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UPPER AND LOWER BOUNDS FOR THE NUMERICAL RADIUS WITH AN APPLICATION TO INVOLUTION OPERATORS

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ABSTRACT. New upper and lower bounds for the numerical radii of Hilbert space operators are given. An application to involution operators is also provided.

1. Introduction. Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$. For $A \in \mathcal{B}(\mathcal{H})$, let $r(A)$, $w(A)$ and $\|A\|$ denote the spectral radius, the numerical radius and the operator norm of A, respectively. Also, let $m(A)$ be the nonnegative number defined by

$$m(A) = \inf_{\|x\|=1} |\langle Ax, x \rangle| .$$

Recall that $w(A) = \sup\limits_{\|x\|=1} |\langle Ax, x \rangle|$. It is well-known that, for every $A \in \mathcal{B}(\mathcal{H})$,

$$r(A) \le w(A) \tag{1.1}$$

with equality if A is normal. Moreover, $w(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the operator norm $\|\cdot\|$. In fact, for every $A \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}\|A\| \le w(A) \le \|A\| . \tag{1.2}$$

The inequalities in (1.2) are sharp. The first inequality becomes an equality if $A^2 = 0$. The second inequality becomes an equality if A is normal. Another basic fact about the numerical radius is the *power*

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inequality, which asserts that

$$w\left(A^n\right) \le w^n\left(A\right) \quad \text{for } n = 1, 2, \dots . \tag{1.3}$$

For proofs and more facts about the numerical radius, we refer the reader to **[2, 3]**.

Kittaneh has shown in **[5, 7]**, respectively, that, if $A \in \mathcal{B}(\mathcal{H})$, then

$$w(A) \le \frac{1}{2}\left(\|A\| + \left\|A^2\right\|^{1/2}\right) \tag{1.4}$$

and

$$\frac{1}{4}\left\||A|^2 + |A^*|^2\right\| \le w^2(A) \le \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|. \tag{1.5}$$

Obviously, the inequality (1.4) is sharper than the second inequality in (1.2), and the inequalities (1.5) refine the inequalities (1.2).

In Section 2, we establish a considerable improvement of the inequalities (1.2), which also refines the inequalities (1.4) and (1.5) of Kittaneh.

In Section 3, we utilize the main result obtained in Section 2 to compute the numerical radii of involution operators and compute the operator norms of their real and imaginary parts.

2. Upper and lower bounds for the numerical radius.

In order to achieve our goal, we need the following three lemmas. The first lemma is well known (see, e.g., **[9]**). It gives a useful characterization of the numerical radius.

Lemma 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$w(A) = \sup_{\theta \in \mathbb{R}} \left\|Re\left(e^{i\theta}A\right)\right\|. \tag{2.1}$$

The second lemma, which can be found in **[4]**, gives estimations of the operator norms of 2×2 operator matrices, regarded as operators on $\mathcal{H}_1 \oplus \mathcal{H}_2$.

Lemma 2.2. *Let $A \in \mathcal{B}(\mathcal{H}_1)$, $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $D \in \mathcal{B}(\mathcal{H}_2)$. Then*

$$\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix} \right\|.$$

Here $\mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$ is the space of all bounded linear operators from $\mathcal{H}_j$ to $\mathcal{H}_i$.

The third lemma contains a special case of a more general inequality for sums of positive operators that is sharper than the triangle inequality. See [**6**].

Lemma 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\left\| |A|^2 + |A^*|^2 \right\| \leq \left\| A^2 \right\| + \|A\|^2 . \tag{2.2}$$

Now, we are ready to present our new improvement of the inequalities (1.2).

Theorem 2.4. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} \frac{1}{2}\sqrt{\left\| |A|^2 + |A^*|^2 \right\| + 2m\left(A^2\right)} &\leq w\left(A\right) \\ &\leq \frac{1}{2}\sqrt{\left\| |A|^2 + |A^*|^2 \right\| + 2w\left(A^2\right)}. \end{aligned} \tag{2.3}$$

Proof. Let x be a unit vector in $\mathcal{H}$, and let ψ be a real number such that $e^{2i\psi}\langle A^2x, x\rangle = |\langle A^2x, x\rangle|$. Then we have

$$\begin{aligned} w\left(A\right) &\geq \left\| \mathrm{Re}\left(e^{i\psi}A\right) \right\| = \frac{1}{2}\left\| e^{i\psi}A + e^{-i\psi}A^* \right\| \\ &= \frac{1}{2}\left\| \left(e^{i\psi}A + e^{-i\psi}A^*\right)^2 \right\|^{\frac{1}{2}} \\ &= \frac{1}{2}\sqrt{\left\| |A|^2 + |A^*|^2 + 2\mathrm{Re}\left(e^{2i\psi}A^2\right) \right\|} \\ &\geq \frac{1}{2}\sqrt{\left| \left\langle \left(|A|^2 + |A^*|^2 + 2\mathrm{Re}\left(e^{2i\psi}A^2\right)\right)x, x \right\rangle \right|} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sqrt{\left|\left\langle (|A|^2+|A^*|^2)x,x\right\rangle + 2\left\langle \mathrm{Re}\,(e^{2i\psi}A^2)x,x\right\rangle\right|} \\
&= \frac{1}{2}\sqrt{\left|\left\langle (|A|^2+|A^*|^2)x,x\right\rangle + 2\mathrm{Re}\,(e^{2i\psi}\left\langle A^2x,x\right\rangle)\right|} \\
&= \frac{1}{2}\sqrt{\left\langle (|A|^2+|A^*|^2)x,x\right\rangle + 2\left|\left\langle A^2x,x\right\rangle\right|} \\
&\geq \frac{1}{2}\sqrt{\left\langle (|A|^2+|A^*|^2)x,x\right\rangle + 2m(A^2)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
w\,(A) &\geq \frac{1}{2}\sup_{\|x\|=1}\sqrt{\left\langle \left(|A|^2+|A^*|^2\right)x,x\right\rangle + 2m\,(A^2)} \\
&= \frac{1}{2}\sqrt{\left\||A|^2+|A^*|^2\right\| + 2m\,(A^2)},
\end{aligned}
$$

which proves the first inequality in (2.3).

To prove the second inequality in (2.3), note that, by Lemma 1, we have

$$
\begin{aligned}
w\,(A) &= \sup_{\theta\in\mathbb{R}}\left\|\mathrm{Re}\,(e^{i\theta}A)\right\| \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|e^{i\theta}A+e^{-i\theta}A^*\right\| \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|(e^{i\theta}A+e^{-i\theta}A^*)^2\right\|^{1/2} \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\||A|^2+|A^*|^2+2\mathrm{Re}\,(e^{2i\theta}A^2)\right\|^{1/2} \\
&\leq \frac{1}{2}\sqrt{\left\||A|^2+|A^*|^2\right\| + 2\sup_{\theta\in\mathbb{R}}\left\|\mathrm{Re}\,(e^{2i\theta}A^2)\right\|} \\
&= \frac{1}{2}\sqrt{\left\||A|^2+|A^*|^2\right\| + 2w(A^2)},
\end{aligned}
$$

which proves the second inequality in (2.3) and completes the proof of the theorem. □

Remark 2.5. Using the power inequality (1.3) and the second inequality in (2.3), we have

$$w^2(A) \le 2w^2(A) - w\left(A^2\right) \le \frac{1}{2}\left\| |A|^2 + |A^*|^2 \right\|,$$

which shows that the second inequality in (2.3) is sharper than the second inequality in (1.5). Obviously, the first inequality in (2.3) is sharper than the first inequality in (1.5).

The following corollary shows that the second inequality in (2.3) is sharper than the inequality (1.4).

Corollary 2.6. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} w(A) &\le \frac{1}{2}\sqrt{\|A\|^2 + \|A^2\| + 2w(A^2)} \\ &\le \frac{1}{2}\sqrt{\|A\|^2 + 3\|A^2\|} \\ &\le \frac{1}{2}\left(\|A\| + \|A^2\|^{1/2}\right). \end{aligned} \tag{2.4}$$

Proof. The first inequality in (2.4) follows directly from the second inequality in (2.3) and Lemma 2.3. The second inequality in (2.4) follows from the first inequality in (2.4) by noting that $w(A^2) \le \|A^2\|$.

To prove the last inequality in (2.4), note that

$$\|A^2\| = \|A^2\|^{1/2}\|A^2\|^{1/2} \le \|A\|\,\|A^2\|^{1/2},$$

and so

$$\begin{aligned} \frac{1}{2}\sqrt{\|A\|^2 + 3\|A^2\|} &= \frac{1}{2}\sqrt{\|A\|^2 + 2\|A^2\| + \|A^2\|} \\ &\le \frac{1}{2}\sqrt{\|A\|^2 + 2\|A\|\,\|A^2\|^{1/2} + \|A^2\|} \\ &= \frac{1}{2}\sqrt{\left(\|A\| + \|A^2\|^{1/2}\right)^2} \\ &= \frac{1}{2}\left(\|A\| + \|A^2\|^{1/2}\right), \end{aligned}$$

as required. □

3. The numerical radii of involution operators. Recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is said to be idempotent if $A^2 = A$, 2-nilpotent if $A^2 = 0$ and an involution if $A^2 = I$.

Furuta [**1**] has used the polar decomposition to compute the numerical radii of idempotent and 2-nilpotent operators. In fact, Furuta has proved the following theorem.

Theorem 3.1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then the following statements hold*:

(i) If A is an idempotent operator such that $A \neq 0$, then

$$w(A) = \|\operatorname{Re} A\| = \frac{1}{2}\left(1 + \|A\|\right)$$

and

$$\|\operatorname{Im} A\| = \frac{1}{2}\left(\|A\|^2 - 1\right)^{1/2}.$$

(ii) If A is a 2-nilpotent operator such that $A \neq 0$, then

$$w(A) = \|\operatorname{Re} A\| = \|\operatorname{Im} A\| = \frac{1}{2}\|A\|.$$

It follows from the inequalities (2.3) that, if $A \in \mathcal{B}(\mathcal{H})$ is an involution operator, then

$$w(A) = \frac{1}{2}\sqrt{\left\||A|^2 + |A^*|^2\right\| + 2}. \tag{3.1}$$

In this section, we apply relation (3.1) to compute the numerical radii of involution operators, and compute the operator norms of their real and imaginary parts. To achieve our goal, we need the following lemma in which we compute the operator norms of certain 2×2 operator matrices. A special case of this lemma has been proved by Paul and Bag [**8**]. Our approach here is different from theirs.

Lemma 3.2. *Let*

$$A = \begin{bmatrix} aI & T \\ 0 & bI \end{bmatrix}$$

be an operator on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $T \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, and let a, b be complex numbers. Then

$$\|A\| = \frac{1}{\sqrt{2}} \sqrt{|a|^2 + |b|^2 + \|T\|^2 + \sqrt{\left(|a|^2 + |b|^2 + \|T\|^2\right)^2 - 4|a|^2|b|^2}}.$$

Proof. By Lemma 2, we have

$$\|A\| \le \left\| \begin{bmatrix} |a| & \|T\| \\ 0 & |b| \end{bmatrix} \right\| = \left\| \begin{bmatrix} |a| & \|T\| \\ 0 & |b| \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\| \tag{3.2}$$

for some nonnegative real numbers α, β with $\alpha^2 + \beta^2 = 1$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of unit vectors in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, such that $\lim_{n\to\infty} |\langle Ty_n, x_n\rangle| = \|T\|$. For $n \in \mathbb{N}$, let θ_n be a real number such that $\overline{a}\langle Ty_n, x_n\rangle = e^{i\theta_n}|a||\langle Ty_n, x_n\rangle|$. Consider the sequence

$$\{z_n\} = \left\{ \begin{bmatrix} \alpha e^{i\theta_n} x_n \\ \beta y_n \end{bmatrix} \right\}.$$

It is easy to see that $\{z_n\}$ is a sequence of unit vectors in $\mathcal{H}_1 \oplus \mathcal{H}_2$ and

$$\begin{aligned}
\|Az_n\| &= \left\| \begin{bmatrix} \alpha a e^{i\theta_n} x_n + \beta T y_n \\ \beta b y_n \end{bmatrix} \right\| \\
&= \sqrt{\|\alpha a e^{i\theta_n} x_n + \beta T y_n\|^2 + \|\beta b y_n\|^2} \\
&= \sqrt{\alpha^2 |a|^2 + \beta^2 \|Ty_n\|^2 + 2\alpha\beta \mathrm{Re}\, (e^{-i\theta_n} \overline{a} \langle Ty_n, x_n\rangle) + \beta^2 |b|^2} \\
&= \sqrt{\alpha^2 |a|^2 + \beta^2 \|Ty_n\|^2 + 2\alpha\beta |a| |\langle Ty_n, x_n\rangle| + \beta^2 |b|^2} \\
&\longrightarrow \sqrt{\alpha^2 |a|^2 + \beta^2 \|T\|^2 + 2\alpha\beta |a| \|T\| + \beta^2 |b|^2} \quad (\text{as } n \to \infty) \\
&= \sqrt{(\alpha |a| + \beta \|T\|)^2 + \beta^2 |b|^2} \\
&= \left\| \begin{bmatrix} |a| & \|T\| \\ 0 & |b| \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\| = \left\| \begin{bmatrix} |a| & \|T\| \\ 0 & |b| \end{bmatrix} \right\|.
\end{aligned}$$

Thus,

$$\|A\| \ge \left\| \begin{bmatrix} |a| & \|T\| \\ 0 & |b| \end{bmatrix} \right\|. \tag{3.3}$$

By the inequalities (3.2) and (3.3), we deduce that

$$\|A\| = \left\| \begin{bmatrix} |a| & \|T\| \\ 0 & |b| \end{bmatrix} \right\|.$$

But,

$$\begin{aligned}
\left\| \begin{bmatrix} |a| & \|T\| \\ 0 & |b| \end{bmatrix} \right\| &= \left\| \begin{bmatrix} |a| & 0 \\ \|T\| & |b| \end{bmatrix} \begin{bmatrix} |a| & \|T\| \\ 0 & |b| \end{bmatrix} \right\|^{1/2} \\
&\quad (\text{since } \|X\| = \|X^* X\|^{1/2}) \\
&= r^{1/2} \left(\begin{bmatrix} |a| & 0 \\ \|T\| & |b| \end{bmatrix} \begin{bmatrix} |a| & \|T\| \\ 0 & |b| \end{bmatrix} \right) \\
&= r^{1/2} \left(\begin{bmatrix} |a|^2 & |a|\,\|T\| \\ |a|\,\|T\| & |b|^2 + \|T\|^2 \end{bmatrix} \right) \\
&= \frac{1}{\sqrt{2}} \sqrt{|a|^2 + |b|^2 + \|T\|^2 + \sqrt{\left(|a|^2 + |b|^2 + \|T\|^2\right)^2 - 4\,|a|^2\,|b|^2}},
\end{aligned}$$

which completes the proof of the lemma. □

Now, we apply relation (3.1) and Lemma 3.2 to prove the following theorem.

Theorem 3.3. *If $A \in \mathcal{B}(\mathcal{H})$ is an involution operator, then the following relations hold*:

(i) $w(A) = \frac{1}{2}(\|A\| + \|A\|^{-1})$.
(ii) $\|\operatorname{Re} A\| = \frac{1}{2}(\|A\| + \|A\|^{-1})$ and $\|\operatorname{Im} A\| = \frac{1}{2}(\|A\| - \|A\|^{-1})$.

Proof. Since $A^2 = I$, A can be represented, with respect to an appropriate decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, as

$$A = \begin{bmatrix} I & T \\ 0 & -I \end{bmatrix},$$

where $T \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$. By relation (3.1), we have

$$\begin{aligned}
w(A) &= \frac{1}{2} \sqrt{\left\| |A|^2 + |A^*|^2 \right\| + 2} \\
&= \frac{1}{2} \left(\left\| \begin{bmatrix} I & 0 \\ T^* & -I \end{bmatrix} \begin{bmatrix} I & T \\ 0 & -I \end{bmatrix} \right. \right.
\end{aligned}$$

$$
\begin{aligned}
&\left. + \begin{bmatrix} I & T \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ T^* & -I \end{bmatrix} \right\| + 2 \Bigg)^{1/2} \\
&= \frac{1}{2}\left(\left\| \begin{bmatrix} I & T \\ T^* & T^*T + I \end{bmatrix} + \begin{bmatrix} TT^* + I & -T \\ -T^* & I \end{bmatrix} \right\| + 2 \right)^{1/2} \\
&= \frac{1}{2}\left(\left\| \begin{bmatrix} TT^* + 2I & 0 \\ 0 & T^*T + 2I \end{bmatrix} \right\| + 2 \right)^{1/2} \\
&= \frac{1}{2}\left(\|TT^* + 2I\| + 2 \right)^{1/2},
\end{aligned}
$$

and so

$$
w(A) = \frac{1}{2}\sqrt{\|T\|^2 + 4}. \tag{3.4}
$$

Also, by Lemma 3.2, we have

$$
\begin{aligned}
\|A\| &= \frac{1}{\sqrt{2}}\sqrt{2 + \|T\|^2 + \sqrt{\|T\|^4 + 4\|T\|^2}} \\
&= \frac{1}{2}\sqrt{\|T\|^2 + 4} + \frac{1}{2}\|T\|.
\end{aligned} \tag{3.5}
$$

From relations (3.4) and (3.5), we conclude that

$$
w(A) = \frac{1}{2}\left(\|A\| + \|A\|^{-1} \right),
$$

which proves part (i).

To prove part (ii), note that

$$
\begin{aligned}
\|\operatorname{Re} A\| &= \left\| \begin{bmatrix} I & \frac{T}{2} \\ \frac{T^*}{2} & -I \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} I & \frac{T}{2} \\ \frac{T^*}{2} & -I \end{bmatrix} \begin{bmatrix} I & \frac{T}{2} \\ \frac{T^*}{2} & -I \end{bmatrix} \right\|^{1/2} \\
&= \left\| \begin{bmatrix} \frac{TT^*}{4} + I & 0 \\ 0 & \frac{T^*T}{4} + I \end{bmatrix} \right\|^{1/2} \\
&= \frac{1}{2}\sqrt{\|T\|^2 + 4}
\end{aligned}
$$

and

$$\begin{aligned}\|\operatorname{Im} A\| &= \left\| \begin{bmatrix} 0 & \frac{T}{2i} \\ -\frac{T^*}{2i} & 0 \end{bmatrix} \right\| \\ &= \frac{1}{2} \|T\| .\end{aligned}$$

Thus,

$$\|\operatorname{Re} A\| = w(A) = \frac{1}{2}\left(\|A\| + \|A\|^{-1}\right)$$

and

$$\begin{aligned}\|\mathrm{Im}A\| &= \|A\| - w(A) \\ &= \|A\| - \frac{1}{2}\left(\|A\| + \|A\|^{-1}\right) \\ &= \frac{1}{2}\left(\|A\| - \|A\|^{-1}\right),\end{aligned}$$

which proves part (ii) and completes the proof of the theorem. $\square$

We remark here that the finite-dimensional version of part (i) in Theorem 3.3 has been given in [**8**] using a completely different argument.

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