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NEW LINEARIZATION FORMULAE FOR THE PRODUCTS OF CHEBYSHEV POLYNOMIALS OF THIRD AND FOURTH KINDS

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ABSTRACT. This paper deals with the problem of finding two new closed formulae for linearization coefficients of two special nonsymmetric cases for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ corresponding to the parameters' values $\beta = -\alpha = \pm 1/2$. From these two formulae, the linearization coefficients of the products of Chebyshev polynomials of the third and fourth kinds are established. Based on using algorithmic methods, such as the algorithms by Zeilberger, Petkovsek and Van-Hoeij, and two certain Whipple's transformations, six new closed formulae for summing certain terminating hypergeometric functions of unit argument are given.

1. Introduction. Chebyshev polynomials have become increasingly important in numerical analysis, from both theoretical and practical points of view. There are four kinds of Chebyshev polynomials. The majority of books and research papers dealing with specific orthogonal polynomials of Chebyshev family contain mainly results of Chebyshev polynomials of first and second kinds $T_n(x)$ and $U_n(x)$ and their numerous uses in different applications, see for example, [**5, 8, 9, 15, 22**]. However, there is only a very limited body of literature on Chebyshev polynomials of third and fourth kinds $V_n(x)$ and $W_n(x)$, either from the theoretical or practical points of view and on their uses in various applications. Recently, Doha and Abd-Elhameed in [**13**] have introduced new closed formulae explicitly expressing the integrals of third and fourth kinds Chebyshev polynomials of any degree that has been integrated an arbitrary number of times in terms of third and fourth kinds Chebyshev polynomials themselves. Gautschi [**18**], Mason

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[23] and Notaris **[25]** used Chebyshev polynomials of third and fourth kinds in approximation and other applications. The interested reader in Chebyshev polynomials of third and fourth kinds is referred to the excellent book of Mason and Handscomb **[24]**.

If we were asked for a "pecking order" of these four Chebyshev polynomials $T_n(x)$, $U_n(x)$, $V_n(x)$ and $W_n(x)$, then we would say that $T_n(x)$ is the most important and versatile. Moreover, $T_n(x)$ generally leads to the simplest formulae, whereas results for the other polynomials may involve slight complications. However, all four polynomials have their role. For example, $U_n(x)$ is useful in numerical integration (see **[23]**), while $V_n(x)$ and $W_n(x)$ can be useful in situations in which singularities occur at one end point ($+1$ or -1) but not at the other (see **[24]**).

The general linearization problem consists of finding the coefficients g_{ijk} in the expansion of the product of two polynomials $p_i(x)$ and $q_j(x)$ in terms of an arbitrary sequence of orthogonal polynomials $\{y_k(x)\}$, i.e.,

$$p_i(x)\,q_j(x) = \sum_{k=0}^{i+j} g_{ijk}\,y_k(x). \tag{1.1}$$

As particular cases of problem (1.1) we have the following two important problems:

(i) The standard linearization or Clebsh-Gordan-type problem which consists of finding the coefficients $L_{ij}(k)$ in the expansion of the product of two polynomials $p_i(x)$ and $p_j(x)$ in terms of the sequence $\{p_n\}_{n\geq 0}$, i.e.,

$$p_i(x)\,p_j(x) = \sum_{k=0}^{i+j} L_{ij}(k)\,p_k(x). \tag{1.2}$$

(ii) The connection problem, which consists of finding the coefficients c_{ik} such that

$$p_i(x) = \sum_{k=0}^{i} c_{ik}\,y_k(x).$$

The two problems of linearizing products of orthogonal polynomials and the connection coefficients, in general, and of ultraspherical and

Jacobi polynomials, in particular, have been studied by a large number of authors, (see, among them, [**6, 10, 11, 12, 14, 21, 28, 29, 30**].

Linearization problems are frequently encountered in many applications; for instance, the case in which $i = j$ in the standard linearization formula (1.2) is often required to evaluate the logarithmic potentials of orthogonal polynomials appearing in the calculation of the position and momentum information entropies of quantum systems (see Dehesa et al. [**7**]). This motivates our interest in these problems.

The standard linearization problem associated to Jacobi polynomials and to establish the conditions of nonnegativity of the linearization coefficients has been studied by many authors (see, for instance, [**3, 16, 17, 19, 26**]).

The main aim of this paper is to establish new simple closed formulae for linearization of the product of two Jacobi polynomials for certain special parameters. Also, the formulae for the linearization of the products of Chebyshev polynomials of third and fourth kinds are deduced. To the best of our knowledge all of these formulae are completely new and are traceless in literature.

The paper is organized as follows. In Section 2, we give some relevant properties of Jacobi polynomials. In Section 3, and with the aid of some symbolic algebra such as the algorithms by Zeilberger, Petkovsek and Van-Hoeij, the linearization formulae for Chebyshev polynomials of third and fourth kinds are given. Making use of the results obtained in Section 3, two formulae expressing the squares of Chebyshev polynomials of third and fourth kinds in terms of Chebyshev polynomials of third and fourth kinds themselves are given in Section 4. Moreover, in this section, and with the aid of two certain Whipple's transformations, reduction formulae for two terminating balanced hypergeometric functions of the type ${}_4F_3(1)$, and two terminating very well-poised hypergeometric functions of the type ${}_7F_6(1)$ are deduced.

2. Some properties of classical Jacobi polynomials. The classical Jacobi polynomials associated with the real parameters ($\alpha > -1, \beta > -1$) (see [**1, 2**]) are a sequence of polynomials

$$P_n^{(\alpha,\beta)}(x), \quad x \in [-1,1],\ n = 0,1,2,\ldots,$$

each respectively of degree n. For our present purposes, it is more convenient to introduce the normalized orthogonal polynomials

$$R_n^{(\alpha,\beta)}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}.$$

This means that

$$R_n^{(\alpha,\beta)}(x) = \frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} P_n^{(\alpha,\beta)}(x). \tag{2.1}$$

The polynomials $R_n^{(\alpha,\beta)}(x)$ satisfy the orthogonality relation,

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\, R_m^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(x)\, dx = \begin{cases} 0 & m \neq n, \\ h_n & m = n, \end{cases} \tag{2.2}$$

where

$$h_n = \frac{2^{\lambda}\, n!\, \Gamma(n+\beta+1)\, [\Gamma(\alpha+1)]^2}{(2n+\lambda)\, \Gamma(n+\lambda)\, \Gamma(n+\alpha+1)}, \tag{2.3}$$

and $\lambda = \alpha + \beta + 1$.

Of these polynomials, the most commonly used are the ultraspherical polynomials $C_n^{(\alpha)}(x)$, the Chebyshev polynomials of first and second kinds $T_n(x)$, $U_n(x)$ and the Legendre polynomials $L_n(x)$. These orthogonal polynomials are interrelated to the normalized Jacobi polynomials by the following relations:

$$\begin{aligned} C_n^{(\alpha)}(x) &= R_n^{(\alpha-1/2,\alpha-1/2)}(x), \\ T_n(x) &= R_n^{(-1/2,-1/2)}(x), \\ U_n(x) &= (n+1)\, R_n^{(1/2,1/2)}(x), \\ L_n(x) &= R_n^{(0,0)}(x). \end{aligned}$$

The Chebyshev polynomials $V_n(x)$ and $W_n(x)$ of the third and fourth kinds are polynomials in x defined, respectively, by (see **[24]**)

$$V_n(x) = \frac{\cos(n+1/2)\theta}{\cos(\theta/2)}$$

and

$$W_n(x) = \frac{\sin(n+(1/2))\theta}{\sin(\theta/2)},$$

when $x = \cos\theta$.

These two kinds of polynomials are, in fact, rescalings of two particular Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ for the two nonsymmetric special cases $\beta = -\alpha = \pm 1/2$. They are given explicitly by

$$V_n(x) = \frac{2^{2n}}{\binom{2n}{n}} P_n^{(-1/2,1/2)}(x) \tag{2.4}$$

and

$$W_n(x) = \frac{2^{2n}}{\binom{2n}{n}} P_n^{(1/2,-1/2)}(x). \tag{2.5}$$

It is readily seen that

$$W_n(x) = (-1)^n V_n(-x), \tag{2.6}$$

and, therefore, it is sufficient to establish properties and relations for $V_n(x)$, and then deduce analogous properties and relations for $W_n(x)$ (replacing x by $-x$).

Making use of formulae (2.1), (2.4) and (2.5) enables one to show that

$$V_n(x) = R_n^{(-1/2,1/2)}(x),$$

and

$$W_n(x) = (2n+1)\, R_n^{(1/2,-1/2)}(x).$$

Several other properties for the polynomials $V_n(x)$ and $W_n(x)$, can be found in [**24**].

3. Linearization formulae for $V_n(x)$ and $W_n(x)$. In this section, we are interested in establishing new linearization formulae for Chebyshev polynomials of third and fourth kinds.

3.1. Linearization coefficients of Jacobi polynomials. Now, consider the linearization of products of two normalized Jacobi polynomials $R_m^{(\alpha,\beta)}(x)$ and $R_n^{(\alpha,\beta)}(x)$. Rahman [**26**] discussed the following linearization problem

$$R_m^{(\alpha,\beta)}(x)\,R_n^{(\alpha,\beta)}(x) = \sum_{k=|n-m|}^{n+m} g(k,m,n;\alpha,\beta)\,R_k^{(\alpha,\beta)}(x). \tag{3.1}$$

Using the orthogonality relation (2.2), the linearization coefficients $g(k,m,n;\alpha,\beta)$ can be expressed in terms of the integral of the product of three Jacobi polynomials. Explicitly,

$$g(k,m,n;\alpha,\beta)=\frac{1}{h_k}\int_{-1}^{1}(1-x)^{\alpha}\,(1+x)^{\beta}\,R_k^{(\alpha,\beta)}(x)\,R_m^{(\alpha,\beta)}(x)\,R_n^{(\alpha,\beta)}(x)\,dx, \tag{3.2}$$

where h_k is given by (2.3).

Thus, the evaluation of the integral in (3.2) is equivalent to the linearization problem (3.1). The solution of the linearization problem (3.1) is given in **[26]** by the following theorem.

Theorem 3.1. *For all $n \geq m$,* **[26]** *proved that*

$$R_m^{(\alpha,\beta)}(x)\,R_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{2m} L_{m,n,j}\,R_{n-m+j}^{(\alpha,\beta)}(x), \tag{3.3}$$

where

$$\begin{aligned} L_{m,n,j} &= \frac{n!\,\Gamma(\alpha+1)\,(2n-2m+2j+\lambda)}{(n-m)!\,j!\,\Gamma(\lambda+2n+1)} \\ &\times \frac{\Gamma(n-m+j+\alpha+1)\,\Gamma(\beta+n+1)\,\Gamma(\lambda+2m)\,\Gamma(\lambda+2n-2m+j)}{\Gamma(n-m+j+\beta+1)\,\Gamma(\lambda+m)\,\Gamma(\alpha+n-m+1)\,\Gamma(\alpha+m+1)} \\ &\times \sum_{r=0}^{j} \frac{(-j)_r\,(j+2n-2m+\lambda)_r}{r!\,(\lambda+2n+1)_r} \\ &\times {}_4F_3\left(\begin{matrix} -r,\ r+2n-2m+1,\ -m,\ -m-\beta \\ n-m+1,\ n-m+\alpha+1, -2m-\alpha-\beta \end{matrix} \,\middle|\, 1 \right). \end{aligned} \tag{3.4}$$

It is worthwhile noting here that although the balanced ${}_4F_3(1)$ in this formula is terminated, it cannot be summed in closed form except for certain special values of its parameters. This motivates our interest to study, in particular, the two cases correspond to $\beta = -\alpha = \pm 1/2$.

3.2. Linearization formula for $V_n(x)$. To obtain the linearization formula of the product of two Chebyshev polynomials of the third kind, the following two lemmas are needed.

Lemma 3.2. *For all $r, n, m \in \mathbb{Z}^{\geq 0}$ and $r \leq 2m$, we have*

$$ {}_4F_3\left(\left. \begin{array}{c} -r,\ r+2n-2m+1,\ -m,\ -m-\frac{1}{2} \\ n-m+1,\ n-m+\frac{1}{2},\ -2m \end{array} \right| 1 \right) = \frac{(2n+r+1)!\,(2n-2m)!}{(2n-2m+r)!\,(2n+1)!}. \tag{3.5}$$

Proof. If we set

$$ {}_4F_3\left(\left. \begin{array}{c} -r,\ r+2n-2m+1,\ -m,\ -m-\frac{1}{2} \\ n-m+1,\ n-m+\frac{1}{2},\ -2m \end{array} \right| 1 \right) = G(r,n,m), $$

then with the aid of algorithmic methods such as the algorithms by Zeilberger, Petkovsek and Van-Hoeij (see, for instance, [**20**]), via the `Maple` software, and in particular, `sumrecursion` command, $G(r,n,m)$, satisfies the following recurrence relation

$$ \begin{aligned} &(r+1)(r+2n+2)\,G(r,n,m) \\ &\quad + 2\Big[(2m(n+r+2) - (r+1)(2n+r+2)\Big]\,G(r+1,n,m) \\ &\qquad - (2m-r-1)(r+2n-2m+2)\,G(r+2,n,m) = 0, \end{aligned} \tag{3.6}$$

with the initial values

$$ G(0,n,m) = 1,\ G(1,n,m) = \frac{2(n+1)}{2\,n-2\,m+1}. \tag{3.7}$$

The exact solution of this recurrence relation is given explicitly by

$$ G(r,n,m) = \frac{(2n+r+1)!\,(2n-2m)!}{(2n-2m+r)!\,(2n+1)!}. $$ □

Remark 3.3. It is worthwhile noting here that the exact solution of the recurrence relation (3.6) can be obtained with the aid of Petkovsek's algorithm (see [**20**]), or with the improved version of [**31**]. Moreover, and in this respect, one may use the package in `Maple` called `LREtools[hypergeomsols]`.

Lemma 3.4. *For all $j, r, s \in \mathbb{Z}^{\geq 0}$, we have*

$$\sum_{r=0}^{j} \frac{(-j)_r \, (r+s+1)_j}{r!} = (-1)^j \, j!\,.$$

Proof. Let

$$\sum_{r=0}^{j} \frac{(-j)_r \, (r+s+1)_j}{r!} = M(j, r, s).$$

Then, making use of symbolic algebra such as the algorithms of Zeilberger, Petkovsek and Van-Hoeij, $M(j, r, s)$ satisfies the recurrence relation

$$M(j+1, r, s) + (j+1)\, M(j, r, s) = 0, \qquad M(0, r, s) = 1,$$

which can immediately be solved to give

$$M(r, s, j) = (-1)^j \, j!\,.$$

Now, based on Theorem 3.1 and Lemmas 3.2 and 3.4, we give the following linearization formula for the product of $V_m(x)\, V_n(x)$. □

Theorem 3.5. *For any nonnegative integers m and n, we have*

$$V_m(x)\, V_n(x) = \sum_{k=|n-m|}^{n+m} (-1)^{k+m+n}\, V_k(x). \tag{3.8}$$

Proof. If we substitute by $\alpha = -1/2$ and $\beta = 1/2$ into relation (3.3), then we get

$$R_m^{(-1/2,1/2)}(x)\, R_n^{(-1/2,1/2)}(x) = \sum_{j=0}^{2m} L_{m,n,j}\, R_{n-m+j}^{(-1/2,1/2)}(x), \quad n \geq m,$$

where

$$
\begin{aligned}
L_{m,n,j} &= \frac{(j+2n-2m)!}{j!\,(2n-2m)!} \\
&\quad\times \sum_{r=0}^{j} \frac{(-j)_r\,(j+2n-2m+1)_r}{r!\,(2n+2)_r} \\
&\quad\times {}_4F_3\left(\begin{matrix} -r,\ r+2\,n-2\,m+1,\ -m,\ -m-\frac{1}{2} \\ n-m+1, n-m+\frac{1}{2}, -2m \end{matrix} \,\middle|\, 1 \right),
\end{aligned} \tag{3.9}
$$

and if we make use of Lemma 3.2, then the linearization coefficients $L_{m,n,j}$ in equation (3.9) take the simpler form

$$
\begin{aligned}
L_{m,n,j} =& \frac{(2n+1)!\,(n-m)!\,\Gamma(n-m+1/2)}{j!\,(2n-2m)!\,n!\,\Gamma(n+3/2)} \\
&\times \sum_{r=0}^{j} \frac{(-j)_r (j+2n-2m+r)!\,\Gamma(n+(r/2)+1)\,\Gamma(n+(r/2)+(3/2))}{r!\,(2n+r+1)!\,\Gamma(n-m+(r/2)+1)\,\Gamma(n-m+(r/2)+(1/2))}.
\end{aligned} \tag{3.10}
$$

Now, the application of the Legendre duplication formula (see [27]) enables one to get

$$
\begin{aligned}
&\frac{\Gamma(n+(r/2)+1)\,\Gamma(n+(r/2)+(3/2))}{\Gamma(n-m+(r/2)+1)\,\Gamma(n-m+(r/2)+(1/2))} \\
&\qquad = \frac{(2n+r+1)!}{2^{2m+1}\,(2n-2m+r)!},
\end{aligned} \tag{3.11}
$$

and

$$
\frac{(2n+1)!\,(n-m)!\,\Gamma(n-m+1/2)}{(2n-2m)!\,n!\,\Gamma(n+3/2)} = 2^{2m+1}, \tag{3.12}
$$

and hence, the substitution of relations (3.11) and (3.12) into formula (3.10) gives

$$
L_{m,n,j} = \frac{1}{j!} \sum_{r=0}^{j} \frac{(-j)_r (r+2n-2m+1)_j}{r!}.
$$

If we make use of Lemma 3.4 with $s = 2n-2m \geq 0$, then we obtain the following linearization formula

$$
R_m^{(-1/2,1/2)}(x)\, R_n^{(-1/2,1/2)}(x) = \sum_{j=0}^{2m} (-1)^j\, R_{n-m+j}^{(-1/2,1/2)}(x), \quad n \geq m,
$$

which immediately yields

$$V_m(x)\,V_n(x) = \sum_{k=|n-m|}^{n+m} (-1)^{k+m+n}\,V_k(x), \quad \text{for all } n,m \in \mathbb{Z}^{\geq 0}.$$

Thus, the proof of Theorem 3.5 is complete. $\square$

3.3. Linearization formula for $W_n(x)$. Similar analysis to that performed in subsection 3.2 can be made to obtain a linearization formula for the product $W_m(x)\,W_n(x)$. However, and based on any suitable symbolic algebra such as the algorithms by Zeilberger, Petkovsek and Van-Hoeij, the following two lemmas can be proved.

Lemma 3.6. *For all $r,n,m \in \mathbb{Z}^{\geq 0}$ and $r \leq 2m$, we have*

$$(3.13)\quad {}_4F_3\left(\left.\begin{array}{l} -r,\ r+2n-2m+1,\ -m,\ -m+\frac{1}{2} \\ n-m+1,\ n-m+\frac{3}{2},\ -2m \end{array}\right| 1\right) = \frac{(2n+r+1)!\,(2n-2m+1)!}{(2n-2m+2r+1)\,(2n-2m+r)!\,(2n+1)!}.$$

Proof. If we set

$${}_4F_3\left(\left.\begin{array}{l} -r,\ r+2n-2m+1,\ -m,\ -m+\frac{1}{2} \\ n-m+1,\ n-m+\frac{3}{2},\ -2m \end{array}\right| 1\right) = G(r,n,m),$$

then the recurrence relation satisfied by $G(r,n,m)$ is given by

$$\begin{aligned}&(r+1)(2n+r+2)(2\,n-2\,m+2\,r+1)\,G(r,n,m)\\ &\quad+2(2\,n-2\,m+2\,r+3)\Big[2m(n+r+2)-(r+1)(2\,n+r+2)\Big]\\ &\qquad G(r+1,n,m)-(2\,m-r-1)(2\,n-2\,m+r+2)\\ &\qquad\qquad\times(2\,n-2\,m+2\,r+5)\,G(r+2,n,m)=0,\end{aligned}$$

with the initial values

$$G(0,n,m)=1, \qquad G(1,n,m)=\frac{2(n+1)}{2\,n-2\,m+3}.$$

The exact solution of this recurrence relation is given explicitly by

$$G(r,n,m) = \frac{(2n+r+1)!\,(2n-2m+1)!}{(2n-2m+2r+1)\,(2n-2m+r)!\,(2n+1)!}. \qquad \square$$

Lemma 3.7. *For all $j, r, s \in \mathbb{Z}^{\geq 0}$, we have*

$$\sum_{r=0}^{j} \frac{(-j)_r\,(j+s+1)_r}{r!\,(s+r)!\,(s+2r+1)} = \frac{j!}{(s+2j+1)\,(s+j)!}.$$

Proof. Let

$$\sum_{r=0}^{j} \frac{(-j)_r\,(j+s+1)_r}{r!\,(s+r)!\,(s+2r+1)} = M(j,r,s).$$

Then making use of a suitable symbolic algorithm such as the algorithms by Zeilberger, Petkovsek and Van-Hoeij, $M(j,r,s)$ satisfies the recurrence relation

$$(j+s+1)\,(2j+s+3)\,M(j+1,r,s) - (j+1)\,(2j+s+1)\,M(j,r,s) = 0,$$

$$M(0,r,s) = \frac{1}{(s+1)!},$$

which can be solved immediately to give

$$M(r,s,j) = \frac{j!}{(s+2j+1)\,(s+j)!}. \qquad \square$$

Now, the linearization of product of $W_m(x)\,W_n(x)$ is given in the following theorem.

Theorem 3.8. *For any nonnegative integers n and m, we have*

$$W_m(x)\,W_n(x) = \sum_{k=|n-m|}^{n+m} W_k(x). \tag{3.14}$$

Proof. With the aid of Theorem 3.1 and the two Lemmas 3.6 and 3.7, and after some manipulation, we get

$$R_m^{(1/2,-1/2)}(x)\,R_n^{(1/2,-1/2)}(x) = \sum_{j=0}^{2m} \frac{(2n-2m+2j+1)}{(2n+1)(2m+1)}\;R_{n-m+j}^{(1/2,-1/2)}(x),$$

then if we make use of the identity

$$R_n^{(1/2,-1/2)}(x) = \frac{1}{2n+1}\, W_n(x),$$

formula (3.14) is obtained. □

Remark 3.9. Theorem 3.8 can be obtained as a direct consequence of Theorem 3.5, with the aid of formula (2.6).

3.4. Comparison with the results in [6]. Reference [**6**] is integral to our manuscript. The authors in [**6**] discussed the general linearization problem

$$P_i^{(\lambda,\delta)}(x)\, P_j^{(\mu,\gamma)}(x) = \sum_{k=0}^{i+j} L_{ij}(k)\, P_k^{(\alpha,\beta)}(x), \tag{3.15}$$

where $P_i^{(\lambda,\delta)}(x)$ is the Jacobi polynomial of degree i. In fact, they obtained an expression for the linearization coefficients $L_{ij}(k)$ in terms of the Kampé de Fériet function $F^{2:2}_{2:1}$ which is represented as a double hypergeometric function. Comparing the results obtained in subsections 3.2 and 3.3 with those obtained in [**6**], it is worth pointing out the following:

- Although linearization formula (3.15) is general, the coefficients $L_{ij}(k)$ are not easy to reduce in simple forms even for special choices of the involved parameters in (3.15). For the choice corresponding to $\alpha = \mu + \lambda$ and $\beta = \delta + \gamma$, the authors in [**6**] developed a new formula for the linearization coefficients $L_{ij}(k)$ which is expressed in terms of a product of two certain terminating hypergeometric functions of the type ${}_3F_2(1)$. For the particular choice corresponding to $\lambda = \delta = \mu = \gamma$, one of the resulting ${}_3F_2(1)$ cannot be reduced to any hypergeometric term, while the second can be reduced with the aid of Petkovesk's algorithm.
- The two developed linearization formulae (3.8) and (3.14) are equivalent to those obtained from (3.15) with the following two choices of parameters:
  (i) $\lambda = \mu = \alpha = -1/2$, $\delta = \gamma = \beta = 1/2$.
  (ii) $\lambda = \mu = \alpha = 1/2$, $\delta = \gamma = \beta = -1/2$, but if we use (3.15), the two resulting linearization formulae are

expressed in terms of $F^{2:2}_{2:1}$, which cannot be reduced in simple linearization formulae. So, we follow an alternative approach to obtain (3.8) and (3.14), that is, we make use of the linearization formula (3.3) of Rahman [**26**].

4. New trigonometric identities and transformation formulae.

4.1. Squares formulae for $V_n(x)$ and $W_n(x)$. As special cases of the linearization formulae (3.8) and (3.14), the expressions for the squares of $V_n(x)$ and $W_n(x)$ in terms of $V_n(x)$ and $W_n(x)$ themselves are given in the following corollary.

Corollary 4.1. *The squares of $V_n(x)$ and $W_n(x)$ are given by the following relations*:

$$V_n^2(x) = \sum_{k=0}^{2n} (-1)^k\, V_k(x), \tag{4.1}$$

and

$$W_n^2(x) = \sum_{k=0}^{2n} W_k(x). \tag{4.2}$$

Remark 4.2. It is worth noting that relations (4.1) and (4.2) are equivalent to the following two trigonometric identities:

$$\sum_{k=0}^{2n} (-1)^k \cos\left(k + \frac{1}{2}\right)\theta = \frac{\cos^2(n+1/2)\,\theta}{\cos\theta/2} \tag{4.3}$$

$$\sum_{k=0}^{2n} \sin\left(k + \frac{1}{2}\right)\theta = \frac{\sin^2(n+1/2)\theta}{\sin\theta/2}. \tag{4.4}$$

4.2. Reduction formulae for certain hypergeometric functions of a unit argument. In this section, new reduction formulae for summing four terminating hypergeometric series are deduced. These reduction formulae are obtained by making use of the following two

Whipple's transformations: the first gives a connection between a two terminating balanced ${}_4F_3(1)$ series, and the second gives a connection between a terminating balanced ${}_4F_3(1)$ series and a terminating very well-poised ${}_7F_6(1)$ series.

Theorem 4.3. *For any positive integer r, we have*

$$
{}_4F_3\left(\begin{matrix}-r,\ a,\ b\,,c\\ d,e,f\end{matrix}\ \middle|\ 1\right)=\frac{(e-a)_r\,(f-a)_r}{(e)_r\,(f)_r}
{}_4F_3\left(\begin{matrix}-r,\ a,\ d-b\,,d-c\\ d,a+1-r-e,a+1-r-f\end{matrix}\ \middle|\ 1\right),
$$

where $-r+a+b+c+1=d+e+f$.

(For the proof, see Andrews et al. [**2**]).

Theorem 4.4. *For any positive integer r, we have*

$$
{}_4F_3\left(\begin{matrix}-r,\ x,\ y\,,z\\ u,v,w\end{matrix}\ \middle|\ 1\right)=\frac{(u-y)_r\,(u-z)_r}{(u)_r\,(u-y-z)_r}
\times{}_7F_6\left(\begin{matrix}-r,\ a,\ 1+\frac{a}{2}\,,w-x\,,v-x\,,y\,,z\\ 1+a+r,\ \frac{a}{2}\,,v\,,w\,,1+a-y\,,1+a-z\end{matrix}\ \middle|\ 1\right),\tag{4.5}
$$

where $a=y+z-u-r=w+v-x-1$.

(For the proof, see Bailey [**4**]).

Now, we can state and prove the following two theorems:

Theorem 4.5. *For all $r,s,n\in\mathbb{Z}^{\geq 0}$ and $r\geq 2n-2s$, the following two reduction formulae hold.*

$$
{}_4F_3\left(\begin{matrix}-r,\ r+2s+1,\ n+1,\ n+\frac{3}{2}\\ s+1,\ s+\frac{3}{2},\ 2n+2\end{matrix}\ \middle|\ 1\right)=\frac{(2s-2n)_r\,(2\,s+1)!}{(2r+2s+1)\,(r+2s)!}\tag{4.6}
$$

and

$$
{}_4F_3\left(\begin{matrix}-r,\ r+2s+1,\ n+1,\ n+\frac{1}{2}\\ s+1,\ s+\frac{1}{2},\ 2n+2\end{matrix}\ \middle|\ 1\right)=\frac{(2s-2n)_r\,(2\,s)!}{(r+2s)!}.\tag{4.7}
$$

Proof. If we make use of the Whipple's transformation stated in Theorem 4.3, then we have the following two transformation formulae:

$$(4.8)\quad {}_4F_3\left(\begin{matrix} -r,\ r+2s+1,\ n+1,\ n+\frac{3}{2} \\ s+1,\ s+\frac{3}{2},\ 2n+2 \end{matrix}\,\middle|\, 1\right) = \frac{(s+1/2)_r\,(2s-2n)_r}{(-s-r-1/2)_r\,(-2n-r-1)_r} \times {}_4F_3\left(\begin{matrix} -r,\ r+2s+1,\ s-n,\ s-n-\frac{1}{2} \\ s+1,\ s+\frac{1}{2},\ 2s-2n \end{matrix}\,\middle|\, 1\right),$$

and

$$(4.9)\quad {}_4F_3\left(\begin{matrix} -r,\ r+2s+1,\ n+1,\ n+\frac{1}{2} \\ s+1,\ s+\frac{1}{2},\ 2n+2 \end{matrix}\,\middle|\, 1\right) = \frac{(s+3/2)_r\,(2s-2n)_r}{(-s-r+1/2)_r\,(-2n-r-1)_r} \times {}_4F_3\left(\begin{matrix} -r,\ r+2s+1,\ s-n,\ s-n+\frac{1}{2} \\ s+1,\ s+\frac{3}{2},\ 2s-2n \end{matrix}\,\middle|\, 1\right).$$

Now, the two reduction formulae (4.6) and (4.7) can be immediately obtained after the application of Lemmas 3.2 and 3.6 on the right hand sides of the two transformation formulae (4.8) and (4.9). $\square$

Theorem 4.6. *For all $n, r, s \in \mathbb{Z}^{\geq 0}$ and $r \geq 2n-2s$, the following two reduction formulae hold*:

(4.10)
$$ {}_7F_6\left(\begin{matrix} -r, -2n-r+s-\frac{3}{2}, -n-\frac{r}{2}+\frac{s}{2}+\frac{1}{4}, -2n-r-1, -r-s-\frac{1}{2}, s-n, -n+s-\frac{1}{2} \\ -2n+s-\frac{1}{2}, -n-\frac{r}{2}+\frac{s}{2}-\frac{3}{4}, s+\frac{1}{2}, 2s-2n, -n-r-\frac{1}{2}, -n-r \end{matrix}\,\middle|\, 1\right) = \frac{(2s)!\,(2n+r+1)!\,(s+1)_r\,\left(2n-s+\frac{3}{2}\right)_r}{(2n+1)!\,(r+2s)!\,(n+1)_r\,\left(n+\frac{3}{2}\right)_r},$$

and

(4.11)
$$ {}_7F_6\left(\begin{matrix} -r, -2n-r+s-\frac{1}{2}, -n-\frac{r}{2}+\frac{s}{2}+\frac{3}{4}, -2n-r-1, -r-s+\frac{1}{2}, s-n, -n+s+\frac{1}{2} \\ -2n+s+\frac{1}{2}, -n-\frac{r}{2}+\frac{s}{2}-\frac{1}{4}, s+\frac{3}{2}, 2s-2n, -n-r+\frac{1}{2}, -n-r \end{matrix}\,\middle|\, 1\right) = \frac{(2s+1)!\,(2n+r+1)!\,(s+1)_r\,\left(2n-s+\frac{1}{2}\right)_r}{(2n+1)!\,(2r+2s+1)\,(r+2s)!\,\left(n+\frac{1}{2}\right)_r\,(n+1)_r}.$$

Proof. The two reduction formulae (4.10) and (4.11), can be obtained with the aid of Theorem 4.4, and the two reduction formulae (3.5) and (3.13). □

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