\log q}\\ &= \lim_{x\to\infty}\frac{\log(q^x-1)-\log(q-1)+(q^x\log q)/(1-q^x)+1/2\log q}{\log(q^{x+1}-1)-\log(q-1)+(q^x\log q)/(1-q^x)-1/2\log q}\\ &= \lim_{x\to\infty}\frac{x\log q+\log(1-q^{-x})-\log(q-1)+(\log q)/(q^{-x}-1)+1/2\log q}{(x+1)\log q+\log(1-q^{-x-1})-\log(q-1)+(\log q)/(q^{-x}-1)-1/2\log q}.\end{aligned}$$

Using L'Hopital's rule yields

$$\begin{aligned}\lim_{x\to\infty}\frac{I_0(x;q)}{I_1(x;q)} &= \lim_{x\to\infty}\frac{\log q+(q^{-x})/(1-q^{-x})+(q^{-x}\log^2 q)/(q^{-x}-1)^2}{\log q+(q^{-x-1})/(1-q^{-x-1})+(q^{-x}\log^2 q)/(q^{-x}-1)^2}\\ &= \frac{\log q}{\log q}=1.\end{aligned}$$ $\square$

Lemma 2.5. *For all $q>0$, the function,*

$$F_q(x)=\log[x]_q-\log[x+1/2]_q-\frac{1}{2}\frac{q^x\log q}{1-q^x}, \tag{2.6}$$

is strictly positive for all $x>0$.

Proof. When $0 < q < 1$, the series expansion of the logarithm function and binomial theorem give

$$F_q(x) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{q^{xk}}{k}g(y), \quad y = q^k,$$

where $g(y) = 2\sqrt{y} - \log y - 2$, which can be read as

$$g(y) = 2\sqrt{y}\sum_{n=2}^{\infty}\frac{\log^n(1/y)}{n!}\left(\frac{1}{2}\right)^n(n-1) > 0.$$

Hence, $F_q(x) > 0$ for $0 < q < 1$ and $x > 0$. When $q \geq 1$, it is not difficult to see that $F_q(x) = F_{q^{-1}}(x)$ which concludes that $F_q(x) > 0$ for all $q > 0$ and $x > 0$. $\square$

Lemma 2.6. *For all $q > 0$, the function*

$$G_q(x) = \frac{1}{2}\log[x+1/2]_q + \frac{1}{2}\log[x]_q + \frac{3}{4}\frac{q^x\log q}{1-q^x} - \psi_q(x), \tag{2.7}$$

is strictly positive for all $x > 0$.

Proof. The function $G_q(x)$ can be represented by using (1.4) as

$$G_q(x) = -\frac{1}{4}\sum_{k=1}^{\infty}\frac{q^{xk}}{k(1-q^k)}h(y), \quad y = q^k;\ 0 < q < 1,$$

where

$$h(y) = \log y + 3y\log y + 2\sqrt{y}(1-y) + 2(1-y),$$

which can be read as

$$h(y) = -y\sqrt{y}\sum_{n=3}^{\infty}\frac{\log^n(1/y)}{n!} \left\{\left(\frac{3}{2}\right)^{n-1}(n-3) + \left(\frac{1}{2}\right)^{n-1}(3n+1) - 2\right\} < 0,$$

which reveals that $G_q(x) > 0$ for $0 < q < 1$ and $x > 0$. When $q \geq 1$, (1.7) gives $G_q(x) = G_{q^{-1}}(x)$, and so $G_q(x) > 0$ for $q > 0$ and $x > 0$. This ends the proof. $\square$

3. The main results. In this section, we are seeking to derive an infinite family of approximations of the q-digamma function and, as a consequence, new inequalities for the q-digamma function. The infinite family $I_a(x;q) : a \in [0,1]$ can be used as approximating functions for $\psi_q(x)$. We show that, for all $a \in [0,1]$, I_a is asymptotically equivalent to $\psi_q(x)$ for $q > 0$ and is a good pointwise approximation. In order to present our proofs, we will use the lemmas proved in the previous section and the same technique used in **[5]**.

Theorem 3.1. *For every $x, q \in \mathbb{R}_+$, there exists at least one real number $a \in [0,1]$ such that*

$$\psi_q(x) = I_a(x;q). \tag{3.1}$$

Proof. The intermediate value theorem states that, for each value between the least upper bound and greatest lower bound of the image of a continuous function, there is at least one point in its domain which the function maps to that value. It is clear that the function $a \mapsto I_a(x;q)$ is a continuous function for all $a \in [0,1]$. From equation (1.9) and Lemma 2.1, along with the identity **[7]**,

$$\psi_q(x+1) = \psi_q(x) - \frac{q^x \log q}{1-q^x}. \tag{3.2}$$

we have

$$I_0(x;q) \le \psi_q(x) \le I_1(x;q). \tag{3.3}$$

According to the intermediate value theorem, we conclude that at least one real number $a \in [0,1]$ exists such that $\psi_q(x) = I_a(x;q)$. This concludes the proof. $\square$

As in **[5]**, we will use the notation $f \sim g$ on $\mathbb{R}_+$ to denote that the functions f and g are asymptotic.

Theorem 3.2. *For all $a \in [0,1]$ and $q > 0$, then $\psi_q(x) \sim I_a(x;q)$ on $\mathbb{R}_+$.*

Proof. Dividing the inequality (3.3) by $I_1(x;q)$ which is a positive for all $x \in [2,\infty)$, see Lemma 2.3, would yield

$$\frac{I_0(x;q)}{I_1(x;q)} \le \frac{\psi_q(x)}{I_1(x;q)} \le 1.$$

Exploiting equation (2.5) gives

$$\lim_{x\to\infty} \frac{\psi_q(x)}{I_1(x;q)} = 1,$$

and thus,

$$\psi_q(x) \sim I_1(x;q) \quad \text{on } \mathbb{R}_+. \tag{3.4}$$

Similarly, we can deduce that

$$\lim_{x\to\infty} \frac{\psi_q(x)}{I_0(x;q)} = 1,$$

and thus,

$$\psi_q(x) \sim I_0(x;q) \quad \text{on } \mathbb{R}_+. \tag{3.5}$$

In view of Lemmas 2.2 and 2.3, we see, for all $x \ge 2$, $q > 0$ and for all $a \in [0,1]$, that

$$0 < I_0(x;q) \le I_a(x;q) \le I_1(x;q), \tag{3.6}$$

and $\psi_q(x) > 0$ for all $x \ge 2$ and $q > 0$. Therefore,

$$\frac{\psi_q(x)}{I_1(x;q)} \le \frac{\psi_q(x)}{I_a(x;q)} \le \frac{\psi_q(x)}{I_0(x;q)}. \tag{3.7}$$

In view of equations (3.4), (3.5) and (3.7), we conclude that

$$\lim_{x\to\infty} \frac{\psi_q(x)}{I_a(x;q)} = 1,$$

and thus,

$$\psi_q(x) \sim I_a(x;q) \quad \text{on } \mathbb{R}_+.$$

This ends the proof. □

We are now interested in studying the error of the approximation.

Definition 3.3. Suppose that $x \in [2, \infty)$, $q \in \mathbb{R}_+$ and $a \in [0, 1]$. We define

$$E_a(x; q) = \psi_q(x) - I_a(x; q) \tag{3.8}$$

as the error of the approximation $\psi_q(x) \approx I_a(x; q)$.

Theorem 3.4. *For any $a \in [0, 1]$ and $q \in \mathbb{R}_+$, the error $E_a(x; q)$ approaches zero as $x \to \infty$, and therefore $\psi_q(x) \approx I_a(x; q)$ for relatively large x.*

Proof. In view of equations (3.3) and (3.6), we deduce for $a \in [0, 1]$, $q \in \mathbb{R}_+$ and $x \in [2, \infty)$ that

$$|\psi_q(x) - I_a(x; q)| \leq I_1(x; q) - I_0(x; q),$$

or equivalently,

$$0 \leq |E_a(x; q)| \leq \log\left(\frac{1 - q^{x+1}}{1 - q^x}\right) - H(q - 1) \log q. \tag{3.9}$$

Taking the limits as $x \to \infty$ when $0 < q < 1$ gives

$$\lim_{x\to\infty} |E_a(x; q)| = \lim_{x\to\infty} \log\left(\frac{1 - q^{x+1}}{1 - q^x}\right) = 0, \tag{3.10}$$

and when $q \geq 1$, gives

$$\begin{aligned} \lim_{x\to\infty} |E_a(x; q)| &= \lim_{x\to\infty} \log\left(\frac{q^{x+1} - 1}{q^x - 1}\right) - \log q \\ &= \lim_{x\to\infty} \log\left(\frac{1 - q^{-x-1}}{1 - q^{-x}}\right) = 0. \end{aligned} \tag{3.11}$$

By virtue of equations (3.10) and (3.11) we conclude, for $q > 0$, that

$$\lim_{x\to\infty} E_a(x; q) = 0. \qquad \square$$

Theorem 3.5. *For any $x \in [2, \infty)$, $a \in [0, 1]$ and $q \in \mathbb{R}_+$, we have*

(i) *the errors $E_a(x; q)$ are uniformly bounded between $-\log(3/2)$ and $\log(3/2)$.*

(ii)

$$\psi_q(x) = \log[x+a]_q + \frac{q^x \log q}{1-q^x} - \left(\frac{1}{2}+a\right)H(q-1)\log q$$
$$+ O\left(\log\left(q^{-H(q-1)}\left(1+\frac{q^x}{[x]_q}\right)\right)\right).$$

Proof. Define the function

$$\alpha(x,q) = \log[x+1]_q - \log[x]_q - H(q-1)\log q.$$

Differentiation gives, for all $q>0$,

$$\frac{\partial}{\partial x}\alpha(x,q) = -\frac{q^{x+1}\log q}{1-q^{x+1}} + \frac{q^x \log q}{1-q^x} = \frac{q^x(1-q)\log q}{(1-q^{x+1})(1-q^x)} < 0,$$

which reveals that the function $\alpha(x,q)$ is decreasing on $[2,\infty)$ for all $q>0$ with a maximum of

$$\alpha(2,q) = \log(1+q+q^2) - \log(1+q) - H(q-1)\log q.$$

When $0<q<1$, we get

$$\frac{d}{dq}\alpha(2,q) = \frac{q^2+2q}{(1+q)(1+q+q^2)} > 0,$$

which shows that $\alpha(2,q)$ is increasing function on $(0,1)$ onto $(0,\log(3/2))$. When $q>1$, we get

$$\frac{d}{dq}\alpha(2,q) = \frac{-1-2q}{q(1+q)(1+q+q^2)} < 0,$$

which reveals that $\alpha(2,q)$ is a decreasing function on $(1,\infty)$ onto $(0,\log(3/2))$.

In view of the previous information, we can conclude that the function $\alpha(x,q)$ is bounded between zero and $\log(3/2)$. Therefore, $|E_q(x;q)| < \log(3/2)$. The second case follows immediately from equations (1.9) and (3.9). $\square$

Remark 3.6. It is worth mentioning that Moak [4] proved the following approximation for the q-digamma function:

$$\psi_q(x) = \log[x]_q + \frac{1}{2}\frac{q^x \log q}{1-q^x} + O\left(\frac{q^x \log^2 q}{(1-q^x)^2}\right)$$

holds for all $q > 0$ and $x > 0$ and so $\psi_q(x) \sim I(x;q)$ on $\mathbb{R}_+$ where

$$I(x;q) = \log[x]_q + \frac{1}{2}\frac{q^x \log q}{1-q^x}. \tag{3.12}$$

In the following theorem, we will show that the approximation $I_{1/2}(x;q)$ of $\psi_q(x)$ is better than the approximation (of Moak) $I(x;q)$ of $\psi_q(x)$ for all $q > 0$ and $x > 0$.

Theorem 3.7. *For all $x, q \in \mathbb{R}_+$ such that $x \geq 2$, the error of approximation $I_{1/2}(x;q)$ of $\psi_q(x)$ is less than the error of approximation $I(x;q)$ of $\psi_q(x)$.*

Proof. Lemma 2.5 gives that

$$-\log[x]_q - 1/2\frac{q^x \log q}{1-q^x} < -\log[x+1/2]_q - \frac{q^x \log q}{1-q^x},$$

which can be rewritten as

$$\psi_q(x) - \log[x]_q - \frac{1}{2}\frac{q^x \log q}{1-q^x} < \psi_q(x) - \log[x+1/2]_q - \frac{q^x \log q}{1-q^x},$$

or equivalently,

$$\psi_q(x) - I(x;q) < \psi_q(x) - I_{1/2}(x;q). \tag{3.13}$$

Also, from Lemma 2.6, we have

$$\psi_q(x) - \log[x+1/2]_q - \frac{q^x \log q}{1-q^x} < \log[x]_q + \frac{1}{2}\frac{q^x \log q}{1-q^x} - \psi_q(x),$$

which is equivalent to

$$\psi_q(x) - I_{1/2}(x;q) < I(x;q) - \psi_q(x). \tag{3.14}$$

Combining equations (3.13) and (3.14) yields

$$|\psi_q(x) - I_{1/2}(x;q)| < |\psi_q(x) - I(x;q)|.$$

Equivalently,

$$|E_{1/2}(x;q)| < |\psi_q(x) - I(x;q)|.$$

□

Acknowledgments. I am very grateful to the referee(s) for valuable comments and useful suggestions, which greatly helped to improve the presentation and the quality of the paper.

REFERENCES

1. H. Alzer and A.Z. Grinshpan, *Inequalities for the gamma and q-gamma functions*, J. Approx. Theor. **144** (2007), 67–83.

2. C. Krattenthaler and H.M. Srivastava, *Summations for basic hypergeometric series involving a q-analogue of the digamma function*, Comp. Math. Appl. **32** (1996), 73–91.

3. T. Mansour and A.SH. Shabani, *Some inequalities for the q-digamma function*, J. Inequal. Pure Appl. Math. **10** (2009), 1–8.

4. D.S. Moak, *The q-analogue of Stirling's formula*, Rocky Mountain J. Math. **14** (1984), 403–413.

5. I. Muqattash and M. Yahdi, *Infinite family of approximations of the digamma function*, Math. Comp. Mod. **43** (2006), 1329–1336.

6. A. Salem, *Some properties and expansions associated with q-digamma function*, Quaest. Math. **36** (2013), 67–77.

7. ———, *Complete monotonicity properties of functions involving q-gamma and q-digamma functions*, Math. Inequal. Appl. **17** (2014), 801-811.

8. ———, *Two classes of bounds for the q-gamma and the q-digamma functions in terms of the q-zeta functions*, Banach J. Math. Anal. **8** (2014), 109–117.

9. ———, *An infinite class of completely monotonic functions involving the q-gamma function*, J. Math. Anal. Appl. **406** (2013), 392–399.

10. ———, *A completely monotonic function involving q-gamma and q-digamma functions*, J. Approx. Theor. **164** (2012), 971–980.

Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
Email address: ahmedsalem74@hotmail.com