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ON 2-SG-SEMISIMPLE RINGS

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ABSTRACT. In this paper, we investigate 2-SG-semisimple rings which are a particular kind of quasi-Frobenius rings over which all modules are periodic of period 2. Namely, we show that local 2-SG-semisimple rings are the same as the known Artinian valuation rings. Also, a relation between Dedekind domains and 2-SG-semisimple rings is established.

1. Introduction. Throughout this paper, all rings are commutative with identity element and all modules are unital. It is convenient to use *m-local* or (simply) *local* to refer to not necessarily Noetherian rings with a unique maximal ideal m. We assume that the reader is familiar with the Gorenstein homological algebra (some references are [**9, 10, 12**]).

For a ring R and a positive integer $n \geq 1$, an R-module M is said to be n-strongly Gorenstein projective (n-SG-projective for short), if there exists an exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0,$$

where each P_i is projective, such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence exact whenever Q is a projective R-module (see [**6**]). The 1-SG-projective module is simply called *strongly Gorenstein projective* (SG-projective for short) (see [**5**]). An extension of these kinds of modules was given in [**3**]. Namely, we have, for integers $n \geq 1$ and $m \geq 0$, a module M is called *(n,m)-SG-projective* if there exists an exact sequence of modules,

$$0 \longrightarrow M \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow M \longrightarrow 0,$$

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where $\mathrm{pd}(Q_i) \leq m$ for $1 \leq i \leq n$, such that $\mathrm{Ext}^i(M, Q) = 0$ for any $i > m$ and for any projective module Q. A general study of rings over which every module is (n, m)-SG-projective was done in [**4**], and such rings are called (n, m)-SG. Thus, as in classical homological dimension, the (n, m)-SG rings with small integers n and m would be of interest. Let us call by n-SG-*semisimple*, for an integer $n \geq 1$, the $(n, 0)$-SG rings. From [**4,** Corollary 2.8], n-SG-semisimple rings are a particular kind of quasi-Frobenius rings. In [**8**], it was proved that a local ring is 1-SG-semisimple if and only if it contains a unique non-trivial ideal.

The aim of this paper is to study 2-SG-semisimple rings. We prove that 2-SG-semisimple is the same as the well-known Artinian serial rings (see Corollary 2.7). Recall that a ring is called *serial* if it is a finite direct product of valuation rings, where a ring (not necessarily a domain) is called *valuation* if the lattice of all its ideals is linearly ordered under inclusions (see, for example, [**11,** pages 10 and 11]). Namely, we prove that a local ring is 2-SG-semisimple if and only if it is an Artinian valuation ring (see Theorem 2.6). Also, a relation between Dedekind domains and 2-SG-semisimple rings is established in Proposition 2.9.

Before starting, we need to recall some useful results about quasi-Frobenius rings (for more details about these kinds of rings, see, for example, [**14**]). The quasi-Frobenius rings have several characterizations, and here, we only need the following ones:

Theorem 1.1 ([**14**], Theorems 1.50, 7.55 and 7.56)**.** *For a ring R, the following are equivalent*:

(i) *R is quasi-Frobenius*;
(ii) *R is Artinian and self-injective*;
(iii) *every projective R-module is injective*;
(iv) *every injective R-module is projective*;
(v) *R is Noetherian and, for every ideal I,* $\mathrm{Ann}\,(\mathrm{Ann}\,(I)) = I$, *where* $\mathrm{Ann}\,(I)$ *denotes the annihilator of I.*

For the local case, we have the following result:

Theorem 1.2 ([**13**], Theorems 221)**.** *Let R be an m-local and zero-dimensional Noetherian ring. The following are equivalent*:

(i) *R is quasi-Frobenius*;
(ii) $\mathrm{Ann}\,(m)$ *is a principal ideal.*

We have the following structural characterization of quasi-Frobenius rings.

Proposition 1.3. *A ring R is quasi-Frobenius if and only if $R = R_1 \times \cdots \times R_n$, where each R_i is a local quasi-Frobenius ring.*

2. Main results. We aim to give an equivalent characterization of 2-SG-semisimple rings. The following leads us to restrict the study to the case of local rings.

Lemma 2.1 ([**4**], Proposition 2.13)**.** *A ring R is* 2-SG-*semisimple if and only if $R = R_1 \times \cdots \times R_n$, where each R_i is a local* 2-SG-*semisimple ring.*

Before giving the main result, we need the following lemmas.

The following result is a characterization of Artinian valuation local rings.

Lemma 2.2 ([**2**], Proposition 8.8)**.** *Let R be an Artinian m-local ring. Then the following assertions are equivalent*:

(i) *every ideal is principal*;
(ii) *the maximal ideal m is principal*;
(iii) *R is a valuation ring.*

In this case every ideal I of R is of the form $a^n R$ where a generates m.

The two results below investigate the 2-SG-projective modules over local quasi-Frobenius rings.

Lemma 2.3. *Let R be a local quasi-Frobenius ring and M a finitely generated R-module. If M is* 2-SG-*projective, then there is an exact sequence $0 \to M \to F_2 \to F_1 \to M \to 0$ where F_1 and F_2 are free and* *finitely generated R-modules. Furthermore, if M is an ideal of R, then the exact sequence can be of the form*:

$$0 \longrightarrow M \longrightarrow R \longrightarrow R^n \longrightarrow M \longrightarrow 0,$$

where n is a positive integer.

Proof. Let M be a finitely generated 2-SG-projective R-module. Then, by [**18,** Theorem 3.14], there exists an exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow F_2 \longrightarrow F_1 \longrightarrow M \longrightarrow 0$$

with F_1 and F_2 are finitely generated projective R-modules. Notice that R is local, so F_1 and F_2 are finitely generated free and the first assertion follows.

Now, suppose that M is an ideal of R. Decomposing the exact sequence $0 \to M \to F_2 \to F_1 \to M \to 0$ to get the short exact sequences: $0 \to M \to F_2 \to K \to 0$ and $0 \to K \to F_1 \to M \to 0$. Since R is quasi-Frobenius, F_1 and R are injective R-modules. Then, we can apply the dual of the horseshoe lemma [**15,** Note after Lemma 6.20] to the short exact sequences above with the canonical one, $0 \to M \to R \to R/M \to 0$, to get the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & R/M & \to & Q & \to & M & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & R & \to & R \oplus F_1 & \to & F_1 & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & M & \to & F_2 & \to & K & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

From the top horizontal sequence, Q is a Gorenstein projective and finitely generated R-module. Then, using the middle vertical sequence, Q has finite projective dimension. This shows, using [**12,** Proposition 2.27], that Q is projective and then free (since R is local). Then, there is a positive integer n such that $Q \cong R^n$. Finally, combining the top horizontal sequence with the left vertical one to get the desired sequence. $\square$

Corollary 2.4. *Let R be a local quasi-Frobenius ring, and let a be a zero-divisor element of R. If the principal ideal aR is* 2-SG*-projective, then* Ann$\,(a)$ *is also principal and there are exact sequences of the form*:

$$0 \longrightarrow aR \longrightarrow R \longrightarrow R \longrightarrow aR \longrightarrow 0$$
$$0 \longrightarrow \mathrm{Ann}\,(a) \longrightarrow R \longrightarrow R \longrightarrow \mathrm{Ann}\,(a) \longrightarrow 0$$
$$0 \longrightarrow R/aR \longrightarrow R \longrightarrow R \longrightarrow R/aR \longrightarrow 0$$

Proof. By Lemma 2.3, we have an exact sequence of the form:

$$0 \longrightarrow R/aR \longrightarrow R^n \longrightarrow aR \longrightarrow 0$$

where n is a positive integer. By the Schanuel lemma [**15,** Theorem 9.4 (i)], the above exact sequence with the following canonical one:

$$0 \longrightarrow \mathrm{Ann}\,(aR) \longrightarrow R \longrightarrow aR \longrightarrow 0$$

implies that $\mathrm{Ann}\,(a) \oplus R^n \cong R/aR \oplus R$. This shows that $\mathrm{Ann}\,(a)$ must be principal and $n = 1$ which help to construct the desired sequences. $\square$

The structure of modules over Artinian serial rings is given by the following well-known result.

Lemma 2.5 ([**11**], Theorems 5.6)**.** *Let R be an Artinian serial ring. Then every R-module is a direct sum of cyclic modules.*

Now we are in position to give the main result.

Theorem 2.6. *An m-local ring R is* 2-SG*-semisimple if and only if it is an Artinian valuation ring.*

Proof. If R is 2-SG-semisimple, then it is quasi-Frobenius (by [**4,** Corollary 2.8]). Then, by Theorem 1.2, $\mathrm{Ann}\,(m)$ is principal. This shows, using Corollary 2.4 and Theorem 1.1, that $m = \mathrm{Ann}\,(\mathrm{Ann}\,(m))$ is principal. Therefore, R is a valuation ring (by Lemma 2.2). Conversely, assume that R is an Artinian valuation ring. Obviously, R is quasi-Frobenius with only principal ideals. Then, for every zero-divisor element a of R, we have the exact sequences $0 \to \mathrm{Ann}\,(a) \to R \to$

$aR \to 0$ and $0 \to aR = \mathrm{Ann}\,(\mathrm{Ann}\,(a)) \to R \to \mathrm{Ann}\,(a) \to 0$. Combining these sequences, we deduce that aR is 2-SG-projective. Then, from Corollary 2.4, the cyclic module R/aR is also 2-SG-projective and so are all cyclic modules including the free ones. Therefore, Lemma 2.5 with [**3,** Proposition 2.3] show that every module is 2-SG-projective and therefore R is 2-SG-semisimple. □

From Lemma 2.1, the structure of 2-SG-semisimple rings is immediately deduced as follows.

Corollary 2.7. *A ring R is* 2-SG-*semisimple if and only if it is an Artinian serial ring.*

To construct examples of 2-SG-semisimple rings, one can use the well-known result that nontrivial factor rings of Dedekind domains are principal Artinian serial rings, which means that nontrivial factor rings of Dedekind domains are 2-SG-semisimple (see, for example, [**17,** Corollary, page 278]). The following result (Proposition 2.9) shows that the Dedekind domains is closely related to the 2-SG-semisimple rings in the sense that the converse of the well-known result above holds true. To prove this result, we use the following lemma.

Lemma 2.8. *Let R be a domain and P a maximal ideal of R which is finitely generated. Then P is invertible if and only if P_P (i.e., PR_P) is a principal ideal of R_P.*

Proof. By [**16,** Theorem 8.4.2], P is invertible if and only if $P_{\mathfrak{m}}$ is principal for any maximal ideal $\mathfrak{m}$ of R. Since P is maximal, $P_{\mathfrak{m}} = R_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ other than P. □

Proposition 2.9. *A domain R is Dedekind if and only if every nontrivial factor ring of R is* 2-SG-*semisimple.*

Proof. If every nontrivial factor ring of R is 2-SG-semisimple, then, by [**13,** Theorem 90], R must be one-dimensional and Noetherian. So, by [**1,** Theorem 3], R must be a Dedekind domain. We give a direct proof here. Let P be a maximal ideal of R, and let a be an element in P which is not in P^2. Since R/P^2 is a QF-ring, by Theorem 1.1, $(\overline{a}) = \mathrm{Ann}\,(\mathrm{Ann}\,(\overline{a}))$. Since $(P/P^2)^2 = 0$, it can be seen that $\operatorname{Ann}(\operatorname{Ann}(\overline{a})) = P/P^2$. Therefore $(\overline{a}) = P/P^2$. So $Ra + P^2 = P$ and by the Nakayama lemma, $P_P = (a)_P$. Thus, by Lemma 2.8, P is invertible, and this means that R is a Dedekind domain.

For the "only if" part, let I be a proper ideal of a Dedekind domain R. Then $I = P_1^{t_1} P_2^{t_2} \cdots P_n^{t_n}$ for some prime ideals $P_1,\ P_2, \cdots, P_n$ and some integers $t_1,\ t_2, \cdots, t_n$. By the Chinese remainder theorem, $R/I \cong R/P_1^{t_1} \bigoplus R/P_2^{t_2} \bigoplus \cdots \bigoplus R/P_n^{t_n}$. In order to show that R/I is 2-SG-semisimple, we only need to prove that $R/P_i^{t_i}$ is such a ring. When $t_i = 1$, the field R/P_i is certainly 2-SG-semisimple. Therefore, we can assume that $t_i > 1$. Since $R/P_i^{t_i}$ is an Artinian local ring, by Lemma 2.2 and Theorem 2.6, it suffices to prove that the maximal ideal $P_i/P_i^{t_i}$ is principal. By [**16,** Corollary 9.8.7], we can choose an element $b \in P_i^{t_i}$ and an element $c \in P_i$ such that $P_i = (b, c)$. Therefore, $P_i/P_i^{t_i} = (c + P_i^{t_i})$ is principal. $\square$

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