ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 46, Number 5, 2016

NILPOTENT TOEPLITZ OPERATORS ON REINHARDT DOMAINS

MEHMET ÇELİK AND YUNUS E. ZEYTUNCU

ABSTRACT. We construct explicit examples of non-trivial nilpotent Toeplitz operators on Bergman spaces of certain Reinhardt domains in $\mathbb{C}^2$.

1. Introduction.

1.1. Set-up and results. Let Ω be a domain in $\mathbb{C}^n$ and $A^2(\Omega)$ denote the Bergman space of Ω. The Bergman projection operator $\mathcal{B}_\Omega$ is the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$. It is an integral operator with the kernel called the *Bergman kernel*, denoted by $B_\Omega(z,w)$. If $\{e_n(z)\}_{n=0}^{\infty}$ is an orthonormal basis for $A^2(\Omega)$, then the Bergman kernel can be represented as

$$B_\Omega(z,w)=\sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}.$$

See **[11]** for the general theory of Bergman spaces.

For a function u on Ω, the Toeplitz operator $T_u : A^2(\Omega)\to A^2(\Omega)$ with the symbol u is defined by $T_u(f)=\mathcal{B}_\Omega(uf)$.

In this note, we are interested in the zero product problem. For two symbols u_1 and u_2, if the product $T_{u_1}T_{u_2}$ is identically zero on $A^2(\Omega)$, then can we claim T_{u_1} or T_{u_2} is identically zero? This is a non-trivial problem, and the answer is not even known when Ω is the unit disc.

Here, we indicate the problem has a different flavor in higher dimensions. In particular, we present a family of Reinhardt domains in $\mathbb{C}^2$

2010 AMS *Mathematics subject classification.* Primary 47B35, Secondary 32A36.

Keywords and phrases. Toeplitz operator, Bergman space, nilpotent operator, Reinhardt domain.

Received by the editors on June 25, 2014, and in revised form on November 20, 2014.

DOI:10.1216/RMJ-2016-46-5-1395

on which not only zero products but nilpotent Toeplitz operators of non-trivial Bergman space Toeplitz operators exist.

Theorem 1.1. *There exist Reinhardt domains in $\mathbb{C}^2$ on whose Bergman spaces there are nilpotent Toeplitz operators.*

Remark 1.2. It becomes clear in the proof that the operators in Theorem 1.1 are also of infinite rank.

1.2. History. The zero product problem on the Hardy space is initiated in [**5**]. It is completely solved in [**3**], where authors established that the product of non-zero Toeplitz operators is never zero. For the intermediate results, before the complete solution, see [**9, 10**] and the references in [**3**].

In [**1**], it is shown that, for the Toeplitz operators on the Bergman space $A^2(\mathbb{D})$ of the unit disc $\mathbb{D}$, the analogue of the Brown-Halmos theorem holds under an additional hypothesis that u and v are bounded and harmonic. Later, the same result is proven for radial symbols in [**2**]. The problem on $\mathbb{D}$, without extra assumptions on the symbols, remains open.

The higher-dimensional cases are studied in [**6, 7, 8**], where the results on the unit disc are extended to the ball or to the polydisk. In these papers, neither non-trivial zero products nor nilpotent Toeplitz operators are observed.

In [**4**], the problem is considered on the Segal-Bargmann space (the space of square integrable entire functions on $\mathbb{C}^n$ with a Gaussian decay weight) and an example of a non-trivial zero product of three Toeplitz operators is constructed. However, no nilpotent Toeplitz operator is observed.

2. Proof of Theorem 1.1. Inspired by the construction in [**12**], we define the following family of domains Ω_m in $\mathbb{C}^2$.

$$X = \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| > e, \quad |z_2| < \frac{1}{|z_1| \log |z_1|} \right\},$$
$$Y_m = \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_2| > 2, \quad \left| |z_1| - \frac{1}{|z_2|} \right| < \frac{1}{|z_2|^m} \right\},$$

$$Z = \Big\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \le e, \quad |z_2| \le 2\Big\},$$

and put

$$\Omega_m = X \cup Y_m \cup Z, \quad m = 1, 2, \dots .$$

Each Ω_m is an unbounded Reinhardt domain with finite volume, see Figure 1.

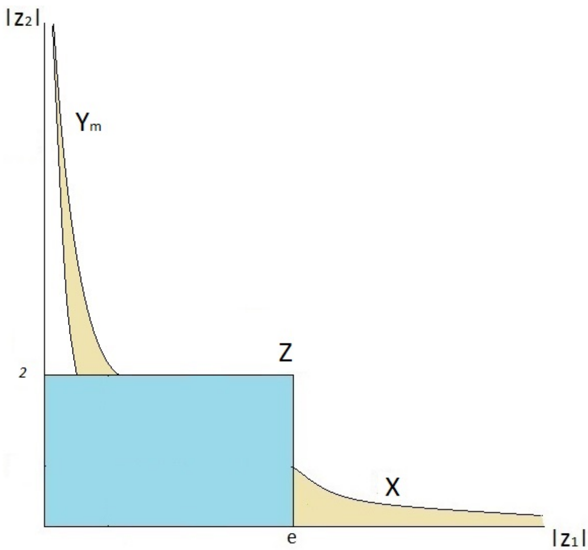

FIGURE 1. Representation of Ω_m in absolute space $\{(r_1, r_2) \in \mathbb{R}^2 \mid r_1 \ge 0$ and $r_2 \ge 0\}$, under the map $\tau : (z_1, z_2) \to (|z_1|, |z_2|)$.

Lemma 2.1. *For a multi-index $\alpha = (\alpha_1, \alpha_2)$, the monomial z^α is in $A^2(\Omega_m)$ if and only if $\alpha_2 \ge \alpha_1 > \alpha_2 - (m-1)/2$.*

Proof. We begin with the calculation on the domain X.

$$\begin{aligned}\int_X |z^\alpha|^2 \, dV(z) &= \int_{|z_1|>e} |z_1|^{2\alpha_1} \int_{|z_2|<1/|z_1| \log |z_1|} |z_2|^{2\alpha_2} \, dA(z_2) \, dA(z_1) \\ &= 4\pi^2 \int_e^\infty r_1^{2\alpha_1+1} \int_0^{1/(r_1 \log(r_1))} r_2^{2\alpha_2+1} dr_2 \, dr_1\end{aligned}$$

$$= \frac{4\pi^2}{2\alpha_2 + 2} \int_e^\infty \frac{r_1^{2\alpha_1+1}}{r_1^{2\alpha_2+2}(\log(r_1))^{2\alpha_2+2}}\, dr_1.$$

We note that, for $k > 0$, the improper integral

$$\int_e^\infty \frac{1}{x^m (\log x)^k}\, dx$$

converges if and only if $m \geq 1$. Therefore, the last integral above (where $k = 2\alpha_2 + 2 > 0$ and $m = 2(\alpha_2 - \alpha_1) + 1$) is finite if and only if $(\alpha_2 - \alpha_1) \geq 0$. In other words,

$$z^\alpha \in A^2(X) \Longleftrightarrow \alpha_2 \geq \alpha_1. \tag{2.1}$$

We continue with the calculation on domain Y_m.

$$\begin{aligned}\int_{Y_m} |z^\alpha|^2\, dV(z) &= \int_{|z_2|>2} |z_2|^{2\alpha_2} \int_{(1/|z_2|)-(1/|z_2|^m)<|z_1|<(1/|z_2|)+(1/|z_2|^m)} \\ &\qquad\qquad \cdot |z_1|^{2\alpha_1}\, dA(z_2)\, dA(z_1) \\ &= 4\pi^2 \int_2^\infty r_2^{2\alpha_2+1} \int_{1/r_2-1/r_2^m}^{1/r_2+1/r_2^m} r_1^{2\alpha_1+1} dr_1 dr_2 \\ &= \frac{4\pi^2}{2\alpha_1+2} \int_2^\infty r_2^{2\alpha_2+1} \Bigg[\left(\frac{1}{r_2} - \frac{1}{r_2^m}\right)^{2\alpha_1+2} \\ &\qquad\qquad - \left(\frac{1}{r_2} + \frac{1}{r_2^m}\right)^{2\alpha_1+2}\Bigg]\, dr_2.\end{aligned}$$

Since $r_2 > 2$, after using the binomial expansion in the brackets, we consider the term $1/r_2$ with the smallest degree as the dominant, which is $1/r_2^{2\alpha_1+1+m}$. The last integral can be estimated by:

$$\int_{Y_m} |z^\alpha|^2\, dV(z) \approx \int_2^\infty r_2^{2\alpha_2+1} \frac{1}{r_2^{2\alpha_1+1+m}}\, dr_2.$$

The integral on the right is finite if and only if $\alpha_1 > \alpha_2 + (1-m)/2$. In other words,

$$z^\alpha \in A^2(Y_m) \Longleftrightarrow \alpha_1 > \alpha_2 + \frac{1-m}{2}. \tag{2.2}$$

Lemma 2.1 follows from equations (2.1) and (2.2). $\square$

Next, we set $m \geq 6$, $\phi = z_1/\overline{z}_1$ and consider T_ϕ on $A^2(\Omega_m)$.

Proposition 2.2. *The following properties hold*:

(i) T_ϕ *is not a zero operator.*
(ii) T_ϕ *does not have finite rank.*
(iii) T_ϕ *is a bounded operator.*
(iv) T_ϕ *is a nilpotent operator of degree* $\lfloor m/4 \rfloor$, *the largest integer less than or equal to* $m/4$.

Remark 2.3. Once we prove Proposition 2.2, we immediately obtain Theorem 1.1. However, it will be clear in the proof that the domain and the operator we present are not unique but part of a family of domains and operators. We leave exploration of more examples to the reader.

Before starting the proof of Proposition 2.2, we define the following lattice for $m \geq 6$:

$$\begin{aligned} R_m &= \left\{ (\alpha_1, \alpha_2) \in \mathbb{N}^2 \mid \alpha_2 \geq \alpha_1 > \alpha_2 - \frac{m-1}{2} \right\} \\ &= \left\{ (\alpha_1, \alpha_2) \in \mathbb{N}^2 \mid \alpha_1 + \frac{m-1}{2} > \alpha_2 \geq \alpha_1 \right\}. \end{aligned}$$

Remark 2.4. Shifting α_1 to the right by a number s greater than or equal to $(m-1)/2$ is enough to put the resulting index $(\alpha_1 + s, \alpha_2)$ out of R_m, that is, if $(\alpha_1, \alpha_2) \in R_m$, then for $s \geq (m-1)/2$, we get $(\alpha_1 + s, \alpha_2) \notin R_m$.

For a multi-index $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$, we set

$$c_\gamma^2 = \int_\Omega |z^\gamma|^2 \, dV(z).$$

Then, on a radially symmetric domain Ω that contains the origin, the set (or a subset of) $\{z^\gamma / c_\gamma\}_{\gamma \in \mathbb{N}^n}$ gives a complete orthonormal basis for $A^2(\Omega)$. Each $f \in A^2(\Omega)$ can be written in the form

$$f(z) = \sum_\gamma f_\gamma \frac{z^\gamma}{c_\gamma},$$

where the sum converges in $A^2(\Omega)$, but also uniformly on compact subset of Ω. For the coefficients f_γ, we have $f_\gamma = \langle f(z), z^\gamma / c_\gamma \rangle_\Omega$.

Proof of Proposition 2.2. Consider T_ϕ on $A^2(\Omega_m)$ for $m \geq 6$. Ω_m is a radially symmetric domain and the monomials with exponents that reside in R_m form a complete system for $A^2(\Omega_m)$. By using the orthogonality of monomials we obtain

$$T_\phi(z^\alpha) = \mathcal{B}_{\Omega_m}\left(\frac{z_1}{\bar{z}_1} \cdot z^\alpha\right) = \sum_{\gamma \in R_m} \left\langle \frac{z_1}{\bar{z}_1} z^\alpha, \frac{z^\gamma}{c_\gamma} \right\rangle \frac{z^\gamma}{c_\gamma} = \frac{c^2_{(\alpha_1+1,\alpha_2)}}{c^2_{(\alpha_1+2,\alpha_2)}} z_1^{\alpha_1+2} z_2^{\alpha_2}. \tag{2.3}$$

On the above summation only $(\gamma_1, \gamma_2) = (\alpha_1 + 2, \alpha_2)$ survives. Moreover, there exist multi-indices (α_1, α_2) in R_m such that $(\alpha_1 + 2, \alpha_2)$ is also in R_m. Therefore, there exists $z^\alpha \in A^2(\Omega_m)$ such that

$$T_\phi(z^\alpha) = \frac{c^2_{(\alpha_1+1,\alpha_2)}}{c^2_{(\alpha_1+2,\alpha_2)}} z_1^{\alpha_1+2} z_2^{\alpha_2} \in A^2(\Omega_m)$$

and T_ϕ is a non-zero operator.

For $m \geq 6$ and $k \in \mathbb{N}$, $z_1^k z_2^{k+2} \in A^2(\Omega_m)$ and

$$T_\phi\left(z_1^k z_2^{k+2}\right) = \frac{c^2_{(k+1,k+2)}}{c^2_{(k+2,k+2)}} z_1^{k+2} z_2^{k+2} \in A^2(\Omega_m).$$

Hence, the range of the operator T_ϕ contains all the monomials of the form $z_1^{k+2} z_2^{k+2}$, and so the range of T_ϕ is infinite-dimensional.

If $g(z_1, z_2) \in A^2(\Omega_m)$, then its series expansion will be

$$g(z_1, z_2) = \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} g_{\alpha_1\alpha_2} \frac{z_2^{\alpha_2} z_1^{\alpha_1}}{c_{(\alpha_1,\alpha_2)}} = \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} \left\langle g(z), \frac{z^\alpha}{c_\alpha} \right\rangle \frac{z^\alpha}{c_\alpha},$$

where

$$r = \begin{cases} m/2 & \text{if } m \text{ is even}, \\ (m-1)/2 & \text{if } m \text{ is odd}. \end{cases}$$

The norm of $g(z_1, z_2)$ is given by

$$\|g\|^2_{A^2(\Omega_m)} = \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} |g_{\alpha_1\alpha_2}|^2, \tag{2.4}$$

and the norm of $T_\phi(g)$ is

$$\|T_\phi(g)\|^2_{A^2(\Omega_m)} = \left\| \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} \left\langle \frac{z_1}{\overline{z}_1} \cdot g(z), \frac{z^\alpha}{c_\alpha} \right\rangle \frac{z^\alpha}{c_\alpha} \right\|^2 \tag{2.5}$$

$$= \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} \left| \left\langle \frac{z_1}{\overline{z}_1} \cdot g(z), \frac{z^\alpha}{c_\alpha} \right\rangle \right|^2$$

$$= \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} \left| \left\langle \frac{z_1}{\overline{z}_1} \cdot \sum_\beta g_\beta \frac{z^\beta}{c_\beta}, \frac{z^\alpha}{c_\alpha} \right\rangle \right|^2$$

$$= \sum_{\alpha_1=2}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} \left| \left\langle \frac{z_1}{\overline{z}_1} \cdot g_{(\alpha_1-2,\alpha_2)} z_1^{\alpha_1-2} z_2^{\alpha_2}, \frac{z^\alpha}{c_\alpha} \right\rangle \right|^2,$$

by orthogonality of monomials

$$= \sum_{\alpha_1=2}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} \left| g_{(\alpha_1-2,\alpha_2)} \frac{c_{(\alpha_1-1,\alpha_2)}}{c_\alpha} \right|^2; \tag{2.6}$$

then we shift the indices

$$= \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=\alpha_1}^{\alpha_1+r-1} |\widetilde{g}_{\alpha_1\alpha_2}|^2, \tag{2.7}$$

where

$$\widetilde{g}_{\alpha_1\alpha_2} = \begin{cases} 0 & \text{if } \alpha_1 = \alpha_2 \text{ or } \alpha_1 = \alpha_2 + 1, \\ (c_{(\alpha_1+1,\alpha_2)}/c_{(\alpha_1+2,\alpha_2)}) g_{\alpha_1\alpha_2} & \text{otherwise.} \end{cases}$$

The ratio $c^2_{(\alpha_1+1,\alpha_2)}/c^2_{(\alpha_1+2,\alpha_2)}$ is uniformly bounded by a constant. Indeed, each integral on X and Y_m has a uniform bound from above (say C_X and C_{Y_m}) because of conditions (2.1) and (2.2). Furthermore, we compute the integrals on the polydisc Z explicitly and estimate as follows:

$$\frac{c^2_{(\alpha_1+1,\alpha_2)}}{c^2_{(\alpha_1+2,\alpha_2)}} \le \frac{C_X + C_{Y_m} + \pi(e^{2\alpha_1+4})/(\alpha_1+2) \cdot \pi(2^{2\alpha_2+2})/(\alpha_2+1)}{\pi(e^{2\alpha_1+2})/(\alpha_1+1) \cdot \pi(2^{2\alpha_2+2})/(\alpha_2+1)} \le \frac{C_X + C_{Y_m}}{\pi^2} + e^2 = C. \tag{2.8}$$

This estimate implies

$$|\widetilde{g}_{\alpha_1\alpha_2}|^2 \leq C \cdot |g_{\alpha_1\alpha_2}|^2, \text{ for all } (\alpha_1,\alpha_2) \in R_m. \tag{2.9}$$

Thus, from equations (2.4), (2.5) and (2.9), it follows that

$$\|T_\phi(g)\|^2_{A^2(\Omega_m)} \leq C \cdot \|g\|^2_{A^2(\Omega_m)}.$$

Finally, we calculate the powers of T_ϕ:

$$\begin{aligned} T^2_\phi(z^\alpha) = T_\phi \cdot T_\phi(z^\alpha) &= T_\phi\left(\frac{c^2_{(\alpha_1+1,\alpha_2)}}{c^2_{(\alpha_1+2,\alpha_2)}} z_1^{\alpha_1+2} z_2^{\alpha_2}\right) \\ &= \frac{c^2_{(\alpha_1+1,\alpha_2)}}{c^2_{(\alpha_1+2,\alpha_2)}} \cdot \frac{c^2_{(\alpha_1+3,\alpha_2)}}{c^2_{(\alpha_1+4,\alpha_2)}} z_1^{\alpha_1+4} z_2^{\alpha_2}. \end{aligned} \tag{2.10}$$

As for the third power,

$$T^3_\phi(z^\alpha) = \frac{c^2_{(\alpha_1+1,\alpha_2)}}{c^2_{(\alpha_1+2,\alpha_2)}} \cdot \frac{c^2_{(\alpha_1+3,\alpha_2)}}{c^2_{(\alpha_1+4,\alpha_2)}} \cdot \frac{c^2_{(\alpha_1+5,\alpha_2)}}{c^2_{(\alpha_1+6,\alpha_2)}} z_1^{\alpha_1+6} z_2^{\alpha_2}. \tag{2.11}$$

Continuing in this fashion, the kth power of the operator is:

$$T^k_\phi(z^\alpha) = \frac{c^2_{(\alpha_1+1,\alpha_2)}}{c^2_{(\alpha_1+2,\alpha_2)}} \cdot \frac{c^2_{(\alpha_1+3,\alpha_2)}}{c^2_{(\alpha_1+4,\alpha_2)}} \cdots \frac{c^2_{(\alpha_1+2k-1,\alpha_2)}}{c^2_{(\alpha_1+2k,\alpha_2)}} z_1^{\alpha_1+2k} z_2^{\alpha_2}. \tag{2.12}$$

In equation (2.12), if $2k < r$, then there exists an $(\alpha_1,\alpha_2) \in R_m$ such that $(\alpha_1 + 2k, \alpha_2) \in R_m$, see the discussion in Remark 2.4, so $z_1^{\alpha_1+2k} z_2^{\alpha_2} \in A^2(\Omega_m)$ and $T^k_\phi \not\equiv 0$ on $A^2(\Omega_m)$.

However, in equation (2.12), if $2k \geq r$, then for all $(\alpha_1,\alpha_2) \in R_m$, we have $(\alpha_1 + 2k, \alpha_2) \notin R_m$ by Remark 2.4, so we see that $z_1^{\alpha_1+2k} z_2^{\alpha_2} \notin A^2(\Omega_m)$ and $T^k_\phi \equiv 0$ on $A^2(\Omega_m)$, that is, T_ϕ is a nilpotent operator of degree k on $A^2(\Omega_m)$. □

We illustrate the main arguments of the proof in the following example.

Example 2.5. Set $m = 9$. Then the monomial $z_1^{\alpha_1} z_2^{\alpha_2}$ is in $A^2(\Omega_9)$ if and only if $\alpha_1 + 4 > \alpha_2 \geq \alpha_1$. The exponents of the monomial in $A^2(\Omega_9)$ are marked on the lattice in Figure 2. It can be noted that T_ϕ acts like a shift on the lattice; it takes $(\alpha_1, \alpha_2 + 2)$ to $(\alpha_1 + 2, \alpha_2 + 2)$. Thus, if T_ϕ is applied on any monomial two times, then the exponent

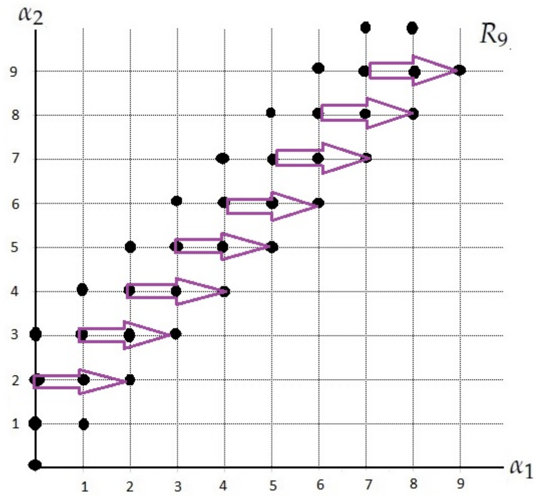

FIGURE 2. Representation of the lattice R_m for $m = 9$ and the action of T_ϕ on R_m.

of the monomial runs out of the lattice R_9, that is, if $z_1^{\alpha_1} z_2^{\alpha_2} \in A^2(\Omega_9)$, then

$$T_\phi \cdot T_\phi(z_1^{\alpha_1} z_2^{\alpha_2}) = \frac{c^2_{(\alpha_1+1,\alpha_2)}}{c^2_{(\alpha_1+2,\alpha_2)}} \cdot \frac{c^2_{(\alpha_1+3,\alpha_2)}}{c^2_{(\alpha_1+4,\alpha_2)}} z_1^{\alpha_1+4} z_2^{\alpha_2} \notin A^2(\Omega_9),$$

and so $T_\phi^2 \equiv 0$ on $A^2(\Omega_9)$.

Acknowledgments. We thank the referees for constructive comments on Proposition 2.2 and also for the useful editorial remarks on the exposition of the article.

REFERENCES

1. Patrick Ahern and Željko Čučković, *Products of Toeplitz operators on the Bergman space*, Illinois J. Math. **45** (2001), 113–121.

2. ______, *A theorem of Brown-Halmos type for Bergman space Toeplitz operators*, J. Funct. Anal. **187** (2001), 200–210.

3. Alexandru Aleman and Dragan Vukotić, *Zero products of Toeplitz operators*, Duke Math. J. **148** (2009), 373–403.

4. Wolfram Bauer and Trieu Le, *Algebraic properties and the finite rank problem for Toeplitz operators on the Segal-Bargmann space*, J. Funct. Anal. **261** (2011), 2617–2640.

5. Arlen Brown and P.R. Halmos, *Algebraic properties of Toeplitz operators*, J. reine angew. Math. **213** (1963/1964), 89–102.

6. Boo Rim Choe and Hyungwoon Koo, *Zero products of Toeplitz operators with harmonic symbols*, J. Funct. Anal. **233** (2006), 307–334.

7. Boo Rim Choe, Hyungwoon Koo and Young Joo Lee, *Zero products of Toeplitz operators with n-harmonic symbols*, Int. Equat. Oper. Theor. **57** (2007), 43–66.

8. Boo Rim Choe, Young Joo Lee, Kyesook Nam and Dechao Zheng, *Products of Bergman space Toeplitz operators on the polydisk*, Math. Ann. **337** (2007), 295–316.

9. Caixing Gu, *Products of several Toeplitz operators*, J. Funct. Anal. **171** (2000), 483–527.

10. Kun Yu Guo, *A problem on products of Toeplitz operators*, Proc. Amer. Math. Soc. **124** (1996), 869–871.

11. Steven G. Krantz, *Function theory of several complex variables*, AMS Chelsea Publishing, Providence, RI, 2001.

12. Jan J.O.O. Wiegerinck, *Domains with finite-dimensional Bergman space*, Math. Z. **187** (1984), 559–562.

Department of Mathematics, Texas A&M University-Commerce, Commerce, TX 75429
Email address: **mehmet.celik@tamuc.edu**

University of Michigan, Department of Mathematics and Statistics, Dearborn, MI 48128
Email address: **zeytuncu@umich.edu**