ARRANGEMENTS OF SPHERES
AND PROJECTIVE SPACES

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ABSTRACT. We develop the theory of arrangements of spheres. Consider a finite collection of codimension-1 subspheres in a positive-dimensional sphere. There are two posets associated with this collection: the poset of faces and the poset of intersections. We also associate a topological space: the complement of the union of tangent bundles of these subspheres in the tangent bundle of the ambient sphere. We call this space the tangent bundle complement. As in the case of hyperplane arrangements the aim of this new notion is to understand the interaction between the combinatorics of the intersections and the topology of the tangent bundle complement. In the present paper, we find a closed form formula for the homotopy type of the complement and express some of its topological invariants in terms of the associated combinatorial information.

1. Introduction. An arrangement of hyperplanes is a finite set $\mathcal{A}$ consisting of codimension-1 subspaces of $\mathbb{R}^l$. These hyperplanes and their intersections induce a polyhedral stratification of $\mathbb{R}^l$. The combinatorial information of an arrangement $\mathcal{A}$ is contained in two posets, namely, the *face poset*, which consists of all the strata, and the *intersection poset*, which contains all possible intersections of hyperplanes in $\mathcal{A}$. A topological space associated with $\mathcal{A}$, denoted $M(\mathcal{A})$, is the complement of the union of the complexified hyperplanes in $\mathbb{C}^l$. It is an open submanifold of $\mathbb{C}^l$ with the homotopy type of a finite-dimensional CW complex [**16,** subsection 5.1]. The study of this complement was initiated in the works of Fox et al. in the 1960's and 1970's (see [**16,** subsection 5.1]).

One of the aspects of the theory of arrangements is to understand the interaction between the combinatorial data of an arrangement and the topology of $M(\mathcal{A})$. For example, the cohomology ring of

the complement, known as the *Orlik-Solomon algebra* is completely determined by the intersection data [**16**, subsection 5.4]. A pioneering result by Salvetti [**18**] states that the homotopy type of the complement is determined by the face poset.

A generalization of hyperplane arrangements was introduced by the author [**9**] where a study of arrangements of codimension-1 submanifolds in a smooth manifold was initiated. In this paper, we focus on a particular example: arrangements of spheres. Given a smooth sphere S^l, we consider a finite collection of codimension-1 subspheres, denoted by $\mathcal{A}$, which satisfy reasonably nice conditions. For example, these subspheres are tamely embedded, their intersections are hyperplane-like, and they induce a stratification of the ambient sphere such that all the strata are contractible. Consequently, one can define face and intersection posets in this context. The topological space associated with such a collection is the complement of the union of tangent bundles of these subspheres in TS^l. We call this space the *tangent bundle complement* and denote it by $M(\mathcal{A})$. We ask the same question: to what extent do the combinatorics of $\mathcal{A}$ help determine the topology of $M(\mathcal{A})$?

We explore this interaction of combinatorics and topology by first describing a regular cell complex that has the homotopy type of $M(\mathcal{A})$. The construction of this complex relies on the order relations in the face poset and is a generalization of the classical Salvetti complex. We then concentrate only on those arrangements which exhibit certain antipodal symmetry. For these so-called *mirrored arrangements* we find a closed form formula for the homotopy type of $M(\mathcal{A})$. We then show that the cohomology groups of $M(\mathcal{A})$ are determined by the intersection data. Moreover, the coholomogy ring of $M(\mathcal{A})$ can be expressed as a direct sum of an Orlik-Solomon algebra and a free abelian group in the top dimension. The rank of this top-dimensional free abelian group is equal to the number of graded pieces in the Orlik-Solomon algebra. We also identify a class of arrangements for which the word problem for $\pi_1(M(\mathcal{A}))$ is solvable.

In the case of mirrored arrangements, as a consequence of the antipodal symmetry, we can define *projective arrangements*, i.e., a finite collection of subspaces homeomorphic to $\mathbb{RP}^{l-1}$ in $\mathbb{RP}^l$. We exploit this antipodal symmetry to its full extent and derive similar results regarding the tangent bundle complement. For example, the antipodal

map helps us to understand the homotopy type of the tangent bundle complement as well as its fundamental group.

An important motivation for studying hyperplane arrangements comes from their natural connection with Coxeter groups and the associated Artin groups. Let W be a finite, irreducible Coxeter group of rank n. It acts linearly (in fact as origin-fixing isometries) on a real vector space V of dimension n. Such a group is generated by reflections and has the following presentation:

$$W = \langle s_1, \ldots, s_n \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1, \text{ for all } i \neq j \text{ and } 2 \leq m_{ij} < \infty \rangle.$$

Its action on V is not free; each reflection in W fixes a hyperplane. The union of these reflecting hyperplanes is the *reflection arrangement*, denoted $\mathcal{A}_W$, associated to W. The complement of these fixed hyperplanes is a disjoint union of open simplicial cones called (*Weyl*) *chambers*. Under the W action, these chambers are permuted freely (see [**7,** Chapter 6]).

Complexifying this situation, we get a finite arrangement of complex hyperplanes in $V \otimes \mathbb{C}$. The complement of the union of these hyperplanes, denoted M_W, is connected and admits a fixed point free action of W. Brieskorn [**3**] showed that the fundamental group of the orbit space N_W has the following presentation:

$$\left\langle s_1, \ldots, s_n \mid \underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}} \quad \text{for all } i \neq j \right\rangle.$$

This group is known as the *Artin group* associated to W and is denoted by A_W. There is a natural surjection from A_W onto W whose kernel is the so-called pure Artin group PA_W. It is the fundamental group of M_W. If W is the symmetric group, i.e., type A Coxeter group, then A_W is the braid group and PA_W is the pure braid group.

Deligne [**8**] showed that the universal cover of N_W is contractible. Hence, N_W is a $K(A_W, 1)$ space. Subsequent study of these groups is influenced greatly by Deligne's work. Some of the important properties of Artin groups were proved by expanding on his ideas, notably the biautomatic nature of these groups [**4**]. Simply put, it says that the

Artin groups have solvable word and conjugacy problems. We refer the reader to [**5**, subsection 1.2] for details and references.

Sphere arrangements with a similar motivation are investigated here. It is well known that the finite subgroups of isometries of a sphere generated by reflections are in fact Coxeter groups (see [**7**, Chapter 10] and [**12**]). Each reflection in this Coxeter transformation group fixes a codimension-1 subsphere giving rise to a spherical arrangement. The complement of this arrangement is a disjoint union of 'spherical' simplices, and they are freely permuted by the action. Since the group acts via isometries, the action extends to the tangent bundle of the sphere. The complement of the union of the tangent bundles of the fixed sub-spheres serves as the analogue of the space M_W introduced above. The Coxeter transformation group acts fixed point freely on this complement. The fundamental group of the orbit space is the desired generalization of Artin groups. The main aim of this paper is to lay topological foundations for the study of these "Artin-like" groups. We illustrate with an example.

Example 1.1. Consider the 1-sphere S^1. In this case, a Coxeter transformation group W is a dihedral group of order $2n$ with the presentation $\langle r, s \mid r^2 = s^2 = (rs)^n = 1 \rangle$. The n reflections in W fix n 0-spheres, i.e., $2n$ points. Declaring one of the chambers as the fundamental chamber all others can be labeled by elements of $W \setminus \{1\}$. The $2n$ points in this arrangement can be labeled by conjugates of the two standard parabolic subgroups W_r and W_s of W. See [**7**, Chapter 5] for details regarding such labeling.

The tangent bundle complement is an infinite cylinder with $2n$ punctures. The Salvetti complex (see subsection 3.6 for its construction) has $2n$ 0-cells with labels $\langle g, g \rangle$ for every $g \in W$. There are $4n$ 1-cells with labels of the form, $\langle hW', h \rangle$, where W' is one of the standard parabolic subgroups and $h \in W$. The reader can verify that the boundary of this 1-cell is $\{\langle h, h \rangle, \langle g, g \rangle\}$ such that $g^{-1}h \in W'$. The 'labeled' Salvetti complex inherits the free W-action on the tangent bundle complement. The orbit complex consists of exactly one 0-cell and two 1-cells with both their end points joined at the 0-cell. It has the homotopy type of wedge of two circles.

The generalized pure Artin group in this case is F_{2n+1}, the free group on $2n + 1$ generators. The generalized Artin group is F_2, and we get

the following exact sequence:

$$1 \longrightarrow F_{2n+1} \hookrightarrow F_2 \twoheadrightarrow W \longrightarrow 1.$$

If we were to denote the generators of F_2 as r and s, then $F_{2n+1} = \pi_1(M(\mathcal{A}))$ has the following presentation:

$$\left\langle r^2, s^2, (rs)^n, rs^2r, \ldots, \underbrace{rsr\cdots}_{n-1}\epsilon^2\underbrace{srs\cdots}_{n-1}, sr^2s, \ldots, \underbrace{srs\cdots}_{n-1}\epsilon^2\underbrace{rsr\cdots}_{n-1} \right\rangle,$$

where ϵ is r or s depending on the parity of n.

In joint work with Das [6] we extend this correspondence to higher-dimensional spheres (in fact, to smooth manifolds). In particular, we show that the homotopy equivalence between the tangent bundle complement and the Salvetti complex is W-equivariant. Further, we explain how the group theoretic data can be used to define this complex. We also describe a presentation for the associated groups. Current work in progress, among other things, focuses on computation of the (twisted) cohomology of $M(\mathcal{A})$ with group ring coefficients.

The paper is organized as follows. Section 2 concerns the preliminaries of hyperplane arrangements. In Section 3, we introduce the new objects of study, arrangements of spheres and the tangent bundle complement. In Section 4, we look at how the combinatorics of intersections determines the topology of the complement. We investigate the fundamental group in Section 5. In Section 6, we look at arrangements of projective spaces.

2. Arrangements of hyperplanes. These arrangements arise naturally in geometric, algebraic and combinatorial instances. In this section, we formally define hyperplane arrangements and the combinatorial data associated with them in the setting that is most relevant to our work.

Definition 2.1. A (real) *arrangement of hyperplanes* is a collection $\mathcal{A} = \{H_1, \ldots, H_k\}$ of finitely many hyperplanes in $\mathbb{R}^l$, $l \geq 1$.

An arrangement is called *central* if the intersection of all the hyperplanes in $\mathcal{A}$ is non-empty. However, we allow our arrangements to be

non-central. For a subset X of $\mathbb{R}^l$, the *restriction* of $\mathcal{A}$ to X is the subarrangement

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\}.$$

Analogously, one may define hyperplane arrangements in $\mathbb{C}^l$, which are called *complex arrangements*. To every real arrangement $\mathcal{A}$ there is an associated complex arrangement $\mathcal{A}_\mathbb{C}$; for every $H \in \mathcal{A}$, there is a hyperplane $H_\mathbb{C} \in \mathcal{A}_\mathbb{C}$ with the same defining equations as H. In this paper, we focus on (complexified) real arrangements of hyperplanes.

Associated with $\mathcal{A}$, there are two posets containing important information regarding the arrangement, namely, the face poset and the intersection poset.

Definition 2.2. The *intersection poset* $L(\mathcal{A})$ of $\mathcal{A}$ is the set of all intersections of hyperplanes, including $\mathbb{R}^l$ itself as the empty intersection, ordered by reverse inclusion.

The intersection poset is ranked with the rank of an element being the codimension of the corresponding intersection. The *rank of an arrangement* $\mathcal{A}$ is defined to be the rank of its intersection poset. An arrangement is said to be *essential* if its rank equals the dimension of the ambient space; without loss of generality, we will from now on assume this to be the case. In general, $L(\mathcal{A})$ is a (meet) semilattice; it is a lattice if and only if the arrangement is central.

Definition 2.3. The *face poset* $\mathcal{F}(\mathcal{A})$ of $\mathcal{A}$ is the set of all faces ordered by topological inclusion: $F \leq G$ if and only if $F \subseteq \overline{G}$.

Codimension-0 faces are called *chambers*. The set of all chambers will be denoted by $\mathcal{C}(\mathcal{A})$. A chamber is called *bounded* if it is a bounded subset of $\mathbb{R}^l$. Two chambers C and D are *adjacent* if they have a common face in their closure.

The topological space associated with a real hyperplane arrangement $\mathcal{A}$ is its *complexified complement* $M(\mathcal{A})$, which is defined as follows.

Definition 2.4.

$$M(\mathcal{A}) := \mathbb{C}^l \setminus \left(\bigcup_{H \in \mathcal{A}} H_\mathbb{C} \right),$$

where $H_{\mathbb{C}}$ is the hyperplane in $\mathbb{C}^l$ with the same defining equation as $H \in \mathcal{A}$.

2.1. The Salvetti complex.
In [18], Salvetti constructed a regular CW-complex which has the homotopy type of the complexified complement. The construction uses the ordering in $\mathcal{F}(\mathcal{A})$.

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^l$. We construct a regular l-complex, called the *Salvetti complex*, and denote it by $\mathrm{Sal}(\mathcal{A})$, by first describing its cells. The k-cells, for $0 \leq k \leq l$, of $\mathrm{Sal}(\mathcal{A})$ are in one-to-one correspondence with the pairs $[F, C]$, where F is a codimension-k face of $\mathcal{A}$ and C is a chamber whose closure contains F.

Since $\mathrm{Sal}(\mathcal{A})$ is regular, all of the attaching maps are homeomorphisms. Hence, it is enough to specify the boundary of each cell. A cell labelled $[F_1, C_1]$ is contained in the boundary of another cell labelled $[F_2, C_2]$ if and only if $F_1 \geq F_2$ in $\mathcal{F}(\mathcal{A})$ and C_1 and C_2 are contained in the same chamber of $\mathcal{A}_{F_1}$. Now we state the seminal result of Salvetti.

Theorem 2.5 (Salvetti [18]). *Let $\mathcal{A}$ be an arrangement of real hyperplanes and $M(\mathcal{A})$ the complement of its complexification inside $\mathbb{C}^l$. Then there is an embedding of $\mathrm{Sal}(\mathcal{A})$ into $M(\mathcal{A})$. Moreover, there is a natural map in the other direction which is a deformation retraction.*

The above construction is generalized by Björner and Ziegler [2] by giving a CW-complex with the homotopy type of the complement of a complex subspace arrangement.

2.2. Cohomology of the complement.
We begin by associating a combinatorially defined algebra, called the *Orlik-Solomon algebra*, to a (complex) hyperplane arrangement.

Let $\mathcal{A}$ be a hyperplane arrangement. For every $1 \leq p \leq n$, call a p-tuple $S = (H_1, \ldots, H_p)$ of hyperplanes to be *independent* if $\dim(H_1 \cap \cdots \cap H_p) = l - p$, and call it *dependent* if the intersection is nonempty and its codimension is strictly less than p. Geometrically, the independence implies that the hyperplanes of S are in general position.

Let E_1 be the free $\mathbb{Z}$-module generated by the elements e_H for every $H \in \mathcal{A}$. Define $E(\mathcal{A})$ to be the exterior algebra on E_1, and let ∂ denote the differential in $E(\mathcal{A})$. For a p-tuple S of hyperplanes we denote by

$\bigcap S$ the intersection of elements in S, and by e_S we mean $e_{H_1} \wedge \cdots \wedge e_{H_p}$. Let $I(\mathcal{A})$ denote the ideal of E generated by

$$\left\{ e_S \mid \bigcap S = \emptyset \right\} \cup \{\partial e_S \mid S \text{ is dependent}\}.$$

Definition 2.6. The *Orlik-Solomon algebra* of a (complex) arrangement $\mathcal{A}$ is the quotient algebra $E(\mathcal{A})/I(\mathcal{A})$ and is denoted by $A(\mathcal{A})$.

The following important theorem shows how cohomology of $M(\mathcal{A})$ depends on the intersection poset. It combines the work of Arnold, Brieskorn, Orlik and Solomon. For details and exact statements of their individual results, see [**16,** Chapter 3, subsection 5.4].

Theorem 2.7. *Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a hyperplane arrangement in $\mathbb{C}^l$. For $H \in \mathcal{A}$, choose a linear form $\alpha_H \in (\mathbb{C}^l)^*$ such that $\ker(\alpha_H) = H$. Then the integral cohomology algebra of the complement is generated by the classes*:

$$\omega_H := \frac{1}{2\pi} \frac{d\alpha_H}{\alpha_H}.$$

The map $\gamma \colon A(\mathcal{A}) \to H^*(M(\mathcal{A}), \mathbb{Z})$, defined by:

$$\gamma(e_H) \longmapsto \omega_H,$$

induces an isomorphism of graded $\mathbb{Z}$-algebras.

This theorem asserts that a presentation of the cohomology algebra of $M(\mathcal{A})$ can be constructed from the data that are encoded by the intersection poset. Let us mention one more theorem that explicitly states the role of the intersection poset in determining the cohomology of the complement. In particular, the result states that there is a finer grading of cohomology groups indexed by the intersections, and the rank of each cohomology group is determined by the Möbius function of the intersection poset (see [**16,** Proposition 3.75 and Lemma 5.91]).

Theorem 2.8. *Let $\mathcal{A}$ be a nonempty complex arrangement. For $X \in L(\mathcal{A})$, let M_X denote the complexified complement of the restricted*

arrangement $\mathcal{A}_X$. We have the following isomorphisms for each $k \geq 0$:

$$\theta_k \colon \bigoplus_{\mathrm{rank}\, X = k} H^k(M_X) \longrightarrow H^k(M)$$

induced by the inclusions $i_X \colon M \hookrightarrow M_X$. Moreover, the rank of each cohomology group is determined by the following formula:

$$\mathrm{rank}\, H^k(M) = \sum_{X \in L_k} (-1)^{\mathrm{rank}(X)} \mu(\mathbb{R}^l, X),$$

where μ is the Möbius function of $L(\mathcal{A})$.

3. Arrangements of spheres. We now introduce arrangements of codimension-1 subspheres in a sphere. First, we isolate essential properties of a hyperplane arrangement:

(1) there are finitely many codimension 1 subspaces, each of which separates $\mathbb{R}^l$ into two components;
(2) there is a polyhedral stratification of the ambient space, and the face poset of this stratification has the homotopy type of the ambient space.

Remark 3.1. Recall that associated to every poset there is an abstract simplicial complex known as the *order complex*. A k-simplex of the order complex corresponds to a k-chain of the poset. By *homotopy type of the poset* we mean the homotopy type of the geometric realization of the associated order complex.

In this section, we first generalize the above properties in the context of spheres. Then we compare our definition with the topological representation of oriented matroids. Finally, we look at the combinatorics of sphere arrangements.

3.1. Codimension-1 tame subspaces of spheres. We start with a generalization of property (1) above. By an l-sphere S^l, we mean a smooth, closed l-manifold homeomorphic to the unit sphere in $\mathbb{R}^{l+1}$. The 0-sphere S^0 consists of two points, and we assume that the empty set is the sphere of dimension -1.

If S is an $(l-1)$-sphere embedded in S^l $(l \geq 2)$ as a closed subset, then $S^l \setminus S$ has two connected components. Hence, codimension-1 subspheres generalize hyperplanes in this respect. In general, codimension-1 subspheres in a sphere could be very difficult to deal with. For example, consider the *Alexander horned sphere*. It is an embedding of S^2 inside S^3 such that one of the connected components of the complement is not even simply connected. In order to avoid such pathological instances we restrict ourselves to a nice class. A codimension-1 subsphere S of S^l is said to be *tame* (or *locally flat*) if, for every $x \in S$, there is a neighborhood U_x of x in S^l such that

$$(U_x, U_x \cap S) \cong (\mathbb{R}^l, \mathbb{R}^{l-1}).$$

For such a subsphere, the following statements are equivalent (see [**17**, Theorem 1.8.2]):

(1) there exists a homeomorphism h of S^l onto the standard unit sphere in $\mathbb{R}^{l+1}$ such that $h(S)$ is the equator cut out by the coordinate hyperplane $x_{l+1} = 0$;

(2) S is homeomorphic to a piecewise-linearly embedded $(l-1)$-subsphere;

(3) the closure of each connected component of $S^l \setminus S$ is homeomorphic to the l-ball.

Tame subspheres need not intersect like hyperplanes. As an example, consider Figure 1 which shows the non-Pappus arrangement on the unit sphere in $\mathbb{R}^3$.
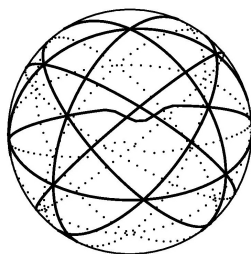

FIGURE 1. Non-Pappus arrangement.

The cone over each of these nine circles is homeomorphic to a plane passing through the origin. Clearly, there is no self-homeomorphism

of S^2 such that the cone over the image of each of these circles is a two-dimensional subspace. This picture can arise as the boundary of a neighborhood of 2-spheres intersecting in a 3-sphere. We would like to avoid such situations as we are interested in dealing with the tangent bundle.

We introduce a notion that will guarantee hyperplane-like intersections of subspheres. But first, some notation. Let $\mathcal{A} = \{S_1, \dots, S_k\}$ be a collection of tame, codimension-1 subspheres in S^l. For every $x \in S^l$ and an open neighborhood V_x of x homeomorphic to $\mathbb{R}^l$, let

$$\mathcal{A}_x := \{S \cap V_x \mid x \in S \in \mathcal{A}\}.$$

Denoting by $\bigcup \mathcal{A}_x$, we mean the union of elements of $\mathcal{A}_x$.

Definition 3.2. Let $\mathcal{A} = \{S_1, \dots, S_k\}$ be a collection of codimension-1, tame sub-spheres of S^l. We say that these sub-spheres have *locally flat intersections* if, for every $x \in S^l$, there exists an open neighborhood V_x and a homeomorphism $\phi \colon V_x \to \mathbb{R}^l$ such that

$$\left(V_x, \bigcup \mathcal{A}_x\right) \cong \left(\mathbb{R}^l, \bigcup \mathcal{A}'\right),$$

where $\mathcal{A}'$ is a central hyperplane arrangement in $\mathbb{R}^l$ with $\phi(x)$ as the common point.

3.2. Cellular stratification. Now we generalize property (2). Let $\mathcal{A} = \{S_1, \dots, S_k\}$ be a collection of codimension-1, tame sub-spheres of S^l with locally flat intersections. Let $\mathcal{L}$ be the set of all possible nonempty intersections of members of $\mathcal{A}$ and $\mathcal{L}^d$ the subset containing codimension-d intersections. We have

$$\bigcup \mathcal{L}^0 = S^l$$

and

$$\bigcup \mathcal{L}^1 = \bigcup_{i=1}^{k} S_i.$$

For each $d \geq 0$, consider the following subset of S^l

$$\mathcal{S}^d(S^l) = \bigcup \mathcal{L}^d \setminus \bigcup \mathcal{L}^{d+1}.$$

Note that each $\mathcal{S}^i(S^l)$ may be disconnected and that the sphere can be expressed as the disjoint union of these connected components. We want these sets to define a 'nice' stratification of S^l; hence, we introduce the language of cellular stratified spaces developed by Tamaki [**21**]. Recall that a subset A of a topological space X is said to be *locally closed* if every $x \in A$ has a neighborhood U in X with $A \cap U$ closed in U.

Definition 3.3. Let X be a topological space and $\mathcal{P}$ a poset. A *stratification* of X indexed by $\mathcal{P}$ is a surjective map $\sigma \colon X \to \mathcal{P}$ satisfying the following properties:

(i) for $p \in \operatorname{Im}(\sigma)$, $e_p := \sigma^{-1}(p)$ is connected and locally closed;
(ii) for $p, q \in \operatorname{Im}(\sigma)$, $e_p \subseteq \overline{e_q}$ if and only if $p \leq q$;
(iii) $e_p \bigcap \overline{e_q} \neq \emptyset \rightarrow e_p \subseteq \overline{e_q}$.

The subspace e_p is called the *stratum with index* p.

One may verify that the boundary of each stratum, $\partial e_p = \overline{e_p} - e_p$ is itself a union of strata. Such a stratification gives a decomposition of X. The indexing poset $\mathcal{P}$ is called the *face poset* of the stratification.

It is now easy to check that the connected components of $\mathcal{S}^i(S^l)$ define a stratification of X. However, it is not desirable to consider arbitrary stratifications. For example, in the case of two non-intersecting circles in S^2, there are three codimension-0 strata and two codimension-1 strata. But the resulting face poset does not have the homotopy type of the 2-sphere. We need to focus on stratifications such that the strata are cells and the incidence relations between the strata recover the homotopy type of S^l. In order to achieve property (iii) we assume that each stratum is a cell and the resulting stratification is a regular CW-complex.

3.3. Definitions and examples. The desired generalization of hyperplane arrangements is the following.

Definition 3.4. Let S^l be a smooth sphere of dimension l. An *arrangement of spheres* is a finite collection $\mathcal{A} = \{S_1, \ldots, S_k\}$ of codimension-1 smooth subspheres in S^l such that:

(i) the S_i's have locally flat intersections (see Definition 3.2);

(ii) for all $I \subseteq \{1, \ldots, k\}$, the intersection $\mathcal{A}_I := \bigcap_{i \in I} S_i$ is a sphere of some dimension;

(iii) if $\mathcal{A}_I \nsubseteq S_j$, for some I and some j, then $\mathcal{A}_I \cap S_j$ is a codimension-1 subsphere in $\mathcal{A}_I$;

(iv) the stratification induced by the intersections of S_i's define the structure of a regular CW-complex.

If there exists a fixed-point free, involutive diffeomorphism f of the sphere such that for each $S \in \mathcal{A}$ we have $f(S) = S$ and $f(x) \neq x = f^2(x)$ for all $x \in S^l$, then we call $\mathcal{A}$ a *centrally symmetric arrangement of spheres.*

As in the case of hyperplane arrangements the combinatorial information associated with sphere arrangements is contained in the two posets which we now define.

Definition 3.5. The *intersection poset* denoted by $L(\mathcal{A})$ is the set of connected components of all possible nonempty intersections of S_i's ordered by reverse inclusion. By convention, $S^l \in L(\mathcal{A})$ is the least element.

The intersection poset is a ranked poset. The rank of each element in $L(\mathcal{A})$ is defined to be the codimension of the corresponding intersection.

Definition 3.6. The intersections of these S_i's in $\mathcal{A}$ define a stratification of S^l. The connected components in each stratum are called *faces.* The collection of all the faces $\mathcal{F}(\mathcal{A})$ ordered by topological inclusion, i.e.,

$$F \leq G \Longleftrightarrow F \subseteq \overline{G}$$

is called the *face poset.* The top-dimensional faces are called *chambers* and the set of all chambers is denoted by $\mathcal{C}(\mathcal{A})$.

For a face F, define its *support* as the least-dimensional intersection containing F. The *dimension of a face* is the dimension of its support. It is straightforward to see that the dimension function makes the face poset a ranked poset.

We now look at two examples of sphere arrangements.

Example 3.7. Let X be the circle S^1, a smooth one-dimensional manifold. The codimension-1 subspheres are the pairs of (diametrically opposite) points in S^1. Consider the arrangement $\mathcal{A} = \{p, q\}$ of two such points. For both of these points there is an open neighborhood which is homeomorphic to an arrangement of a point in $\mathbb{R}$. Figure 2 shows this arrangement and the Hasse diagrams of the face poset and the intersection poset.
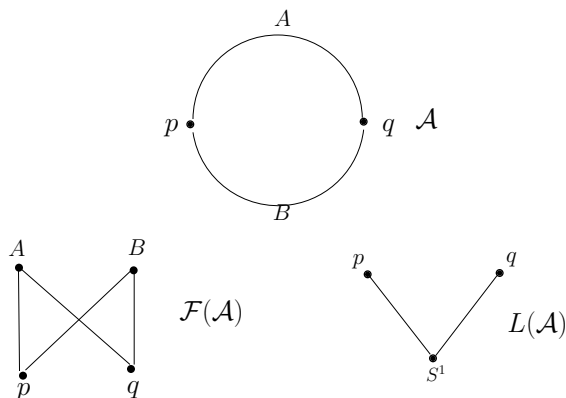

FIGURE 2. Arrangement of two points in a circle.

Example 3.8. As a two-dimensional example, consider an arrangement of two great circles N_1 and N_2 in S^2. Figure 3 shows this arrangement and the related posets. The face poset has two 0-cells, four 1-cells and four 2-cells. Also, note that the geometric realization of the face poset has the homotopy type of S^2.

3.4. Topological representation of oriented matroids. We now explore a connection between sphere and hyperplane arrangements using oriented matroids. The theory of oriented matroids is intimately connected with hyperplane arrangements. This combinatorial structure combines the information contained in face and intersection posets of a hyperplane arrangement. There are several (axiomatic) ways of defining oriented matroids. We refer the reader to the book of Björner et al. [**1**] for various aspects related to oriented matroids. We do
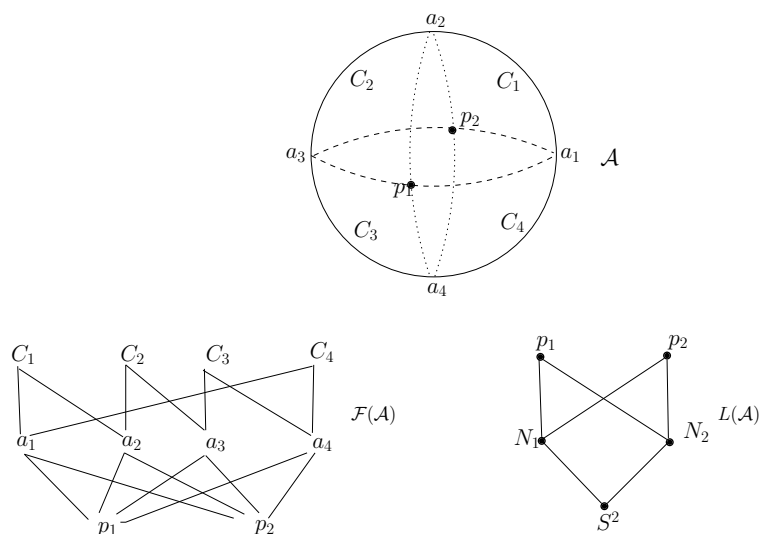
FIGURE 3. Arrangement of 2 circles in a sphere.

not intend to define and explain the properties of oriented matroids. Our aim is to compare their topological representation with the sphere arrangements.

The oriented matroids which correspond to hyperplane arrangements are known as the *realizable oriented matroids*. There are oriented matroids that do not correspond to hyperplane arrangements (e.g., the non-Pappus configuration). Hence, for a long time, an important question in this field was to come up with the right topological model for oriented matroids. This was settled by Folkman and Lawrence [10]. The Folkman-Lawrence topological representation theorem states that, in general, oriented matroids correspond to certain collections of finitely many topological spheres and balls. These so called pseudo-arrangements not only describe oriented matroids in the same way that $\mathbb{R}^l$, and collections of half spaces describe an obvious combinatorial structure, but there is a one-to-one correspondence between such arrangements and the oriented matroids.

In their original formulation Folkman and Lawrence introduced arrangements of pseudo-hemispheres. Much simplification of their ideas

was achieved by Mandel in his thesis [**14**]. He defined the notion of *sphere systems*, which we now state.

Definition 3.9. A finite multi-set $\mathcal{A} = \{S_e \mid e \in E\}$ of codimension-1, tame subspheres in S^l is called a *sphere system* if the following conditions hold:

(i) $S_A := \bigcap_{e \in A} S_e$ is a sphere, for all $A \subset E$.
(ii) If $S_A \nsubseteq S_e$ for $A \subset E$, $e \in E$, and S_e^+ and S_e^- are the two sides of S_e, then $S_A \cap S_e$ is a subsphere in S_A with sides $S_A \cap S_e^+$ and $S_A \cap S_e^-$.

A sphere system is said to be *essential* if the intersection of all the sub-spheres is empty. It can be shown that the stratification of S^l induced by an essential sphere system defines regular CW decomposition of the sphere [**1**, Proposition 5.1.5]. The topological representation theorem states that (loop-free) oriented matroids of rank $l + 1$ (up to reorientation and isomorphism) are in one-to-one correspondence with centrally symmetric, essential sphere systems in S^l. However, the sphere arrangements that we want to deal with are not general enough to represent oriented matroids. Note the differences between the definition of a sphere system and Definition 3.9.

(1) We assume that all intersections are locally flat,
(2) the arrangement is *repetition-free*, i.e., every subsphere appears exactly once.

Given a central and essential arrangement of hyperplanes, consider its intersection with the unit sphere; these intersections define a centrally symmetric sphere arrangement in the sense of Definition 3.9. However, the converse need not be true. Figure 1 shows an arrangement of nine pseudo-circles which is a sphere arrangement in the sense of Definition 3.9, but it does not arise as an intersection with a central hyperplane arrangement.

The rank 3 oriented matroids can be realized as centrally symmetric, repetition-free, essential sphere systems in S^2. The reader can verify that such sphere systems are arrangements of spheres (since intersections of pseudo-circles are locally-flat). Consequently, there is a one-to-one correspondence between rank 3 oriented matroids and

the sphere arrangements in S^2. However, not all higher rank oriented matroids can be realized using sphere arrangements. For example, one can construct a sphere system in S^3 such that there exists at least one intersection whose spherical neighborhood looks like Figure 1.

Remark 3.10. We would like to clarify the distinction between Definitions 3.4 and 3.9. The concept of the sphere system is more general than that of an arrangement of spheres. It is clear that every sphere arrangement is a sphere system. However, the converse is not true in general. There are examples of oriented matroids (say, of rank 4) which do not correspond to any sphere arrangement in S^3. In light of this observation it would be an interesting problem to obtain a combinatorial characterization of (non-realizable) oriented matroids which correspond to sphere arrangements.

A *pseudo-hyperplane* is a tame embedding of a codimension-1 subspace in $\mathbb{R}^l$. Equivalently, it is the cone over a tame subsphere. An arrangement of pseudo-hyperplanes, intuitively, can be constructed by taking the cone over a sphere system.

Definition 3.11. A finite collection $\mathcal{B} = \{H_1, \ldots, H_n\}$ of pseudo-hyperplanes in $\mathbb{R}^l$ is called an *arrangement of pseudo-hyperplanes* if:

  (i) For every $A \subseteq \{1, \ldots, n\}$, the set $H_A := \bigcap_{i \in A} H_i$ is either empty or homeomorphic to some $\mathbb{R}^k$ for $0 \leq k \leq l$.
  (ii) For every $j \notin A$, either $H_A \subseteq H_j$, or $H_j \cap H_A$ is a locally flat embedding of a codimension-1 subspace of H_A.

We say that a pseudo-hyperplane arrangement is *locally flat* if all the (nonempty) intersections are locally flat.

The construction of the Salvetti complex and the Orlik-Solomon algebra hold true in case of pseudo-hyperplane arrangements. In fact, the Orlik-Solomon algebra associated to a pseudo-hyperplane arrangement is isomorphic to the cohomology algebra of the corresponding Salvetti complex (see [**1**, subsection 2.5] and [**2**, Section 7] for details).

3.5. Combinatorics of sphere arrangements. We now take a closer look at the combinatorics of incidence relations among the faces.

Let $\mathcal{A}$ denote a sphere arrangement in S^l. A hypersphere S in $\mathcal{A}$ is said to *separate* two chambers C and D if they are contained in the distinct connected components of $S^l \setminus S$. For two chambers C and D the set of all the hyperspheres that separate these two chambers is denoted by $R(C, D)$. The following lemma is now evident.

Lemma 3.12. *Let $\mathcal{A}$ be an arrangement of spheres in S^l, an l-sphere. Let C_1, C_2 and C_3 be three chambers of this arrangement. Then,*

$$R(C_1, C_3) = [R(C_1, C_2) \setminus R(C_2, C_3)] \cup [R(C_2, C_3) \setminus R(C_2, C_1)].$$

The *distance between two chambers* is defined as the cardinality of $R(C, D)$ and denoted by $d(C, D)$. Given a face F and a chamber C of a sphere arrangement $\mathcal{A}$, define the action of F on C as follows.

Definition 3.13. A face F acts on a chamber C to produce another chamber $F \circ C$ satisfying:

(i) $F \subseteq \overline{F \circ C}$,
(ii) $d(C, F \circ C) = \min\{d(C, C') \mid C' \in \mathcal{C}(\mathcal{A}), F \subseteq \overline{C'}\}$.

Lemma 3.14. *With the same notation as above, the chamber $F \circ C$ always exists and is unique.*

Proof. Clear. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It is easy to check that, if C is a chamber and F and F' are two faces such that $F' \geq F$, then $F' \circ (F \circ C) = F' \circ C$. Moreover, if $F \leq C$, then $F \circ C = C$.

3.6. The tangent bundle complement. Recall that, for a real hyperplane arrangement $\mathcal{A}$, the complexified complement $M(\mathcal{A})$ is the complement of the union of complexified hyperplanes inside the complexified ambient vector space (Definition 2.4). If one were to forget the complex structure on $\mathbb{C}^l$, then, topologically, it is just the tangent bundle of $\mathbb{R}^l$. The same is true for a hyperplane H and its complexification $H_{\mathbb{C}}$. Hence, the complexified complement of a hyperplane arrangement can also be considered as a complement inside the tangent bundle. We use this topological viewpoint to define a generalization of $M(\mathcal{A})$ for sphere arrangements.

Definition 3.15. As before, let $\mathcal{A} = \{N_1, \ldots, N_k\}$ be a sphere arrangement in S^l. Let TS^l denote the tangent bundle of S^l, and let

$$T\mathcal{A} := \bigcup_{i=1}^{k} TN_i.$$

The *tangent bundle complement* of $\mathcal{A}$ is defined as

$$M(\mathcal{A}) := TS^l \setminus T\mathcal{A}.$$

The above space was introduced in [**9**, Chapter 3] in the context of submanifold arrangements.

We now construct a regular CW-complex, in the spirit of Salvetti's construction, that has the homotopy type of the tangent bundle complement. We denote by $(S^l, \mathcal{F}(\mathcal{A}))$ the regular cell structure of S^l induced by $\mathcal{A}$. We are interested in the dual cell structure which is obtained as follows. For every face F, fix a point $x(F) \in F$, call it the *barycenter* of F. Note that $\overline{F}$ is homeomorphic to an appropriate-dimensional disc B_F. Then there exists a regular cell structure of B_F whose face poset is isomorphic to that of $\overline{F}$. For every $G < F$, the barycenter $x(G)$ determines a point y_G of B_F. Moreover, if

$$\gamma := G_0 < \cdots < G_k$$

is a chain of faces of F, then form a simplex γ_B of B which is the convex hull of the vertices $y_{G_0}, \ldots y_{G_k}$. Denote by $\Delta(\gamma)$ the image of γ_B under the given homeomorphism. Note that $\Delta(\gamma)$ need not be the convex hull of $x(G_0), \ldots x(G_k)$. Finally, denote the union of all those $\Delta(\gamma)$'s which arise from chains ending in F by F^*, and call it the *dual cell* of F. The collection of all the dual cells defines a regular cell structure since link of each vertex is a sphere. We denote by $(S^l, \mathcal{F}^*(\mathcal{A}))$ this dual cell structure. Here, $\mathcal{F}^*(\mathcal{A})$ is the face poset of this cell structure with the partial order $\preceq$. Note that $\mathcal{F}^*(\mathcal{A})$ is dual poset of $\mathcal{F}(A)$, i.e., $G^* \preceq F^*$ if and only if $F \leq G$. Every k-face in $(S^l, \mathcal{F}(\mathcal{A}))$ corresponds to an $(l - k)$-cell in $(S^l, \mathcal{F}^*(\mathcal{A}))$ for $0 \leq k \leq l$.

For the sake of notational simplicity we will denote the dual cell complex by $\mathcal{F}^*(\mathcal{A})$ (and by $\mathcal{F}^*$ if the context is clear). Note that a 0-cell C^* is a vertex of a k-cell F^* in $\mathcal{F}^*$ if and only if the closure $\overline{C}$ of the corresponding chamber contains the $(l - k)$-face F. The action of the faces on chambers that was introduced in Definition 3.15 is also

valid for the dual cells. The symbol $F^* \circ C^*$ will denote the vertex of F^* which is dual to the unique chamber closest to C.

Given a sphere arrangement $\mathcal{A}$ in S^l construct a regular l-complex $\mathrm{Sal}(\mathcal{A})$ as follows:

The 0-cells of $\mathrm{Sal}(\mathcal{A})$ correspond to 0-cells of $\mathcal{F}^*$, which we denote by the pairs $\langle C^*; C^* \rangle$.

For each 1-cell $F^* \in \mathcal{F}^*$ with vertices C_1^* and C_2^*, take two homeomorphic copies of F^* denoted by $\langle F^*; C_1^* \rangle$ and $\langle F^*; C_2^* \rangle$. Attach these two 1-cells in $\mathrm{Sal}(\mathcal{A})_0$ (the 0-skeleton) such that

$$\partial \langle F^*; C_i^* \rangle = \{\langle C_1^*; C_1^* \rangle, \langle C_2^*; C_2^* \rangle\},$$

for $i = 1, 2$. We put an orientation on the 1-skeleton $\mathrm{Sal}(\mathcal{A})_1$ by directing each 1-cell $\langle F^*; C^* \rangle$ such that the initial vertex is $\langle C^*; C^* \rangle$.

By induction, assume that we have constructed the $(k-1)$-skeleton of $\mathrm{Sal}(\mathcal{A})$, $1 \leq k - 1 < l$. To each k-cell $G^* \in \mathcal{F}^*$ and to each of its vertices C^* assign a k-cell $\langle G^*; C^* \rangle$ whose face poset is isomorphic to that of G^*. Let $\phi(G^*, C^*): \partial \langle G^*; C^* \rangle \to \mathrm{Sal}(\mathcal{A})_{k-1}$ be the same characteristic map that identifies a $(k-1)$-cell $H^* \subseteq \partial G^*$ with the k-cell $\langle H^*; H^* \circ C^* \rangle \subseteq \partial \langle G^*; C^* \rangle$. Extend the map $\phi(G^*, C^*)$ to the whole of $\langle G^*; C^* \rangle$ and use it as the attaching map, hence obtaining the k-skeleton. The boundary of every k-cell is given by

$$(3.1) \qquad \partial \langle F^*; C^* \rangle = \bigcup_{G^* \prec F^*} \langle G^*; G^* \circ C^* \rangle .$$

Now we state the theorem which justifies the construction of this cell complex.

Theorem 3.16. *The regular* CW*-complex* $\mathrm{Sal}(\mathcal{A})$ *constructed above has the homotopy type of the tangent bundle complement* $M(\mathcal{A})$.

Proof. This is a special case of [**9**, Theorem 3.3.7]. We only sketch the proof here. The first step is to identify an open cover of the sphere indexed by the faces. There are three key properties which these open sets satisfy. First, for every face F, the corresponding open set V_F is a regular neighborhood of F. Second, for another face F', the intersection $V_F \cap F' \neq \emptyset$ if and only if $F \leq F'$. Finally, all of these open sets and their non-empty intersections are contractible. Now, for any point on

the manifold, the tangent space at that point contains a arrangement of hyperplanes combinatorially equivalent to the local arrangement. Using the local trivialization, one can construct an open covering of $M(\mathcal{A})$ which is indexed by the pairs

$$\{(F, C) \in \mathcal{F}(\mathcal{A}) \times \mathcal{C}(\mathcal{A}) \mid F \leq C\}$$

such that each of the open sets is contractible and so are their intersections. This type of open covering satisfies the hypothesis of the nerve lemma. The final step is to establish the condition where two such open sets have a non-empty intersection, thus providing an isomorphism between the nerve of this open cover and the face poset of the Salvetti complex constructed above. $\qquad \square$

Example 3.17. As an example, consider the arrangement of two points in a circle (Example 3.7). The left side of Figure 4 illustrates the arrangement with the induced dual cell structure drawn using dotted lines. The right hand side shows the associated Salvetti complex with cell labeling.
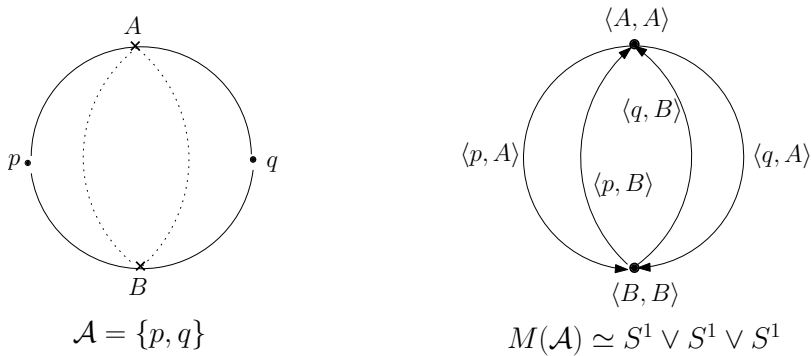

FIGURE 4. Arrangement in S^1 and the associated Salvetti complex.

We now look at some obvious properties of the above-defined CW structure and also infer some more information about the tangent bundle complement.

Theorem 3.18. *Let $\mathcal{A}$ be a sphere arrangement in S^l, and let $\mathrm{Sal}(\mathcal{A})$ denote the associated Salvetti complex. Then:*

(i) *there is a natural cellular map $\psi \colon \mathrm{Sal}(\mathcal{A}) \to \mathcal{F}^*(\mathcal{A})$ given by $\langle F^*, C^* \rangle \mapsto F^*$. The restriction of ψ to the 0-skeleton is a bijection and, in general,*

$$\psi^{-1}(F^*) = \{ \langle F^*, C^* \rangle, C \in \mathcal{C}(\mathcal{A}) \mid C^* \preceq F^* \}.$$

(ii) *For every chamber C, there is a cellular map $\iota_C \colon \mathcal{F}^*(\mathcal{A}) \to \mathrm{Sal}(\mathcal{A})$ taking F^* to $\langle F^*, F^* \circ C^* \rangle$ which is an embedding of $\mathcal{F}^*(\mathcal{A})$ into $\mathrm{Sal}(\mathcal{A})$, and*

$$\mathrm{Sal}(\mathcal{A}) = \bigcup_{C \in \mathcal{C}(\mathcal{A})} \iota_C(\mathcal{F}^*).$$

(iii) *The absolute value of the Euler characteristic of $M(\mathcal{A})$ is the number of chambers.*

(iv) *Let $T\mathcal{A}$ denote the union of the tangent bundles of the submanifolds in $\mathcal{A}$. Then,*

$$\mathrm{rank}\, \widetilde{H}^i(TS^l, T\mathcal{A}) = \begin{cases} |\chi(M(\mathcal{A}))| & \text{if } i = l \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proofs of (i) and (ii) are fairly straightforward. It follows that S^l is homeomorphic to a retract of $M(\mathcal{A})$.

We prove (3) by explicitly counting cells in the Salvetti complex. The Euler characteristic of a CW-complex is equal to the alternating sum of the number of cells of each dimension. Given a k-dimensional dual cell F^* there are as many as $|\{C \in \mathcal{C}(\mathcal{A}) \mid F \leq C\}|$ k-dimensional cells in $\mathrm{Sal}(\mathcal{A})$. Hence, for a 0-cell $\langle C^*, C^* \rangle \in \mathrm{Sal}(\mathcal{A})$, the number of k-cells of $\mathrm{Sal}(\mathcal{A})$ with this particular vertex is equal to the number of k-cells of $\mathcal{F}^*(\mathcal{A})$ that contain C^*. The alternating sum of the number of cells that contain a particular vertex C^* of $\mathcal{F}^*(\mathcal{A})$ is equal to $1 - \chi(\mathrm{Lk}\,(C^*))$, where $\mathrm{Lk}\,(C^*)$ is the link of C^* in $\mathcal{F}^*$. Applying this, we obtain,

$$\chi(\mathrm{Sal}(\mathcal{A})) = \sum_{C \in \mathcal{C}(\mathcal{A})} (1 - \chi(\mathrm{Lk}\,(C^*))).$$

Since S^l is compact, all the chambers are bounded, we have $\mathrm{Lk}\,(C^*) \simeq S^{l-1}$. Thus,

$$\chi(\mathrm{Sal}(\mathcal{A})) = \sum_{C \in \mathcal{C}(\mathcal{A})} (1 - \chi(\mathrm{Lk}\,(C^*)))$$

$$= \sum_{C \in \mathcal{C}(\mathcal{A})} (1 - [1 + (-1)^{l-1}])$$

$$= (-1)^l \sum_{C \in \mathcal{C}(\mathcal{A})} 1.$$

Hence,

$$\chi(M(\mathcal{A})) = (-1)^l \quad \text{(number of chambers)}.$$

Let $\bigcup \mathcal{A}$ denote the union of hyperspheres in $\mathcal{A}$. Since $\mathcal{A}$ induces a regular cell decomposition, $\bigcup \mathcal{A}$ has the homotopy type of wedge of $(l-1)$-spheres. Claim (iv) follows from the homeomorphism of pairs $(TS^l, T\mathcal{A}) \cong (S^l, \bigcup \mathcal{A})$. □

Corollary 3.19. *Let $\mathcal{A}$ be a spherical arrangement in $S^l, l \geq 2$. Then $M(\mathcal{A})$ cannot be an aspherical space.*

Remark 3.20. Let Y be a positive-dimensional intersection of $\mathcal{A}$. The restriction of $\mathcal{A}$ to Y, i.e., the collection

$$\mathcal{A}_Y := \{S \cap Y \mid S \in \mathcal{A}, Y \nsubseteq S\}$$

defines a sphere arrangement in Y. The reader can check using the inclusion ι_C (its restriction to $\mathcal{F}(\mathcal{A}_Y)$) from Theorem 3.18 that Y is homeomorphic to a retract of $\mathrm{Sal}(\mathcal{A})$.

4. Topology of the complement. The aim of this section is to investigate how the combinatorics of the associated posets affect the topology of the tangent bundle complement. Our investigation is based on a simple observation; if, for a given centrally symmetric spherical arrangement, there is an equator generically intersecting the subspheres, then the restriction of the arrangement to the two hemispheres (i.e., components of the complement of the equator) gives combinatorially identical pseudo-hyperplane arrangements. We claim that these restricted pseudo-arrangements play a central role in understanding the topology of the complement. We identify a class of sphere arrangements

for which it is easy to derive a closed form formula for the homotopy type of the complement. Then we establish a connection between the intersection poset and the cohomology groups.

4.1. The homotopy type of the complement. First we look at the arrangements in S^1. An arrangement in S^1 consists of n copies of S^0, i.e., $2n$ points. The tangent bundle complement of such an arrangement is homeomorphic to the infinite cylinder with $2n$ punctures. Thus, we have the following theorem.

Theorem 4.1. *Let $\mathcal{A}$ be an arrangement of 0-spheres in S^1. If $|\mathcal{A}| = n$, then*

$$M(\mathcal{A}) \simeq \bigvee_{2n+1} S^1.$$

From now on, we assume that all our spheres are simply connected. We say that two arrangements are *combinatorially isomorphic* if their corresponding face posets and intersection posets are isomorphic.

Let $\mathcal{A}$ be an arrangement of pseudo-circles in S^2. Then, as a consequence of *Levi's enlargement lemma* [**1**, Proposition 6.4.3], there exists a pseudo-circle $S_0 \notin \mathcal{A}$ such that it is the equator with respect to the given antipodal map, and it meets every member of $\mathcal{A}$ in exactly one point. Let S_0^+ and S_0^- denote the connected components of $S^2 \setminus S_0$. As the equator S_0 intersects every $S \in \mathcal{A}$ generically, $S \cap S_0^+$ is a pseudo-line in $S_0^+ \cong \mathbb{R}^2$, respectively, for S_0^-. Denote by

$$\mathcal{A}^+ := \mathcal{A} \mid S_0^+, \qquad \mathcal{A}^- := \mathcal{A} \mid S_0^-$$

the pseudo-line arrangements in their respective hemispheres. Then $\mathcal{A}^+$ and $\mathcal{A}^-$ are combinatorially isomorphic arrangements of pseudo-lines.

For the rest of this section we assume that $\mathcal{A}$ is centrally symmetric, and there exists a hypersphere S_0 in a general position such that the restriction of $\mathcal{A}$ to both the hemispheres (obtained by deleting S_0) results in combinatorially isomorphic, locally-flat pseudo-hyperplane arrangements (all the intersections are locally flat). This assumption motivates the following definition.

Definition 4.2. A sphere arrangement $\mathcal{A}$ in S^l is said to be *mirrored* if it is centrally symmetric and there exists a pseudo-sphere $S_0 \notin \mathcal{A}$

such that restriction of $\mathcal{A}$ to one of the connected components of $S^l \setminus S_0$ results in a pseudo-hyperplane arrangement combinatorially isomorphic to the restriction of $\mathcal{A}$ to the other connected component.

Note that all of the sphere arrangements in dimensions 1 and 2 are mirrored. Not all sphere arrangements in higher dimension are mirrored since the enlargement lemma fails in general. We refer the reader to [1, Proposition 10.4.5] for an example of pseudo-sphere arrangements in S^3 such that there is no equator in general position. However, our assumption is not too restrictive; for example, arrangements corresponding to realizable oriented matroids are mirrored. If $\mathcal{A}$ is a mirrored sphere arrangement, then we denote by $\mathcal{A}^+$ the restriction of $\mathcal{A}$ to one of the hemispheres and by $\mathcal{A}^-$ the restriction to the other. We note here that many properties of hyperplane arrangements that we are interested in are also true for pseudo-hyperplane arrangements. For example, if all the intersections are locally flat, then the construction of the associated Salvetti complex is the same as described in Section 2.1, and it has the homotopy type of the tangent bundle complement [9, Theorem 3.3.7]

Here are two well-known facts that we need.

Lemma 4.3. *If (Y, A) is a* CW *pair such that the inclusion $A \hookrightarrow Y$ is null homotopic, then $Y/A \simeq Y \vee SA$, where SA is the suspension of A.*

Proof. See [13, Chapter 0]. □

Lemma 4.4. *Let $\mathcal{B}$ be an essential and non-central arrangement of pseudo-hyperplanes in $\mathbb{R}^l$ with locally flat intersections. Then the cell complex which is dual to the induced stratification is regular and homeomorphic to a closed ball of dimension l.*

Proof. See [18, Lemma 9] and [19, Proposition 9]. □

Let C be a chamber of $\mathcal{B}$ and, for any other chamber D, let $\mathcal{F}(C, D)$ denote the set of all of those faces F such that $F \subseteq H$ if and only if $H \notin R(C, D)$. Clearly, all the chambers of $\mathcal{B}$ are in $\mathcal{F}(C, D)$, and it is a disconnected set. In fact, if $l + 1 \leq |R(C, D)| \leq |\mathcal{A}|$, then $\mathcal{F}(C, D)$ is just the set of all chambers. Moreover, note that, if $F \in \mathcal{F}(C, D)$ and G are some face such that $F \subseteq \overline{G}$, then $G \in \mathcal{F}(C, D)$. Let Q

denote the regular cell complex dual to the stratification induced by $\mathcal{B}$, and let $\mathcal{F}(C, D)^*$ denote the dual of $\mathcal{F}(C, D)$. It is straightforward to verify that each connected component of $\mathcal{F}(C, D)^*$ is contractible and deformation retracts onto a subset of ∂Q.

For a chamber C of $\mathcal{B}$, let ι_C denote the inclusion that takes a dual cell F^* to the cell $\langle F^*, F^* \circ C^* \rangle$ of the associated Salvetti complex. This inclusion is the same as the one introduced in Theorem 3.18. In fact, most of Theorem 3.18 is true for pseudo-hyperplane arrangements (with appropriate modification in claim (iii)). For two distinct chambers C and C' of $\mathcal{B}$, we define the following subset of the associated Salvetti complex:

$$I(C, C') := \operatorname{Im}(\iota_C) \cap \operatorname{Im}(\iota_{C'}).$$

Lemma 4.5. *The subset $I(C, C')$ is non-empty and disconnected.*

Proof. First, observe that a cell $\langle G^*, D^* \rangle \in I(C, C')$ if and only if $D^* = G^* \circ C^* = G^* \circ (C')^*$, which means that there is no hyperplane H which contains G and separates C and C'. None of the cells obtained from a hyperplane $H \in R(C, C')$ can belong to $I(C, C')$, and hence, since all the vertices of the Salvetti complex are in $I(C, C')$, the set is disconnected. In fact, $\mathcal{F}(C, C')^* \cong I(C, C')$. $\qquad\square$

Theorem 4.6. *Let $\mathcal{A}$ be a mirrored sphere arrangement in S^l. Let $\mathcal{A}^+$ and $\mathcal{A}^-$ be the pseudo-hyperplane arrangements in the two hemispheres and $\mathcal{C}(\mathcal{A}^+)$ the set of chambers of $\mathcal{A}^+$. Then the tangent bundle complement:*

$$M(\mathcal{A}) \simeq \operatorname{Sal}(\mathcal{A}^-) \vee \bigvee_{|\mathcal{C}(\mathcal{A}^+)|} S^l.$$

Proof. The assumption that the arrangement is mirrored implies that $\mathcal{A}^+$ and $\mathcal{A}^-$ are combinatorially isomorphic, non-central, essential pseudo-hyperplane arrangements with locally flat intersections. Let $C \in \mathcal{C}(\mathcal{A}^+)$, and let Q denote the dual cell complex $(S_0^+, \mathcal{F}^*(\mathcal{A}^+))$. Define the map $\iota_C^+ \colon Q \hookrightarrow \operatorname{Sal}(\mathcal{A})$ as:

$$F^* \longmapsto \langle F^*, F^* \circ C^* \rangle.$$

Claim 1. The image of the map ι_C^+ in $\operatorname{Sal}(\mathcal{A})$ is homeomorphic to Q.

Observe that ι_C^+ is the restriction of the map ι_C defined in Theorem 3.18, which is an embedding. Hence, ι_C^+ maps Q homeomorphically onto its image.

Thus, ι_C^+ is the characteristic map which attaches the boundary ∂Q to the $(l-1)$-skeleton of $\mathrm{Sal}(\mathcal{A}^-)$. For notational simplicity, let j_C denote the restriction of ι_C^+ to ∂Q.

Claim 2. The image of j_C is the boundary of an l-cell in $\mathrm{Sal}(\mathcal{A}^-)$.

Consider the subcomplex of $\mathrm{Sal}(\mathcal{A}^-)$ given by the cells $\{\langle F^*, F^* \circ C^* \rangle \mid F \in \mathcal{F}^*(\mathcal{A}^-)\}$. By Lemma 4.3, this subcomplex is homeomorphic to the closed l-ball. The boundary of this closed ball is precisely the image of j_C.

The characteristic map ι_C^+ is the extension of j_C to the cone over ∂Q (which is Q). Hence, j_C is null homotopic. Let C' be any other chamber. In view of Lemma 4.3 the intersection set $I(C, C') \subseteq \mathrm{Sal}(\mathcal{A}^+)$ retracts onto the boundary $\mathrm{Im}\,(j_C) \cup \mathrm{Im}\,(j_{C'})$. Then it follows from Lemma 4.3 that $\mathrm{Sal}(\mathcal{A}^-) \cup \mathrm{Im}\,\iota_C^+(Q)$ has the homotopy type of $\mathrm{Sal}(\mathcal{A}^-) \vee S^l$. Repeating these arguments for every chamber of $\mathcal{A}^+$ establishes the theorem. $\qquad\square$

We state the following obvious corollary for the sake of completeness.

Corollary 4.7. *Let $\mathcal{A}$ be a mirrored sphere arrangement in S^l. With notation as before, we have*:

$$\pi_1(M(\mathcal{A})) \cong \pi_1(M(\mathcal{A}^-)).$$

Example 4.8. Consider the arrangement of two circles in S^2 introduced in Example 3.8. It is clear that the arrangement $\mathcal{A}^-$ in this case is the arrangement of two lines in $\mathbb{R}^2$ that intersect in a single point. Hence,

$$M(\mathcal{A}) \simeq T^2 \vee S^2 \vee S^2 \vee S^2 \vee S^2.$$

The Salvetti complex consists of four 0-cells, eight 1-cells and eight 2-cells. The 2-torus T^2 in the above formula corresponds to $M(\mathcal{A}^-)$.

Example 4.9. Finally, consider the arrangement of three S^2s in S^3 that intersect like coordinate hyperplanes in $\mathbb{R}^3$. The $\mathcal{A}^-$ in this case is the arrangement of coordinate hyperplanes hence $\mathrm{Sal}(\mathcal{A}^-) \simeq T^3$,

the 3-torus. This arrangement has eight chambers. So we have the following:

$$M(\mathcal{A}) \simeq T^3 \vee \bigvee_8 S^3.$$

Example 4.10. Consider the arrangement of three circles in S^2 that intersect in the general position. This arrangement arises as the intersection of S^2 with coordinate hyperplanes in $\mathbb{R}^3$. In this case, $\mathcal{A}^-$ is the arrangement of three lines in general position. Figure 5 shows $\mathcal{A}^+$ and the dual cell complex Q.
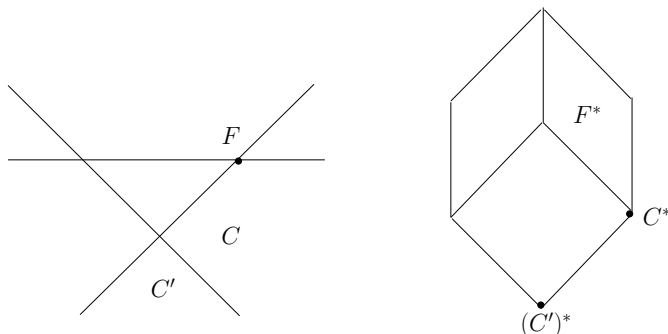

FIGURE 5. Restricted arrangement $\mathcal{A}^+$ and the associated dual complex Q.

The intersection $I(C, C')$ (see Lemma 4.5 contains the 2-cell $\langle F^*, C^* \rangle$ $= \langle F^*, (C')^* \rangle$. The boundaries of $\iota_C^+(Q)$ and $\iota_{C'}^+(Q)$ collapse to a point; hence, $\mathrm{Sal}(\mathcal{A}^-) \cup \iota_C^+(Q) \cup \iota_{C'}^+(Q)$ has the homotopy type of $\mathrm{Sal}(\mathcal{A}^-) \vee S^2 \vee S^2$. In general, the tangent bundle complement has the homotopy type of $\mathrm{Sal}(\mathcal{A}^-) \vee \bigvee_7 S^2$.

4.2. Cohomology of the complement. We now establish a relationship between the cohomology of the tangent bundle complement and the intersection poset. Let $\mathcal{A}$ be a mirrored sphere arrangement in S^l and let $\mathcal{A}^+$ be the pseudo-hyperplane arrangement in the positive hemisphere. Let L and L^+ denote the corresponding intersection posets. Observe that the map from L to L^+ that sends $Y \in L$ to

$Y \mid S_0^+ =: Y^+$ is one-to-one up to rank $l - 1$. If L_{l-1} and L_{l-1}^+ denote the sub-posets consisting of elements of rank less than or equal to $l - 1$, then the previous map is a poset isomorphism. For notational simplicity, we use M^- for $M(\mathcal{A}^-)$.

Theorem 4.11. *With notation as above, we have the following*

$$\operatorname{rank} H^i(M, \mathbb{Z}) = \begin{cases} \displaystyle\sum_{\substack{Y \in L \\ \operatorname{rank}(Y)=i}} |\mu(S^l, Y)| & \text{for } 0 \leq i < l, \\ \displaystyle\sum_{Y \in L} |\mu(S^l, Y)| & \text{for } i = l, \end{cases}$$

where μ is the Möbius function of the intersection poset.

Proof. We use Theorems 2.8 and 4.6 in order to prove the assertion by considering two cases.

Case 1. $1 \leq i < l$.

$$\begin{aligned}
\operatorname{rank}(H^i(M)) &= \operatorname{rank}(H^i(M^-)) + \sum_{|\mathcal{C}(\mathcal{A}^+)|} \operatorname{rank}(H^i(S^l)) \\
&= \operatorname{rank}(H^i(M^-)) + 0 \\
&= \sum_{\operatorname{rank}(Y^-)=i} (-1)^{\operatorname{rank} Y^-} \mu(\mathbb{R}^l, Y^-) \\
&= \sum_{\operatorname{rank}(Y)=i} |\mu(S^l, Y)|.
\end{aligned}$$

The last equality follows from the fact that each Y is a sphere of dimension $l - i$.

Case 2. $i = l$.

$$\begin{aligned}
\operatorname{rank}(H^l(M)) &= \operatorname{rank}(H^l(M^-)) + \sum_{|\mathcal{C}(\mathcal{A}^+)|} \operatorname{rank}(H^l(S^l)) \\
&= \sum_{\operatorname{rank}(Y^-)=l} |\mu(Y^-)| + |\mathcal{C}(\mathcal{A}^+)| \\
&= \sum_{\operatorname{rank}(Y^-)=l} |\mu(Y^-)| + \sum_{Y^+ \in L^+} |\mu(Y^+)| = \sum_{Y \in L} |\mu(Y)|.
\end{aligned}$$

The third equality follows from the expression for the number of chambers of a hyperplane arrangement. The last equality is true because the number of rank l elements in L is twice the corresponding number in L^-. □

Remark 4.12. One may call the cohomology algebra $H^*(M(\mathcal{A}), \mathbb{Z})$ the *spherical Orlik-Solomon algebra*. As stated in the introduction, for mirrored arrangements, the spherical OS-algebra is the direct sum of a graded and ungraded OS-algebras. The ungraded OS-algebra sits in the top dimension. It should be interesting to figure out the structure of the spherical OS-algebra in the general case. Remark 3.20 implies that $H^*(M(\mathcal{A}), \mathbb{Z})$ is a finitely generated $H^*(S^i, \mathbb{Z})$ module for $1 \le i \le l$.

5. The fundamental group. We turn our attention to the fundamental group of $M(\mathcal{A})$. The aim of this section is to identify a class of sphere arrangements for which the word problem for $\pi_1(M(\mathcal{A}))$ is solvable. We do this by carefully analyzing the so-called arrangement groupoid (in fact, the fundamental groupoid) and the path category of the associated Salvetti complex. This type of analysis goes back to the seminal work of Deligne [**8**].

In [**19**], Salvetti introduced a certain class of cell complexes which were called *metrical-hemisphere complexes* (MH-complexes for short). The combinatorial properties of the face poset of these MH-complexes resemble those of zonotopes. He proved that these properties enable one to construct a Salvetti complex similar to the description at the beginning of the previous section (see [**19**, Section 2]). We should note here that MH-complexes are quite general and need not correspond to any type of arrangement. However, Salvetti focused on a certain class of MH-complexes, which he called MH*-complexes. His main results are the following: Given an MH*-complex, if the path category of the associated Salvetti complex admits a calculus of fractions, then the fundamental group has a solvable word problem [**19**, Theorem 27]. If, in addition, the dual of the given MH*-complex is a simplicial subdivision of a closed manifold, then, assuming an additional technical condition, the universal cover of the Salvetti complex is contractible [**19**, Theorem 33].

We make use of these ideas in the current context. Instead of proving that the dual cell decomposition induced by a sphere arrangement is an

MH*-complex, we explicitly provide calculations that give a solution to the word problem. As a result, most of the arguments in this section are straightforward generalizations of the results in [**19**]. We have reproduced the proofs for the benefit of the reader.

An oriented 1-skeleton of a regular CW-complex can be thought of as a directed graph without loops. The 0-cells are the vertices, and 1-cells are the edges. A *path* in such a cell complex is a sequence of consecutive edges, and its *length* is the number of edges. A *minimal path* is a path of the shortest length among all the paths with the same end points. Given a path, we also define its *initial vertex* and *terminal vertex* in the obvious way. Finally, by a *positive path*, we mean a path all of whose edges are traversed according to their orientation. We use the notation introduced in subsection 3.6.

Lemma 5.1. *Let $\mathcal{A}$ be an arrangement of spheres in S^l, $l \geq 2$. Then any two minimal positive paths in the 1-skeleton of $\mathrm{Sal}(\mathcal{A})$ that have the same initial as well as terminal vertices are homotopic relative to $\{0, 1\}$.*

Proof. Given two positive minimal paths α and β in $\mathrm{Sal}(\mathcal{A})$ with the initial vertex $\langle C^*, C^* \rangle$ and the terminal vertex $\langle D^*, D^* \rangle$, apply the retraction map ψ defined in Theorem 3.18. Hence, we get two paths in $\mathcal{F}^*$ and no two edges of α, β are sent to the same edge in $\mathcal{F}^*$. The conclusion follows from the observation that these image paths are contained in $\iota_C(\mathcal{F}^*)$, which is simply connected. $\square$

Given an arrangement $\mathcal{A}$, we denote by $\mathcal{G}^+(\mathcal{A})$ the associated *positive category*, i.e., the category of directed paths in the Salvetti complex $\mathrm{Sal}(\mathcal{A})$. The objects of this category are the vertices of the Salvetti complex, and morphisms are directed homotopy classes of positive paths (two such paths are connected by a sequence of substitutions of minimal positive paths). For a path α, its equivalence class in $\mathcal{G}^+$ is denoted by $[\alpha]_+$.

We denote by $\mathcal{G}(\mathcal{A})$ the arrangement groupoid of $\mathcal{A}$. It is, in fact, the fundamental groupoid of the associated Salvetti complex. For a path α, its equivalence class in $\mathcal{G}(\mathcal{A})$ is denoted by $[\alpha]$. Since $\mathcal{G}(\mathcal{A})$ is the category of fractions of $\mathcal{G}(\mathcal{A})^+$, we denote by $J\colon \mathcal{G}(\mathcal{A})^+ \to \mathcal{G}(\mathcal{A})$ the associated canonical functor. We refer the reader to [**11**] for relevant

terminology from the calculus of fractions. The reference to $\mathcal{A}$ from $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}^+(\mathcal{A})$ is dropped when the context is clear.

For notational simplicity, we write F for F^*, i.e., we do not differentiate between a face in $\mathcal{F}(\mathcal{A})$ and its dual cell in $\mathcal{F}^*(\mathcal{A})$. Given a path α, we denote by $(\pm a_1, \ldots, \pm a_n)$ the sequence of 1-cells in $\mathrm{Sal}(\mathcal{A})$ that are traversed by α either according to or opposite to their orientation depending on the sign.

Theorem 5.2. *Let $\mathcal{A}$ be a centrally symmetric arrangement of spheres in S^l, $l \geq 2$. Then, the associated canonical functor $J\colon \mathcal{G}^+ \to \mathcal{G}$ is faithful on the class of minimal positive paths.*

Proof. We already know that, if α and β are two minimal positive paths with the same end points, then $[\alpha] = [\beta]$ in $\mathcal{G}$. Hence, it is enough to show that $[\alpha]_+ = [\beta]_+$. We argue along the lines of the proof of [**19,** Theorem 20]. Since each $S \in \mathcal{A}$ is centrally symmetric around the origin, the antipodal map induces a fixed point free cellular action on the faces of $\mathcal{A}$.

Suppose $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$ are two minimal positive paths that start at C and end at D. We proceed by induction on n, cases $n = 0, 1$ being trivial. Assume that the statement is true for all minimal positive paths with the same end points and of length strictly less than n. If $a_1 = b_1$, then we are done by induction.

Hence, assume that $a := a_1$ and $b := b_1$ are distinct and are dual to the hyperspheres S_a and S_b, respectively. We have that $S_a, S_b \in R(C, D)$ (the set of hyperspheres separating C and D) and that $S_a \cap S_b \cong S^{l-2}$. Recall that $S \in R(C, D)$ if and only if C is contained in one of the connected components of $S^l \setminus S$ and D is contained in the other. For $S \notin R(C, D)$, let $X_S(C, D)$ denote the closure of the connected component of $S^l \setminus S$ that contains both C and D, and let

$$\mathcal{H}(C, D) = \bigcap_{S \notin R(C,D)} X_S(C, D).$$

Claim 1. The set $\mathcal{H}(C, D)$ is either connected or empty. There are two cases: either $R(C, D) = \mathcal{A}$ or $R(C, D) \subsetneq \mathcal{A}$. In the first case, it is clear that $\mathcal{H}(C, D)$ is empty since each $X_S(C, D)$ is empty. As for the

second case $\mathcal{H}(C, D)$ is an intersection of closed balls which is again a closed ball. See [**1,** Lemma 5.1.9] and [**10,** page 217].

Claim 2. If $\mathcal{H}(C, D) \neq \emptyset$, then $\mathcal{H}(C, D) \cap S_a \cap S_b \neq \emptyset$. Let S_a^+, S_a^- (respectively, S_b^+, S_b^-) denote the (closures of the) connected components of $S_a \setminus (S_a \cap S_b)$ (respectively, $S_b \setminus (S_a \cap S_b)$). Without loss of generality, assume that S_a^+ and S_b^+ intersect $\overline{C}$. This implies

$$\mathcal{H}(C, D) \cap S_a^+ \neq \emptyset \neq \mathcal{H}(C, D) \cap S_b^+.$$

A similar argument using $\overline{D}$ establishes the claim.

Hence, the set $\mathcal{H}(C, D) \cap S_a \cap S_b$ contains a codimension-2 face say F^2. Let C' denote $F^2 * C$. Let γ_0 be a minimal positive path from C' to D. Also, there exist two minimal positive paths γ_1 and γ_2 such that γ_1 starts at $a_1 \circ C$ and γ_2 starts at $b_1 \circ C$ so that both of them end at C'. Using this, we can construct two new minimal positive paths, $\eta = a_1 \gamma_1 \gamma_0$ and $\eta' = b_1 \gamma_2 \gamma_0$. The paths α and η are minimal positive with the same end points and share the same first edge. Hence, by induction, $[\alpha]_+ = [\eta]_+$ if $\gamma \neq 0$. For the same reasons, $[\beta]_+ = [\eta']_+$. If $C' \neq D$, then the path γ_0 is of nonzero length and, again by induction, $[a_1 \gamma_1]_+ = [b_1 \gamma_2]_+$ implying $[\eta]_+ = [\eta']_+$. Now the transitivity of the equivalence relation proves the theorem when $\mathcal{H}(C, D) \neq \emptyset$.

The cases in which either $C' = D$ or $\mathcal{H}(C, D) = \emptyset$ can be treated similarly. $\square$

For a centrally symmetric arrangement $\mathcal{A}$, we indicate by $[\mu(C \to D)]$ the unique equivalence class (in $\mathcal{G}^+(\mathcal{A})$ or $\mathcal{G}(\mathcal{A})$) determined by a minimal positive path from a chamber C to another chamber D.

Definition 5.3. A sphere arrangement $\mathcal{A}$ in S^l is said to have the *involution property* if there exists a graph automorphism $\phi \colon \mathcal{F}_1^* \to \mathcal{F}_1^*$ of the dual 1-skeleton (considered as a graph) satisfying:

  (i) ϕ is an involution (which induces involution on the vertices as well as the edges);
 (ii) for every vertex C, $d(C, \phi(C)) = \max\{d(C, D) \mid D \in \mathcal{C}(\mathcal{A})\}$;
 (iii) $d(C, \phi(C)) = d(C, D) + d(D, \phi(C))$ for every vertex C and D.

Lemma 5.4. *Let $\mathcal{A}$ be a centrally symmetric sphere arrangement in S^l. Then $\mathcal{A}$ has the involution property.*

Proof. The antipodal action on S^l provides the required graph automorphism on the 1-skeleton of the associated Salvetti complex. $\square$

The image of either a vertex or an edge under ϕ will be denoted by the superscript #, for example, $C^\# := \phi(C)$.

Lemma 5.5. *If $\mathcal{A}$ is a sphere arrangement with the involution property then*

(i) $d(C, C^\#) = |\mathcal{A}|$ *for all* $C \in \mathcal{C}(\mathcal{A})$;
(ii) $d(C, D) = d(C^\#, D^\#)$ *for all* C, D.

Proof. Using Definition 5.3 (iii), we have:

$$d(C, C^\#) = d(C, D) + d(D, C^\#), \tag{5.1}$$

$$d(C, C^\#) = d(C, D^\#) + d(D^\#, C^\#). \tag{5.2}$$

Adding equations (5.1) and (5.2), we obtain

$$2d(C, C^\#) = 2d(D, D^\#).$$

Without loss of generality, assume that $d(C, C^\#) = |\mathcal{A}| - 1$. Hence, there is a hypersphere $S \in \mathcal{A}$ such that C and $C^\#$ are on the same side with respect to S. Choose a chamber D such that $S \in R(C, D)$ and is adjacent to $C^\#$. Hence, $d(C, D) > d(C, C^\#)$, which contradicts equation (5.1). Consequently, no such S exists. We call the number $d(C, C^\#) = |\mathcal{A}|$, the *diameter of* S^l (with respect to $\mathcal{A}$).

Now, subtracting $d(D, D^\#) = d(D, C^\#) + d(C^\#, D^\#)$ from equation (5.1), we get

$$0 = d(C, D) - d(C^\#, D^\#),$$

which proves (ii). $\square$

This involution also preserves the positive equivalence on paths as proved in the next lemma.

Lemma 5.6. *If $\mathcal{A}$ is a sphere arrangement with the involution property, then the involution ϕ induces a functor on $\mathcal{G}^+$ which is also an involution.*

Proof. There is a bijection between the set of edge-paths of $\mathcal{F}_1^*$ and the set of all positive paths in $\mathrm{Sal}(\mathcal{A})_1$ given by $\langle F, C \rangle \mapsto F$. Extend the given involution to $\mathrm{Sal}(\mathcal{A})_1$ by sending $\langle F, C \rangle$ to $\langle F^\#, C^\# \rangle$. Under this involution, a positive path $\alpha = (a_1, \ldots, a_n)$ goes to a positive path

$$\alpha^\# := (a_1^\#, \ldots, a_n^\#).$$

If γ_1 and γ_2 are two minimal positive paths contained entirely in the boundary of a 2-cell of $\mathrm{Sal}(\mathcal{A})$, then so are $\gamma_1^\#$ and $\gamma_2^\#$. Therefore, $[\gamma_1]_+ = [\gamma_2]_+ \Rightarrow [\gamma_1^\#]_+ = [\gamma_2^\#]_+$. $\qquad\square$

We define a *positive* loop based at $\langle C, C \rangle$ as:

$$\delta(C) := \mu(C \to C^\#)\mu(C^\# \to C).$$

Note that the equivalence class of $\delta(C)$ in both $\mathcal{G}^+$ and $\mathcal{G}$ is unique. By $\delta^k(C)$, we mean that the positive loop is traversed k times according to the orientation of the edges if $k > 0$ and in the reverse direction if $k < 0$. We will say that a positive path α begins (or ends) with a positive path α' if and only if $\alpha = \alpha'\beta(= \beta\alpha')$ for some positive path β.

Lemma 5.7. *Let $\mathcal{A}$ be a sphere arrangement with the involution property and α a positive path from C to D. Then:*

  (i) *$[\alpha][\mu(D \to D^\#)] = [\mu(C \to C^\#)][\alpha^\#]$;*
  (ii) *if, for a chamber D', β is some positive path from C to D', then $\alpha\delta^n(D)$ begins with β;*
  (iii) *if $[\gamma] \in \mathcal{G}(C, D)$, then there exists an $n \in \mathbb{N}$ and a positive path γ' such that*

$$[\gamma] = [\delta^{-n}(C)][\gamma'].$$

Proof. For (i), we use induction on the length of α. In fact, it is enough to assume that $\alpha = \mu(C \to C_1)$ such that $d(C, C_1) = 1$. Thus:

$$\begin{aligned}
\alpha\mu(C_1 \to C_1^\#) &= \mu(C \to C_1)\mu(C_1 \to C_1^\#) \\
&\overset{+}{\sim} \mu(C \to C_1)\mu(C_1 \to C^\#)\mu(C^\# \to C_1^\#) \\
&\overset{+}{\sim} \mu(C \to C^\#)\mu(C^\# \to C_1^\#) \\
&\overset{+}{\sim} \mu(C \to C^\#)\alpha^\#.
\end{aligned}$$

By the same arguments, the following stronger statement is true:

$$(5.3) \qquad [\alpha][\delta^k(D)] = [\delta^k(C)][\alpha], \quad k \geq 1.$$

For (ii), let $\beta = (b_1, \ldots, b_n)$, where b_i is an edge from B_{i-1} to B_i ($B_0 = C$, $B_n = D'$). Observe that $\beta\mu(B_n \to B_{n-1}^{\#}) = (b_1, \ldots, b_{n-1})\mu(B_{n-1} \to B_{n-1}^{\#})$. By induction on n, assume that there exists a positive path η from B_{n-1} to D such that:

$$(b_1, \ldots, b_{n-1})\eta = \alpha\delta^{n-1}(D).$$

Using (i), we get

$$\beta\mu(B_n \to B_{n-1}^{\#})\eta^{\#} = (b_1, \ldots, b_{n-1})\eta\delta(D) = \alpha\delta^n(D),$$

which proves (ii).

Let γ be a path from C to D. Assume $\gamma = (\epsilon_1 a_1, \ldots, \epsilon_n a_n)$, where $\epsilon_i \in \{\pm 1\}$ for every i. Let A_i be the terminal vertex of $(\epsilon_1 a_1, \ldots, \epsilon_i a_i)$. Setting

$$k = |\{1 \leq i \leq n \mid \epsilon_i = -1\}|,$$

we prove (iii) by induction on k. The case $k = 0$ is clear since it means that γ is a positive path. Assume that the statement is true for $k - 1$.

For the general case, there exists an index j such that $\epsilon_1 = \cdots = \epsilon_{j-1} = 1$ and $\epsilon_j = -1$. We have

$$\delta(C)\gamma = \mu(C \to C^{\#})\mu(C^{\#} \to C)(a_1, \ldots, a_{j-1}, -a_j, \epsilon_{j+1}a_{j+1}, \ldots, \epsilon_n a_n)$$
$$\overset{+}{\sim} \mu(C \to C^{\#})a_1^{\#}\mu(A_1^{\#} \to A_1)(a_2, \ldots, a_{j-1}, -a_j,$$
$$\epsilon_{j+1}a_{j+1}, \ldots, \epsilon_n a_n)$$
$$\overset{+}{\sim} \mu(C \to C^{\#})a_1^{\#} \cdots a_{j-1}^{\#}\mu(A_{j-1}^{\#} \to A_{j-1})$$
$$(-a_j, \epsilon_{j+1}a_{j+1}, \ldots, \epsilon_n a_n)$$
$$\overset{+}{\sim} \mu(C \to C^{\#})a_1^{\#} \cdots a_{j-1}^{\#}\mu(A_{j-1}^{\#} \to A_j)(\epsilon_{j+1}a_{j+1}, \ldots, \epsilon_n a_n)$$
$$\overset{+}{\sim} \delta^{1-n}(C)\gamma' \text{ (by the induction hypothesis),}$$

where γ' is a positive path. Hence, $[\gamma] = [\delta^{-n}(C)]\gamma'$. $\qquad \square$

Recall that [**20**, Section 0.5.7], the *word problem* for a group G, is the problem of deciding whether or not an arbitrary word w in G is the identity of G. The word problem for G is *solvable* if there exists an algorithm to determine whether $w = 1_G$, or equivalently, if there exists an algorithm to determine when two arbitrary words represent the same element of G.

Theorem 5.8. *Let $\mathcal{A}$ be a sphere arrangement with the involution property. If the canonical functor $J \colon \mathcal{G}^+(\mathcal{A}) \to \mathcal{G}(\mathcal{A})$ is faithful, then the word problem for $\pi_1(M(\mathcal{A}))$ is solvable.*

Proof. Let $[\alpha]$ and $[\beta]$ be two loops in $\pi_1(\mathrm{Sal}(\mathcal{A}))$ based at a vertex $\langle C, C \rangle$. Then, according to Lemma 5.7, there is a finite algorithm to write:
$$[\beta] = [\delta^{-k}(C)][\beta'], \qquad [\alpha] = [\delta^{-k}(C)][\alpha'],$$
where β' and α' are positive loops based at $\langle C, C \rangle$. Hence, $[\alpha] = [\beta]$ if and only if $[\alpha']_+ = [\beta']_+$. The theorem follows because there are only finitely many positive paths of given length from which to choose.   □

Remark 5.9. For simplicial arrangements of hyperplanes it was proven by Deligne that the canonical functor J is injective [**8**, Proposition 1.19]. In particular, it was used to prove that the word and conjugacy problems for Artin groups are solvable [**8**, subsection 4.20]. This was further generalized by Salvetti to include the case of simplicial arrangements of pseudohyperplanes in [**19**, Theorem 31].

6. Arrangements of projective spaces. We now look at arrangements in projective spaces. Without loss of generality, let S^l be the unit sphere in $\mathbb{R}^{l+1}$ and $\mathbb{P}^l$ the projective space with $a \colon S^l \to \mathbb{P}^l$ being the covering map. We consider a finite collection of subspaces that are homeomorphic to $\mathbb{P}^{l-1}$. We define the projective arrangements as follows.

Definition 6.1. A finite collection $\mathcal{A} = \{H_1, \ldots, H_n\}$ of codimension-1 projective spaces is called an *arrangement of projective spaces* (or a *projective arrangement*) if and only if $\widetilde{\mathcal{A}} := \{a^{-1}(H) \mid H \in \mathcal{A}\}$ is a centrally symmetric arrangement of spheres in S^l.

As a result of the projective version of the topological representation theorem, the following is an equivalent definition of projective arrangements. A finite collection of codimension-1 projective spaces (tamely embedded) in $\mathbb{P}^l$ is a projective arrangement if the non-empty intersection of every sub-collection is locally flat and homeomorphic to a lower-dimensional projective space.

The homotopy type of the tangent bundle complement associated to a projective arrangement is easier to understand because of the antipodal action.

Theorem 6.2. *Let $\mathcal{A}$ be a projective arrangement in $\mathbb{P}^l$ and $\widetilde{\mathcal{A}}$ the corresponding centrally symmetric sphere arrangement in S^l. Then the antipodal map on the sphere extends to its tangent bundle, and*

$$M(\mathcal{A}) \cong M(\widetilde{\mathcal{A}})/((x,v) \sim a(x,v)).$$

Proof. If (x,v) is a point in the tangent bundle of S^l, extend the antipodal map in the obvious way, $a(x,v) = (-x,-v)$. We now prove that the space $M(\widetilde{\mathcal{A}})$ is a covering space of $M(\mathcal{A})$. This follows from the fact that $a\colon TS^l \to T\mathbb{P}^l$ is a covering map for every l.

Note that the antipodal map is cellular on the faces of the arrangement. Consequently, it induces a cellular map on $\mathrm{Sal}(\widetilde{\mathcal{A}})$ by sending a cell $\langle F,C \rangle$ to $\langle a(F), a(C) \rangle$, and we get a cell structure for $\mathrm{Sal}(\mathcal{A})$. Hence, $\pi_1(M(\widetilde{\mathcal{A}}))$ is an index 2 subgroup of $\pi_1(M(\mathcal{A}))$. $\square$

Example 6.3. Here is a simple example. Consider the arrangement of a pair of diametrically opposite points $\{p_1, p_2, q_1, q_2\}$ in S^1. The arrangement breaks S^1 into four chambers: A_1, A_2, B_1 and B_2; the chambers A_1 and B_1 are diametrically opposite to A_2 and B_2, respectively. The antipodal action is cellular, and it identifies the 0-cells p_i's to p, q_i's to q and the 1-cells A_i's to A, B_i's to B. Giving us the 'projective' arrangement of two $\mathbb{P}^0$'s in $\mathbb{P}^1$. Figure 6 shows the Salvetti complex of the sphere arrangement on the left and the Salvetti complex of 'projective' arrangement on the right.

Given a projective arrangement $\mathcal{A}$, let $J\colon \mathcal{G}^+ \to \mathcal{G}$ denote the canonical functor between the positive category and the arrangement groupoid. For the corresponding (centrally symmetric) spherical
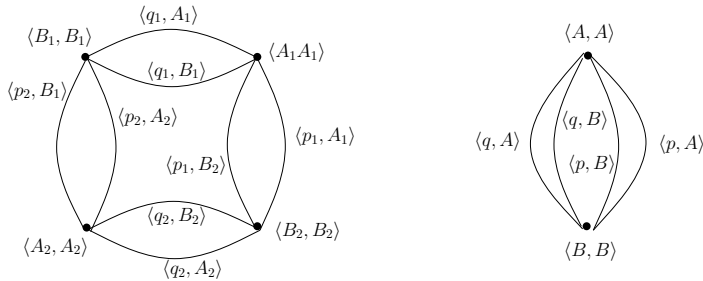
FIGURE 6. Spherical and projective Salvetti complexes.

arrangement $\widetilde{\mathcal{A}}$, let $\widetilde{J}\colon \widetilde{\mathcal{G}}^+ \to \widetilde{\mathcal{G}}$ be the associated canonical functor. Recall that $\widetilde{\mathcal{A}}$ has the involution property (Definition 5.3) and that the antipodal action induces an 'antipodal' functor on $\widetilde{\mathcal{G}}^+$ (Lemma 5.6). Under this functor, an object C (which is a chamber) is mapped to $C^{\#}$ (its antipodal chamber) and a morphism $[\alpha]$ is mapped to $[\alpha^{\#}]$.

Lemma 6.4. *With the notation as above, the following diagram commutes*:

$$
\begin{array}{ccc}
\widetilde{\mathcal{G}}^+ & \xrightarrow{\ \widetilde{J}\ } & \widetilde{\mathcal{G}} \\
{\scriptstyle \Phi^+}\big\downarrow & & \big\downarrow{\scriptstyle \Phi} \\
\mathcal{G}^+ & \xrightarrow[\ J\ ]{} & \mathcal{G},
\end{array}
$$

where Φ^+ identifies antipodal objects and morphisms and Φ is the covering functor for groupoids.

Proof. The proof follows from a simple diagram chase and the fact that S^l is the universal cover of $\mathbb{P}^l$. $\qquad\square$

An immediate consequence of the lemma is the following theorem.

Theorem 6.5. *The restriction of J to the class of minimal positive paths is faithful. Moreover if J is faithful then the word problem for $\pi_1(M(\mathcal{A}))$ is solvable.*

Proof. The first statement follows from the commutativity of the diagram in the previous lemma. If $[\alpha]_+$ is a class of minimally positive paths in $\mathcal{G}^+$, then the class representing either of the α's lift is also minimally positive in $\widetilde{\mathcal{G}}^+$. If there are two distinct classes of minimally positive paths, then first applying $\widetilde{J}$ to their lifts in $\widetilde{\mathcal{G}}^+$ and then applying Φ results in producing two distinct classes of minimally positive paths in $\mathcal{G}$. By the same argument, if $\widetilde{J}$ is faithful, then J is also faithful.

Let $[\alpha]$ be a loop based at a vertex C in $\mathcal{G}$. Let $[\widetilde{\alpha}]$ be the class representing a lift of α which is a loop based at $\widetilde{C}$ (a vertex in the fiber over C). Then, by Lemma 5.7 (iii), we have the following:

$$[\widetilde{\alpha}] = [\delta^{-n}(\widetilde{C})][\widetilde{\alpha}'],$$

where $\delta^{-n}(\widetilde{C}) = \mu(\widetilde{C} \to \widetilde{C}^{\#})\mu(\widetilde{C}^{\#} \to \widetilde{C})$ and $\widetilde{\alpha}'$ is a positive loop based at $\widetilde{C}$. Since Φ is the covering functor, $\Phi([\delta^{-n}(\widetilde{C})]) = [\delta^{-2n}(C)]$; here, $\delta(C)$ is a positive loop based at C which traverses every vertex twice. Let $[\alpha']$ be the image $\Phi([\widetilde{\alpha}'])$ which represents a class of positive loops based at C. Note that choosing another lift of α based at the antipodal point $\widetilde{C}^{\#}$ does not make any difference. Hence, we have proved that any loop in $\mathrm{Sal}(\mathcal{A})$ can be expressed as a composition of a 'special loop' (which traverses each vertex a fixed number of times) and a positive loop. Now the same argument as in the proof of Theorem 5.8 shows that the word problem for $\pi_1(M(\mathcal{A}))$ is solvable. $\square$

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