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FORMAL FIBERS WITH COUNTABLY MANY MAXIMAL ELEMENTS

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ABSTRACT. Let T be a complete local (Noetherian) ring. Let C be a countable set of pairwise incomparable non-maximal prime ideals of T. We find necessary and sufficient conditions for T to be the completion of a local integral domain whose generic formal fiber has maximal elements precisely the elements of C. Furthermore, if the characteristic of T is zero, we provide necessary and sufficient conditions for T to be the completion of an *excellent* local integral domain whose generic formal fiber has maximal elements precisely the elements of C. In addition, for a positive integer k, we construct local integral domains that contain a prime ideal of height k whose formal fiber has countably many maximal elements.

1. Introduction. Suppose T is a complete local (Noetherian) ring, and let G be a set of prime ideals of T. We are interested in determining necessary and sufficient conditions so that T is the completion of a local (Noetherian) domain A where A is a subring of T and

$$\{P \in \operatorname{Spec} T \mid P \cap A = (0)\} = G.$$

Theorem 3.1 in **[2]** answers the above question in the case where the number of maximal elements of G is finite. In this paper, we answer the question in the case where the number of maximal elements of G is countable. The question is still open if the number of maximal elements of G is uncountable.

When we say a ring is *local*, we mean that it is Noetherian and has exactly one maximal ideal. Let A be a local ring and Q a prime ideal of A. The formal fiber of A at Q is defined to be $\operatorname{Spec}(\widehat{A} \otimes_A k(Q))$ where $\widehat{A}$ is the completion of A at its maximal ideal and $k(Q)$ is A_Q/QA_Q. Since there is a one-to-one correspondence between prime ideals of the ring $\widehat{A} \otimes_A k(Q)$ and prime ideals of $\widehat{A}$, whose intersection with A is

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Q, we will abuse notation by saying that if P is a prime ideal of $\widehat{A}$ satisfying $A \cap P = Q$, then P is in the formal fiber of A at Q. If A is an integral domain, the formal fiber of A at (0) is called the generic formal fiber of A. So, by our abuse of notation, the generic formal fiber of A is the set

$$\{P \in \operatorname{Spec} \widehat{A} \mid P \cap A = (0)\}.$$

Our main question, then, can be rephrased as: given a complete local ring T, what sets of prime ideals of T can be realized as the generic formal fiber of a local integral domain whose completion is T?

In [**2**], the following result is proved, answering our main question in the case where the number of maximal elements of the set G is finite.

Theorem 1.1. [**2, Theorem 3.1**]. *Let T be a complete local ring with prime subring Π, maximal ideal M and $G \subseteq \operatorname{Spec} T$ such that G is nonempty and the number of maximal elements of G is finite. Then there exists a local domain A such that $\widehat{A} = T$ and the generic formal fiber of A is exactly the elements of G if and only if T is a field and $G = \{(0)\}$ or the following conditions hold*:

(1) *$M \notin G$, and G contains all the associated prime ideals of T*;
(2) *If $Q \in G$ and $P \in \operatorname{Spec} T$ with $P \subseteq Q$, then $P \in G$*;
(3) *If $Q \in G$, then $Q \cap \Pi = (0)$.*

In Section 2, we show that Theorem 1.1 holds with the condition "the number of maximal elements of G is finite" weakened to "the number of maximal elements of G is countable."

We also consider the case where A is excellent. In other words, if T is a complete local ring and G is a set of prime ideals of T, we are interested in finding necessary and sufficient conditions so that T is the completion of an *excellent* local domain A where A is a subring of T and

$$\{P \in \operatorname{Spec} T \mid P \cap A = (0)\} = G.$$

If T contains the integers, then the following theorem from [**2**] answers the question in the case where the number of maximal elements of G is finite.

Theorem 1.2. [2, Theorem 4.1]. *Let T be a complete local ring containing the integers, let M denote the maximal ideal of T, and let Π denote the prime subring of T. Let $G \subseteq \operatorname{Spec} T$ be such that G is nonempty and the number of maximal elements of G is finite. Then there exists an excellent local domain A with $\widehat{A} = T$ and such that A has generic formal fiber exactly G if and only if T is a field and $G = \{(0)\}$ or the following conditions hold*:

(1) *$M \notin G$, and G contains all the associated prime ideals of T*;
(2) *If $Q \in G$ and $P \in \operatorname{Spec} T$ with $P \subseteq Q$, then $P \in G$*;
(3) *If $Q \in G$, then $Q \cap \Pi = (0)$*;
(4) *T is equidimensional*;
(5) *T_P is a regular local ring for all maximal elements $P \in G$.*

In this paper, we generalize Theorem 1.2 in the same way we generalize Theorem 1.1. In other words, we show that Theorem 1.2 holds with the condition "the number of maximal elements of G is finite" weakened to "the number of maximal elements of G is countable."

Finally, we use our results to control the formal fibers of nonzero prime ideals. There is much less known about constructing rings whose formal fibers at a nonzero prime ideal have a given set of maximal elements. We call a formal fiber semilocal if the number of maximal elements of the formal fiber is finite. In **[1, 3, 5]**, results about semilocal formal fibers at height-one prime ideals are proved. We know of only one paper, however, in which results about semilocal formal fibers at prime ideals of height greater than one are proved. In **[4]**, the authors demonstrate a class of integral domains that have a prime ideal with height greater than one that has a semilocal formal fiber.

In Section 3, we provide a new class of such integral domains. Our class, however, also contains integral domains with nonzero prime ideals whose formal fibers have a countably infinite number of maximal elements. For example, we construct an excellent local domain S such that the completion of S is $\mathbb{Q}[[y_1, y_2, y_3, x_1, x_2, x_3]]$, where $y_1, y_2, y_3, x_1, x_2, x_3$ are indeterminates, and such that the ideal $(x_1, x_2, x_3)S$ is a height 3 prime ideal of S whose formal fiber has maximal elements $\{(y_1 - qy_2, x_1, x_2, x_3) \mid q \in \mathbb{Q}\}$ (see Example 3.5). Our method for constructing these integral domains is different than those in [**4**], and our proofs are much less technical.

All rings in this paper are assumed to be commutative with unity. When we say that a ring is quasi-local, we mean it is a ring with one maximal ideal that is not necessarily Noetherian. As noted earlier, we use the convention that a local ring is a Noetherian ring with exactly one maximal ideal. When we say that (T, M) is a local ring, we mean that T is a local ring with maximal ideal M.

2. The generic formal fiber. The techniques for our proofs are based on those in [**2**]. Suppose that (T, M) is a complete local ring and G is a set of prime ideals of T such that the set of maximal elements of G is countable. We show that the conditions given in Theorem 1.1 are both necessary and sufficient for there to exist a local domain A such that $\widehat{A} = T$, and the generic formal fiber of A is exactly the elements of G. Showing the conditions are necessary is straightforward. The bulk of the proof, therefore, is dedicated to showing the conditions are sufficient. Assuming that the conditions hold, we construct the desired integral domain A. We now outline the ideas for constructing such an A.

We start with the prime subring of T and carefully adjoin elements so that our final ring A satisfies the following conditions.

(1) $\widehat{A} = T$;
(2) If $P \in G$, then $P \cap A = (0)$;
(3) If $P \notin G$, then $P \cap A \neq (0)$.

Since we are assuming that all of the associated prime ideals of T are contained in G, condition (2) gives us that A is an integral domain. The above conditions also imply that the generic formal fiber of A is G. So A will be our desired integral domain.

To satisfy condition (1), we adjoin "enough" elements to the prime subring of T so that $\widehat{A} = T$. But, as we do this, it is clear that to get condition (2) to hold for A, we must be careful never to adjoin a nonzero element of a prime ideal in the set G. Likewise, if P is a prime ideal not in G, then we must, at some point, adjoin a nonzero element of P to get condition (3) to hold. We use [**7,** Proposition 2.1], to ensure that we have adjoined "enough" elements to the prime subring to guarantee $\widehat{A} = T$.

Proposition 2.1. [**7, Proposition 1**]. *If $(A, M \cap A)$ is a quasi-local subring of a complete local ring (T, M), the map $A \to T/M^2$ is onto and $IT \cap A = I$ for every finitely generated ideal I of A, then A is Noetherian and the natural homomorphism $\widehat{A} \to T$ is an isomorphism.*

Our ring A, then, will satisfy the conditions that $A \to T/M^2$ is onto and, if I is a finitely generated ideal of A, then $IT \cap A = I$. Keeping in mind that we cannot adjoin any nonzero elements of prime ideals in G, we introduce the following definition, found in [**2**].

Definition 2.2. Let (T, M) be a complete local ring, and let C be a set of prime ideals of T. Suppose that $(R, R \cap M)$ is a quasi-local subring of T such that $|R| < |T|$ and $R \cap P = (0)$ for every $P \in C$. Then we call R a small C-avoiding subring of T and will denote it by SCA-subring.

Suppose that T is not a field. Letting C be the set of maximal elements of G, it is easy to see that, given our conditions, the prime subring Π of T is localized at $\Pi \cap M$ is an SCA-subring. As we adjoin elements, we ensure that we maintain the properties of SCA-subrings. This will guarantee that we never adjoin a nonzero element of a prime ideal in G. Suppose that R is an SCA-subring of T and let $x \in T$. If $P \in G$ and $x + P \in T/P$ is transcendental over R, then $R[x] \cap P = (0)$. To maintain the SCA-subring properties, then, we adjoin elements x of T such that $x + P \in T/P$ is transcendental over R for all $P \in G$. The construction in [**2**] uses the following lemma to adjoin these transcendental elements.

Lemma 2.3. [**2, Lemma 2.4**]. *Let (T, M) be a complete local ring such that $\dim T \geq 1$, C a finite set of nonmaximal prime ideals of T such that no ideal in C is contained in another ideal of C, and let D be a subset of T such that $|D| < |T|$. Let I be an ideal of T such that $I \not\subseteq P$ for all $P \in C$. Then $I \not\subseteq \bigcup\{r + P \mid r \in D,\ P \in C\}$.*

It is Lemma 2.3 that imposes the condition that the set of maximal elements of the generic formal fiber should be finite in the result in [**2**]. To obtain our result for the case where C is countable, we strengthen Lemma 2.3. To do this, we need the following lemma from [**6**].

Lemma 2.4. [6, Lemma 2]. *Let T be a complete local ring with maximal ideal M, C a countable set of primes in* $\operatorname{Spec} T$ *such that $M \notin C$, and let D be a countable set of elements of T. If I is an ideal of T which is contained in no single P in C, then $I \not\subseteq \bigcup\{r+P \mid r \in D, P \in C\}$.*

We also use [**5,** Lemma 2.2] and [**2,** Lemma 2.2], which we state here.

Lemma 2.5. [5, Lemma 2.2]. *Let (T, M) be a complete local ring of dimension at least one. Let P be a nonmaximal prime ideal of T. Then $|T/P| = |T| \geq |\mathbb{R}|$, where $\mathbb{R}$ denotes the set of real numbers.*

Lemma 2.6. [2, Lemma 2.2]. *Let T be an integral domain and I a nonzero ideal of T. Then $|I| = |T|$.*

We now state and prove our result that strengthens Lemma 2.3. Although we use Lemma 2.7 for a very specific purpose in this paper (that is, to adjoin transcendental elements), it is interesting in its own right as a generalization of the Prime Avoidance lemma.

Lemma 2.7. *Let (T, M) be a complete local ring such that* $\dim T \geq 1$, *let C be a countable set of incomparable nonmaximal prime ideals of T, and let D be a subset of T such that $|D| < |T|$. Let I be an ideal of T such that $I \not\subseteq P$ for all $P \in C$. Then $I \not\subseteq \bigcup\{r + P \mid r \in D, P \in C\}$.*

Proof. The case when C is finite holds by Lemma 2.3, so suppose C is infinite. Since C is countable, we may suppose $C = \{P_i\}_{i=1}^{\infty}$. By Lemma 2.4, with $D = \{0\}$, we have that $I \not\subseteq \cup_{i=1}^{\infty} P_i$. Let $x \in I \setminus \cup_{i=1}^{\infty} P_i$. For $r \in D$ and $P_i \in C$, either $r + P_i \notin \langle x + P_i \rangle$, where $\langle x + P_i \rangle$ is the ideal of T/P_i generated by $x + P_i$, or

$$r + P_i = (x + P_i)(s + P_i)$$

for some $s \in T$. For each $r \in D$ satisfying $r + P_i \in \langle x + P_i \rangle$, choose one such s and define a family of functions $f_i : D \to T$ for all $i \in \mathbb{N}$ as follows:

$$f_i(r) = \begin{cases} 0 & \text{if } r + P_i \notin \langle x + P_i \rangle, \\ s & \text{if } r + P_i = (x + P_i)(s + P_i). \end{cases}$$

Let $S_i = \text{Image}\,(f_i)$, and note that $|T| > |D| \geq |S_i|$ for every i.

We now construct a sequence $\{t_i\}_{i=1}^{\infty}$ of elements of T such that the following properties hold:

(1) $t_n \in \cap_{i=1}^{n-1} P_i \cap M^n$;
(2) $\sum_{i=1}^{n} t_i + P_n \neq s + P_n$ for all $s \in S_n$;
(3) The sequence $\{\sum_{i=1}^{n} t_i\}_{n=1}^{\infty}$ converges in T to $\sum_{i=1}^{\infty} t_i$.

First note that $(M + P_1)/P_1$ is not the zero ideal of T/P_1 since otherwise $M \subseteq P_1$, which is impossible as P_1 is not maximal. By Lemma 2.5, we have $|T| = |T/P_1|$ and, by Lemma 2.6, we have $|T/P_1| = |(M + P_1)/P_1|$. It follows that $|(M + P_1)/P_1| > |S_1|$, and so there exists $t_1 \in M$ such that $t_1 + P_1 \neq s + P_1$ for all $s \in S_1$.

We now inductively construct t_n for $n > 1$. Assume t_i has been defined for all $i < n$ to satisfy conditions (1) and (2) from above. Note that

$$\frac{\cap_{i=1}^{n-1} P_i \cap M^n + P_n}{P_n}$$

is not the zero ideal of T/P_n, and so

$$\left| \frac{\cap_{i=1}^{n-1} P_i \cap M^n + P_n}{P_n} \right| = |T/P_n| > |S_n|.$$

Consider the map $g_n : S_n \to T$ given by $g_n(s) = s - \sum_{i=1}^{n-1} t_i$ for $s \in S_n$. Note that the map g_n is injective since, if $g_n(s) = g_n(s')$, then $s - \sum_{i=1}^{n-1} t_i = s' - \sum_{i=1}^{n-1} t_i$, and adding $\sum_{i=1}^{n-1} t_i$ to both sides gives us $s = s'$. Thus,

$$|g_n(S_n)| = |S_n| < \left| \frac{\cap_{i=1}^{n-1} P_i \cap M^n + P_n}{P_n} \right|.$$

Hence, there exists $t_n \in \cap_{i=1}^{n-1} P_i \cap M^n$ such that, for all $s \in S_n$,

$$t_n + P_n \neq s - \sum_{i=1}^{n-1} t_i + P_n,$$

and so

$$\sum_{i=1}^{n} t_i + P_n \neq s + P_n$$

for all $s \in S_n$. It follows that, for our choice of t_n, conditions (1) and (2) from above are satisfied.

Now, for any $k \in \mathbb{N}$, $\{\sum_{i=k}^{n} t_i\}_{n=k}^{\infty}$ is a Cauchy sequence. Since the series $\sum_{i=k}^{\infty} t_i$ is the limit of $\{\sum_{i=k}^{n} t_i\}_{n=k}^{\infty}$, it follows that $\sum_{i=k}^{\infty} t_i$ converges in T as T is complete.

We now claim $x(\sum_{i=1}^{\infty} t_i) \in I$ and $x(\sum_{i=1}^{\infty} t_i) \notin \bigcup\{r + P_i \mid r \in D, P_i \in C\}$. Clearly, $x(\sum_{i=1}^{\infty} t_i) \in I$ since $x \in I$. Suppose

$$x\biggl(\sum_{i=1}^{\infty} t_i\biggr) \in \bigcup\{r + P_i \mid r \in D, P_i \in C\}.$$

Then, for some $n \geq 1$ and $r \in D$, we have

$$x\biggl(\sum_{i=1}^{\infty} t_i\biggr) \in r + P_n,$$

and so

$$x\biggl(\sum_{i=1}^{\infty} t_i\biggr) + P_n = r + P_n.$$

For each i, since $t_i \in \cap_{j=1}^{i-1} P_j \cap M^i$, we have for $i > n$ that $t_i \in P_n$. Since $\sum_{i=n+1}^{m} t_i \in P_n$ for all $m > n$, we have $\sum_{i=n+1}^{\infty} t_i \in P_n$, as ideals in a complete local ring are closed. Thus, we have

$$x\biggl(\sum_{i=1}^{n} t_i\biggr) + P_n = r + P_n,$$

and, since $r + P_n \in \langle x + P_n \rangle$,

$$x\biggl(\sum_{i=1}^{n} t_i\biggr) + P_n = r + P_n = (x + P_n)(s + P_n).$$

for some $s \in S_n$. Therefore,

$$(x + P_n)\biggl(\sum_{i=1}^{n} t_i + P_n\biggr) = (x + P_n)(s + P_n),$$

and since T/P_n is an integral domain,

$$\sum_{i=1}^{n} t_i + P_n = s + P_n,$$

a contradiction to our choice of t_i's. Hence,

$$x\left(\sum_{i=1}^{\infty} t_i\right) \notin \bigcup\{r + P_i \mid r \in D, P_i \in C\},$$

and so we have that $I \not\subseteq \bigcup\{r + P_i \mid r \in D, P_i \in C\}$. $\square$

With Lemma 2.7, we strengthen the results in [**2**]. As our proofs are quite similar to those found in [**2**], we state our new results here and often refer to the proofs in [**2**], noting the adjustments to the proofs as needed. We begin with the following definition.

Definition 2.8. Let S be a set. Then $\Gamma(S) = \sup(|S|, \aleph_0)$.

The proof of the following lemma is taken almost directly from the proof of [**2,** Lemma 2.5]. The only difference is that we use Lemma 2.7 in place of Lemma 2.3. We include the proof in this paper so that the reader can explicitly see the way Lemma 2.7 is used to adjoin transcendental elements to maintain the property that $R \cap P = (0)$ for all $P \in G$. The set C in the following result should be thought of as the set of maximal elements of G. Lemma 2.9 will be used to ensure that the map $A \longrightarrow T/M^2$ is onto, a condition needed to employ Proposition 2.1.

Lemma 2.9. *Let (T, M) be a complete local ring of dimension at least one. Let C be a countable set of nonmaximal prime ideals of T such that no ideal in C is contained in any other ideal in C. Let J be an ideal of T such that $J \not\subseteq P$ for all $P \in C$. Let R be an SCA-subring of T and $u + J \in T/J$. Then there exists an infinite SCA-subring S of T such that $R \subseteq S \subseteq T$, $\Gamma(S) = \Gamma(R)$, and $u + J$ is in the image of the map $S \to T/J$. Moreover, if $u \in J$, then $S \cap J \neq (0)$.*

Proof. Let $P \in C$. As $R \cap P = (0)$, R embeds into T/P. Let $D_{(P)}$ be a full set of coset representatives of the cosets $t + P \in T/P$ that make $(u + t) + P$ algebraic over R. If R is finite, then the set of elements in T/P that are algebraic over R is countable, while if R is infinite, then the set of elements in T/P that are algebraic over R is equal to the cardinality of R. In either case, we have $|D_{(P)}| < |T|$ since $|R| < |T|$ and, by Lemma 2.5, $|T| \geq |\mathbb{R}|$, where $\mathbb{R}$ denotes the set of real numbers. Let $D = \bigcup_{P \in C} D_{(P)}$ and note that, as C is countable and $|D_{(P)}| < |T|$, we have $|D| < |T|$. Now use Lemma 2.7 with $I = J$ to find an $x \in J$ such that $x \notin \bigcup\{r + P \mid r \in D,\ P \in C\}$. Then $(u+x)+P \in T/P$ is transcendental over R for every $P \in C$. We claim that $S = R[u+x]_{(R[u+x] \cap M)}$ is the desired SCA-subring. It is easy to see that $\Gamma(S) = \Gamma(R)$ and $|S| < |T|$. Now suppose that $f \in R[u+x] \cap P$ for some $P \in C$. Then $f = r_n(u+x)^n + \cdots + r_1(u+x) + r_0 \in P$ where $r_i \in R$. But we chose x such that $(u+x)+P$ is transcendental over R. Therefore, $r_i \in R \cap P = (0)$ for every $i = 1, 2, \ldots, n$, and it follows that $f = 0$. So $S \cap P = (0)$, and we have S is an SCA-subring. Further, if $u \in J$, then $u + x \in J$. Since $(u+x)+P$ is transcendental over R, it must be the case that $u + x \neq 0$ and that S is infinite. It follows that $S \cap J \neq (0)$. □

The following two lemmas will be used to construct A so that $IT \cap A = I$ for every finitely generated ideal I of A. This property will be needed for Proposition 2.1.

Lemma 2.10. *Let (T, M) be a complete local ring of dimension at least one. Let C be a countable set of incomparable nonmaximal prime ideals of T such that, if $Q \in \operatorname{Ass} T$, then $Q \subseteq P$ for some $P \in C$, and let R be an SCA-subring of T. Suppose that I is a finitely generated ideal of R and $c \in IT \cap R$. Then there exists an SCA-subring S of T such that $R \subseteq S \subseteq T$, $\Gamma(S) = \Gamma(R)$, and $c \in IS$.*

Proof. The result follows from the proof of [**2,** Lemma 2.6] using Lemma 2.7 in place of Lemma 2.3. □

Lemma 2.11. *Let (T, M) be a complete local ring of dimension at least one. Let J be an ideal of T with $J \not\subseteq P$ for all $P \in C$, where C is a countable set of incomparable nonmaximal prime ideals of T. Suppose that if $Q \in \operatorname{Ass} T$ then $Q \subseteq P$ for some $P \in C$, and let $u + J \in T/J$. Suppose R is an SCA-subring. Then there exists an SCA-subring S of T such that the following properties hold:*

(1) $R \subseteq S \subseteq T$;
(2) $\Gamma(S) = \Gamma(R)$;
(3) *If $u \in J$, then $S \cap J \neq (0)$;*
(4) *$u + J$ is in the image of the map $S \to T/J$;*

(5) *For every finitely generated ideal I of S, we have $IT \cap S = I$.*

Proof. The result follows from the proof of [**2,** Lemma 2.7] using Lemma 2.9 in place of [**2,** Lemma 2.5] and Lemma 2.10 in place of [**2,** Lemma 2.6]. We note here that the reason why $\Gamma(S) = \Gamma(R)$ is not explained in detail in the proof of [**2,** Lemma 2.7]. We recommend the reader desiring a more detailed argument see the proof of Lemma 2.11 in [**3**]. □

Lemma 2.12. *Let (T, M) be a complete local ring of dimension at least one, and let G be a set of nonmaximal prime ideals of T where G contains the associated prime ideals of T and such that the set of maximal elements of G, call it C, is countable. Moreover, suppose that if $Q \in \operatorname{Spec} T$ with $Q \subseteq P$ for some $P \in G$, then $Q \in G$. Also suppose that, for each prime ideal $P \in G$, P contains no nonzero integers of T. Then there exists a local domain A such that the following properties hold*:

(1) $\widehat{A} = T$;
(2) *If p is a nonzero prime ideal of A, then $T \otimes_A k(p) \cong k(p)$ where $k(p) = A_p/pA_p$*;
(3) *The generic formal fiber of A is exactly the elements of G (and so has maximal elements the elements of C)*;
(4) *If I is a nonzero ideal of A, then A/I is complete.*

Proof. The result follows from the proof of [**2,** Lemma 2.8] using Lemma 2.11 in place of [**2,** Lemma 2.7]. Again, we recommend the reader desiring more details to refer to the proof of Lemma 2.12 in [**3**]. □

We now come to the two main results of our paper. The proofs of Theorems 2.13 and 2.15 are based on the proofs of Theorems 3.1 and 4.1 in [**2**], although we provide fewer details here. The reader interested in more detailed explanations is encouraged to refer to the proofs in [**2**].

Theorem 2.13. *Let (T, M) be a complete local ring with prime subring Π, and let $G \subseteq \operatorname{Spec} T$ be such that G is nonempty and the set of maximal elements of G is countable. Then there exists a local domain*

A such that $\widehat{A} = T$ and the generic formal fiber of A is exactly the elements of G if and only if T is a field and $G = \{(0)\}$ or the following conditions hold:

(1) *$M \notin G$, and G contains all the associated prime ideals of T*;
(2) *If $Q \in G$ and $P \in \operatorname{Spec} T$ with $P \subseteq Q$, then $P \in G$*;
(3) *If $Q \in G$, then $Q \cap \Pi = (0)$.*

Proof. It is not difficult to verify that the conditions are necessary. To see that the conditions are sufficient, first note that if T is a field, then $A = T$ is our desired domain. If T is not a field, then the first condition of the theorem gives us that $\dim T \geq 1$. Lemma 2.12 now gives us the desired domain A. $\square$

Example 2.14. Let $T = \mathbb{Q}[[x, y, z]]$ be the power series ring over $\mathbb{Q}$. Let $C = \{\langle x - qy\rangle \mid q \in \mathbb{Q}\}$, and define $G = C \cup \{\langle 0\rangle\}$. There exists, by Theorem 2.13, a local integral domain A whose completion is T with its generic formal fiber precisely the elements of G.

Theorem 2.15. *Let (T, M) be a complete local ring containing the integers, and let Π denote the prime subring of T. Let $G \subseteq \operatorname{Spec} T$ be such that G is nonempty and the set of maximal elements of G is countable. Then there exists an excellent local domain A with $\widehat{A} = T$, and such that A has generic formal fiber exactly G if and only if T is a field and $G = \{(0)\}$ or the following conditions hold*:

(1) *$M \notin G$, and G contains all the associated prime ideals of T*;
(2) *If $Q \in G$ and $P \in \operatorname{Spec} T$ with $P \subseteq Q$, then $P \in G$*;
(3) *If $Q \in G$, then $Q \cap \Pi = (0)$*;
(4) *T is equidimensional*;
(5) *T_P is a regular local ring for all maximal elements $P \in G$.*

Proof. We note here that the proof is taken almost directly from the proof of Theorem 4.1 in **[2]**. Since this is one of our main theorems we have included the proof here. Assume that T is the completion of an excellent domain A having generic formal fiber exactly G with maximal ideals the maximal elements of G. If $\dim T = 0$, then T is a field and $G = \{(0)\}$. Thus, consider the case where $\dim T \geq 1$. By Theorem 2.13, the first three conditions hold. As A is excellent, it is universally catenary. Hence, A is formally catenary, and it follows that

$A/(0) \cong A$ is formally equidimensional. Thus, the completion, T, is equidimensional.

To see that the fifth condition holds, note that the maximal ideals of $T \otimes_A k((0))$ are the maximal elements of G. Let P be one of these maximal elements. Then $T \otimes_A k((0))$ localized at P is isomorphic to T_P. Since A is excellent, $T \otimes_A k((0))$ is regular, implying that T_P is a regular local ring for every maximal element of G as desired.

Conversely, first suppose that T is a field and $G = \{(0)\}$. Then $A = T$ is the desired domain. So, suppose that T is not a field and that all of the five conditions hold true for some complete local ring T and for some nonempty set G of prime ideals of T such that the number of maximal elements of G is countable. We want to show that there exists an excellent domain A possessing generic formal fiber exactly G. It is not difficult to verify that conditions (1) and (5) imply that T is reduced.

Now, if $\dim T = 0$, then T is a field and we are in the case where $A = T$ is the desired domain. Suppose, on the other hand, that $\dim T \geq 1$. Then use Lemma 2.12 to construct the domain A. We claim that A is excellent with generic formal fiber exactly G. From the construction of A, A has the desired generic formal fiber. To see that A is excellent, suppose that p is a nonzero prime ideal of A. Then, from Lemma 2.12, we have $T \otimes_A k(p) \cong k(p)$. Now let L be a finite field extension of $k(p)$. Then $T \otimes_A L \cong T \otimes_A k(p) \otimes_{k(p)} L \cong k(p) \otimes_{k(p)} L \cong L$. Thus, the fiber over p is geometrically regular. Now T_P is regular by assumption for every maximal element P of G. It follows that $T \otimes_A k((0))$ is regular. Now, since T contains the integers, so does A. It follows that $k((0))$ is a field of characteristic zero, and hence that $T \otimes_A L$ is regular for every finite field extension L of $k((0))$. Thus, all of the formal fibers of A are geometrically regular. Since A is formally equidimensional, it is universally catenary, and thus A is excellent. Hence, A is the desired domain. $\square$

Example 2.16. Let $T = K[[x, y, z]]$, where K is a field containing the rationals, and let $C = \{\langle x - qy \rangle \mid q \in \mathbb{Q}\}$. Define $G = C \cup \{\langle 0 \rangle\}$. It is easily seen that the first three conditions of Theorem 2.15 are satisfied. Since T is an integral domain, it is equidimensional, so condition (4) is satisfied. Since T is a regular local ring, T_P is a regular local ring for all prime ideals P of T. It follows that condition (5) of Theorem 2.15 is satisfied. Thus, there exists an excellent domain A whose completion is T with a generic formal fiber precisely the elements of G.

3. Formal fibers of nonzero prime ideals. Our results from the previous section allow us to construct local domains for which we have control over the formal fiber of a nonzero prime ideal. In particular, we provide a class of local domains whose formal fiber at a height k prime ideal has a specified set of countably many maximal elements. Suppose (T, M) is a complete local ring and C is a nonempty countable set of incomparable prime ideals of T. Let $x_1, x_2, \ldots, x_k$ be indeterminants, and let C' be the set $\{QT[[x_1, x_2, \ldots, x_k]] + (x_1, x_2, \ldots, x_k)T[[x_1, x_2, \ldots, x_k]] \mid Q \in C\} \subset \text{Spec}(T[[x_1, x_2, \ldots, x_k]])$. We give sufficient conditions for there to exist a domain S containing $x_1, x_2, \ldots, x_k$, whose completion is $T[[x_1, x_2, \ldots, x_k]]$ and such that the formal fiber of the height k prime ideal $(x_1, x_2, \ldots, x_k)S$ has maximal elements precisely the elements of C'.

We begin by using Theorem 2.13 to obtain a local domain A with completion T and whose generic formal fiber has maximal elements precisely the elements of C. We then show our desired domain S is $A[x_1, x_2, \ldots, x_k]$ localized at the prime ideal $(A \cap M)A[x_1, x_2, \ldots, x_k] + (x_1, x_2, \ldots, x_k)A[x_1, x_2, \ldots, x_k]$.

Theorem 3.1. *Let (T, M) be a complete local ring with prime subring Π, and let C be a nonempty countable set of incomparable prime ideals of T such that either T is a field and $C = \{(0)\}$, or the following conditions hold:*

(1) *$M \notin C$;*
(2) *If $P \in \text{Ass}\, T$, then $P \subseteq Q$ for some $Q \in C$;*
(3) *If $Q \in C$, then $Q \cap \Pi = (0)$.*

Let k be a positive integer, and let $x_1, x_2, \ldots, x_k$ be indeterminates. Then there exists a local integral domain S such that $x_1, x_2, \ldots, x_k \in S$, the completion of S is $T[[x_1, x_2, \ldots, x_k]]$, and $(x_1, x_2, \ldots, x_k)S$ is a height k prime ideal of S whose formal fiber has maximal elements precisely $\{QT[[x_1, x_2, \ldots, x_k]] + (x_1, x_2, \ldots, x_k)T[[x_1, x_2, \ldots, x_k]] \mid Q \in C\} \subset \text{Spec}\,(T[[x_1, x_2, \ldots, x_k]])$.

Proof. Let $G = \{P \in \operatorname{Spec} T \mid P \subseteq Q \text{ for some } Q \in C\}$. Then, by Theorem 2.13, there is a local domain A such that $\widehat{A} = T$ and the generic formal fiber of A is exactly the elements of G. Let S be the ring $A[x_1, x_2, \ldots, x_k]$ localized at the prime ideal $(A \cap M)A[x_1, x_2, \ldots, x_k] + (x_1, x_2, \ldots, x_k)A[x_1, x_2, \ldots, x_k]$. Clearly, S is an integral domain whose completion is $\widehat{A}[[x_1, x_2, \ldots, x_k]] = T[[x_1, x_2, \ldots, x_k]]$. It is also clear that $(x_1, x_2, \ldots, x_k)S$ is a prime ideal of S. Since $x_1, x_2, \ldots, x_k$ are indeterminates, the height of the prime ideal $(x_1, x_2, \ldots, x_k)S$ is k.

We now show that, for $Q \in C$, we have

$$(QT[[x_1, x_2, \ldots, x_k]] + (x_1, x_2, \ldots, x_k)T[[x_1, x_2, \ldots, x_k]]) \cap S \\ = (x_1, x_2, \ldots, x_k)S.$$

It suffices to show

$$(QT[[x_1, x_2, \ldots, x_k]] + (x_1, x_2, \ldots, x_k)T[[x_1, x_2, \ldots, x_k]]) \\ \cap A[x_1, x_2, \ldots, x_k] \\ = (x_1, x_2, \ldots, x_k)A[x_1, x_2, \ldots x_k].$$

Let

$$f \in (QT[[x_1, x_2, \ldots, x_k]] + (x_1, x_2, \ldots, x_k)T[[x_1, x_2, \ldots, x_k]]) \\ \cap A[x_1, x_2, \ldots, x_k].$$

Since $(QT[[x_1, x_2, \ldots, x_k]] + (x_1, x_2, \ldots, x_k)T[[x_1, x_2, \ldots, x_k]]) \cap T[x_1, x_2, \ldots, x_k] = QT[x_1, x_2, \ldots, x_k] + (x_1, x_2, \ldots, x_k)T[x_1, x_2, \ldots, x_k]$, we have that $f \in QT[x_1, x_2, \ldots, x_k] + (x_1, x_2, \ldots, x_k)T[x_1, x_2, \ldots, x_k]$. So we can write $f = q+g$ where $q \in Q$ and $g \in (x_1, x_2, \ldots, x_k)T[x_1, x_2, \ldots, x_k]$. As $f \in A[x_1, x_2, \ldots, x_k]$, we can write $f = a+h$ where $a \in A$ and $h \in (x_1, x_2, \ldots, x_k)A[x_1, x_2, \ldots x_k]$. Now, the equality $f = q+g = a+h$ holds true in the ring $T[x_1, x_2, \ldots, x_k]$, and so the coefficients of each monomial must be equal. In particular, the constant term on each side must be equal. In other words, $q = a \in Q \cap A = (0)$. It follows that $f = h \in (x_1, x_2, \ldots, x_k)A[x_1, x_2, \ldots x_k]$. Containment in the other direction is clear.

It remains to show that if $J \in \operatorname{Spec}(T[[x_1, x_2, \ldots, x_k]])$ such that $J \cap S = (x_1, x_2, \ldots, x_k)S$, then $J \subseteq QT[[x_1, x_2, \ldots, x_k]] + (x_1, x_2, \ldots, x_k)T[[x_1, x_2, \ldots, x_k]]$ for some $Q \in C$. Now, $(J \cap T) \cap A = J \cap A = (J \cap S) \cap A = (x_1, x_2, \ldots, x_k)S \cap A = (0)$. So

$J \cap T$ is in the generic formal fiber of A. It follows that $J \cap T \subseteq Q$ for some $Q \in C$. Let $f \in J$. Then $f = t + g$ for some $t \in T$ and $g \in (x_1, x_2, \ldots, x_k)T[[x_1, x_2, \ldots, x_k]]$. Since $x_1, x_2, \ldots, x_k \in J$, we have $g \in J$ and so $t \in T \cap J \subseteq Q$. Hence, $f \in QT[[x_1, x_2, \ldots, x_k]] + (x_1, x_2, \ldots, x_k)T[[x_1, x_2, \ldots, x_k]]$, and we have $J \subseteq QT[[x_1, x_2, \ldots, x_k]] + (x_1, x_2, \ldots, x_k)T[[x_1, x_2, \ldots, x_k]]$, as desired. $\square$

Example 3.2. Let $T = \mathbb{Q}[[y_1, y_2, y_3]]/(y_1 y_2)$, and let $C = \{(y_1, y_2 - qy_3) \mid q \in \mathbb{Q}\}$. Then the hypotheses of Theorem 3.1 are satisfied. Using $k = 2$, there exists an integral domain S such that $x_1, x_2 \in S$, $\widehat{S} = \mathbb{Q}[[y_1, y_2, y_3, x_1, x_2]]/(y_1 y_2)$, and $(x_1, x_2)S$ is a height 2 prime ideal of S whose formal fiber has countably many maximal elements, namely, $\{(x_1, x_2, y_1, y_2 - qy_3) \mid q \in \mathbb{Q}\}$.

Example 3.3. We use Theorem 3.1 to show that there is a local domain S whose completion is $\mathbb{C}[[x_1, x_2, x_3, x_4]]$ and such that $x_1, x_2 \in S$ and the formal fiber of $(x_1, x_2)S$ has only one maximal element, namely, $(x_1, x_2, x_3)\mathbb{C}[[x_1, x_2, x_3, x_4]]$. To do this, simply let $T = \mathbb{C}[[x_3, x_4]]$, $C = \{(x_3)\}$, and $k = 2$. Then, by Theorem 3.1, the desired domain S exists.

We now show that, with some extra assumptions on our local ring T and our set C of prime ideals of T, we can force our constructed domain S to be excellent. The proof employs the argument from the proof of Theorem 3.1 but replaces the use of Theorem 2.13 with Theorem 2.15.

Theorem 3.4. *Let (T, M) be a complete local ring containing the integers. Let Π denote the prime subring of T, and let C be a nonempty countable set of incomparable prime ideals of T such that, either T is a field and $C = \{(0)\}$, or the following conditions hold*:

(1) *$M \notin C$*;
(2) *If $P \in \operatorname{Ass} T$, then $P \subseteq Q$ for some $Q \in C$*;
(3) *If $Q \in C$, then $Q \cap \Pi = (0)$*;
(4) *T is equidimensional*;
(5) *T_Q is a regular local ring for all $Q \in C$.*

Let k be a positive integer, and let $x_1, x_2, \ldots, x_k$ be indeterminates. Then there exists an excellent local domain S such that $x_1, x_2, \ldots, x_k \in S$, the completion of S is $T[[x_1, x_2, \ldots, x_k]]$, and $(x_1, x_2, \ldots, x_k)S$ is a height k prime ideal of S whose formal fiber has maximal elements precisely $\{QT[[x_1, x_2, \ldots, x_k]] + (x_1, x_2, \ldots, x_k)T[[x_1, x_2, \ldots, x_k]] \mid Q \in C\} \subset \operatorname{Spec}(T[[x_1, x_2, \ldots, x_k]])$.

Proof. Let $G = \{P \in \operatorname{Spec} T \mid P \subseteq Q \text{ for some } Q \in C\}$. Then, by Theorem 2.15, there is an excellent local domain A such that $\widehat{A} = T$, and the generic formal fiber of A is exactly the elements of G. Let S be the ring $A[x_1, x_2, \ldots, x_k]$ localized at the prime ideal $(A \cap M)A[x_1, x_2, \ldots, x_k] + (x_1, x_2, \ldots, x_k)A[x_1, x_2, \ldots, x_k]$. Then, by the proof of Theorem 3.1, we have that S is a local domain such that $\widehat{S} = T[[x_1, x_2, \ldots, x_k]]$ and $(x_1, x_2, \ldots, x_k)S$ is a height k prime ideal of S whose formal fiber has the desired maximal elements. Noting that excellence is preserved under adjoining finitely many indeterminants and localization, we conclude S is our excellent local domain. $\square$

Example 3.5. Let $T = \mathbb{Q}[[y_1, y_2, y_3]]$, and let $C = \{(y_1 - qy_2) \mid q \in \mathbb{Q}\}$. Then, by Theorem 3.4 using $k = 3$, there exists an excellent local domain S such that the indeterminates x_1, x_2 and x_3 are in S, the completion of S is $\mathbb{Q}[[y_1, y_2, y_3, x_1, x_2, x_3]]$, and the formal fiber of $(x_1, x_2, x_3)S$ has maximal elements $\{(y_1 - qy_2, x_1, x_2, x_3) \mid q \in \mathbb{Q}\}$.

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