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ON CHARACTERIZATIONS OF HOPF HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM WITH COMMUTING OPERATORS

IN-BAE KIM, DONG HO LIM AND HYUNJUNG SONG

ABSTRACT. Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. In this paper we prove that if $R_\xi L_\xi = L_\xi R_\xi$ holds on M, then M is a Hopf hypersurface, where R_ξ and L_ξ denote the structure Jacobi operator and the induced operator from the Lie derivative with respect to the structure vector field ξ, respectively. We characterize such Hopf hypersurfaces of $M_n(c)$.

1. Introduction. A complex n-dimensional Kaeherian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. As is well known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n(\mathbb{C})$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n(\mathbb{C})$, according to $c > 0$, $c = 0$ or $c < 0$.

We consider a real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, g, ξ, η) induced from the Kaehler metric and complex structure J on $M_n(c)$. The structure vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant ([**4**]) and that M is called a *Hopf hypersurface.*

Typical examples of Hopf hypersurfaces in $P_n(\mathbb{C})$ are homogeneous ones, namely, those real hypersurfaces are given as orbits under a subgroup of the projective unitary group $PU(n+1)$. Takagi [**10**] completely classified such hypersurfaces as six model spaces which

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are said to be A_1, A_2, B, C, D and E. On the other hand, real hypersurfaces in $H_n(\mathbb{C})$ have been investigated by Berndt [**1**], Montiel and Romero [**5**], and so on. Berndt [**1**] classified all homogeneous Hopf hyersurfaces in $H_n(\mathbb{C})$ as four model spaces which are said to be A_0, A_1, A_2 and B. If M is a real hypersurface of type A_1 or A_2 in $P_n(\mathbb{C})$ or type A_0, A_1 or A_2 in $H_n(\mathbb{C})$, then M is said to be *of type* A for simplicity.

The induced operator L_ξ on a real hypersurface M from the 2-form $\mathcal{L}_\xi g$ is defined by $(\mathcal{L}_\xi g)(X,Y) = g(L_\xi X, Y)$ for any vector fields X and Y on M, where $\mathcal{L}_\xi$ denotes the operator of the Lie derivative with respect to the structure vector field ξ. This operator L_ξ is given by $L_\xi = \phi A - A\phi$ on M, and the structure vector field ξ is Killing if $L_\xi = 0$. As a typical characterization of real hypersurfaces of type A, the following is due to Okumura [**7**] for $c > 0$, and Montiel and Romero [**5**] for $c < 0$.

Theorem A ([**5, 7**]). *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. It satisfies $L_\xi = 0$ on M if and only if M is locally congruent to one of the model spaces of type* A.

For the curvature tensor field R on a real hypersurface M, we define the Jacobi operator R_X by $R_X = R(\cdot, X)X$ with respect to a unit vector field X. Then we see that R_X is self-adjoint endomorphism of the tangent space. It is related with (the Jacobi vector equation) $\nabla_{\dot\gamma}(\nabla_{\dot\gamma} Y) + R(Y, \dot\gamma)\dot\gamma = 0$ along a geodesic γ on M, where $\dot\gamma$ denotes the velocity vector field of γ. We will call the Jacobi operator R_ξ with respect to the structure vector field ξ a *structure Jacobi operator* on M. Recently, it has been shown that there are no real hypersurfaces in $M_n(c)$ with parallel structure Jacobi operator R_ξ (see [**8**]). Some authors have also studied several conditions on the structured Jacobi operator R_ξ and given some results on the classification of real hypersurfaces of type A in $M_n(c)$ ([**2, 3, 6, 8, 9**], etc). As for the structure Jacobi operator R_ξ and the operator L_ξ, Ki and two of the present authors [**3**] have proved the following.

Theorem B ([**3**]). *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. It satisfies $R_\xi L_\xi = 0$ if and only if M is locally congruent to one of the model spaces of type* A.

In this paper, we shall study a real hypersurface in a nonflat complex space form $M_n(c)$ with commuting operators R_ξ and L_ξ. Namely, we shall prove:

Theorem 1. *Let M be a real hypersurface satisfying $R_\xi L_\xi = L_\xi R_\xi$ in a complex space form $M_n(c)$, $c \neq 0$. Then M is a Hopf hypersurface in $M_n(c)$.*

In Section 5, we shall give another characterization of such a real hypersurface in $M_n(c)$. All manifolds in the present paper are assumed to be connected and of class C^∞ and the real hypersurfaces supposed to be orientable.

2. Preliminaries. Let M be a real hypersurface immersed in a complex space form $M_n(c)$ and N a unit normal vector field of M. By $\widetilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor $\widetilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y on M, where g denotes the Riemannian metric tensor of M induced from $\widetilde{g}$, and A is the shape operator of M in $M_n(c)$. For any vector field X on M we put

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi,$$

where J is the almost complex structure of $M_n(c)$. Then we see that M induces an almost contact metric structure (ϕ, g, ξ, η), that is,

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \ \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad \eta(X) = g(X, \xi) \end{aligned}$$

for any vector fields X and Y on M.

Since the almost complex structure J is parallel, we can verify from the Gauss and Weingarten formulas that

$$\nabla_X \xi = \phi AX, \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.1}$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have the following Gauss and Codazzi equations, respectively:

(2.2)
$$R(X,Y)Z = \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY,$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\} \tag{2.3}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M.

From the Gauss equation (2.2), the structure Jacobi operator R_ξ is given by

$$R_\xi X = R(X,\xi)\xi = \frac{c}{4}\{X - \eta(X)\xi\} + \alpha AX - \eta(AX)A\xi \tag{2.4}$$

for any vector field X on M.

By use of (2.1), we have $(\mathcal{L}_\xi g)(X,Y) = \langle(\phi A - A\phi)X, Y\rangle$ for any vector fields X and Y on M, and hence the induced operator L_ξ from $\mathcal{L}_\xi g$ is given by

$$L_\xi X = (\phi A - A\phi)X. \tag{2.5}$$

Let W be a unit vector field on M with the same direction of the vector field $-\phi\nabla_\xi\xi$, and let μ be the length of the vector field $-\phi\nabla_\xi\xi$ if it does not vanish, and zero (constant function) if it vanishes. Then it follows immediately from (2.1) that

$$A\xi = \alpha\xi + \mu W, \tag{2.6}$$

where $\alpha = \eta(A\xi)$. We notice here that W is orthogonal to ξ. We put

$$\Omega = \{p \in M \mid \mu(p) \neq 0\}. \tag{2.7}$$

Then Ω is an open subset of M.

3. Real hypersurfaces satisfying $R_\xi L_\xi = L_\xi R_\xi$. Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, satisfying $R_\xi L_\xi = L_\xi R_\xi$. In this section, we assume that the open subset Ω given in (2.7) is not empty. Then the above condition together with (2.4), (2.5) and (2.6) implies that

(3.1)
$$\begin{aligned}\alpha(\phi A^2 + A^2\phi - 2A\phi A)X =&\mu\left\{\left(\alpha^2 + \frac{c}{4}\right)w(\phi X) - \alpha w((\phi A - A\phi)X)\right\}\xi\\ &+\mu^2\{\alpha w(\phi X) - w((\phi A - A\phi)X)\}W\\ &+\mu\left\{\left(\alpha^2 + \frac{c}{4}\right)\eta(X) + \alpha\mu w(X)\right\}\phi W\\ &+\mu\{\alpha\eta(X) + \mu w(X)\}(\phi AW - A\phi W)\end{aligned}$$

for any vector field X on Ω, where w is the dual 1-form of the unit vector field W. If we put $X = \xi$ into (3.1) and make use of (2.6), then we have $\alpha \neq 0$ on Ω, and hence

$$A\phi W = -\frac{c}{4\alpha}\phi W. \tag{3.2}$$

Putting $X = W$ into (3.1) and taking account of (2.6) and (3.2) yields

$$\alpha\phi A^2W - 2\alpha A\phi AW - \mu^2\phi AW = \left\{\mu^2\left(\alpha + \frac{c}{4\alpha}\right) - \frac{c^2}{16\alpha}\right\}\phi W, \tag{3.3}$$

and by putting $X = \phi W$ into (3.1), we obtain

$$\alpha A^2W+\frac{c}{2}AW = \mu\left(\alpha^2+\alpha\gamma+\frac{c}{2}\right)\xi+\left\{\mu^2\left(\alpha+\gamma+\frac{c}{4\alpha}\right)-\frac{c^2}{16\alpha}\right\}W, \tag{3.4}$$

where the smooth function γ is defined by $\gamma = g(AW, W)$. If we substitute (3.4) into (3.3), then we have

$$2\alpha A\phi AW + \left(\mu^2 + \frac{c}{2}\right)\phi AW = \mu^2\gamma\phi W. \tag{3.5}$$

Putting $X = AW$ into (3.1) and using (2.6) and (3.2) yields

$$\begin{aligned}&\alpha\phi A^3W + \alpha A^2\phi AW - 2\alpha A\phi A^2W - \mu^2(\alpha + \gamma)\phi AW\\ =&\mu^2\left(\alpha^2 + \alpha\gamma + \frac{c}{2} + \frac{c}{4\alpha}\gamma\right)\phi W.\end{aligned} \tag{3.6}$$

If we substitute (3.4) and (3.5) into (3.6) and make use of (2.6) and (3.2), then we obtain $\phi AW = \gamma\phi W$, or equivalently,

$$AW = \mu\xi + \gamma W \tag{3.7}$$

on Ω. It follows immediately from (3.4) and (3.7) that

$$\left(\frac{c}{4}+\alpha\gamma\right)\left(\frac{c}{4}+\alpha\gamma-\mu^2\right)=0. \tag{3.8}$$

Differentiating the smooth function $\mu = g(A\xi, W)$ along any vector field X on Ω and using (2.1), (2.3), (2.6), (3.2) and (3.7), we have

$$X\mu = g\left((\nabla_\xi A)W + \frac{c}{4\alpha}\gamma\phi W, X\right).$$

Since we have $(\nabla_\xi A)W = \nabla_\xi(\mu\xi+\gamma W) - A\nabla_\xi W$, we see from the equation above that the gradient vector field $\nabla\mu$ of μ is given by

$$\nabla\mu = -(A-\gamma I)\nabla_\xi W + (\xi\mu)\xi + (\xi\gamma)W + \left(\mu^2 + \frac{c}{4\alpha}\gamma\right)\phi W \tag{3.9}$$

on Ω, where I indicates the identity transformation. If we differentiate $\alpha = g(A\xi,\xi)$ along any vector field X and take account of (2.1), (2.3), (2.6), (3.2) and (3.7), then we obtain $\nabla\alpha = (\nabla_\xi A)\xi + (c/2\alpha)\mu\phi W$, and hence

$$\nabla\alpha = \mu\nabla_\xi W + (\xi\alpha)\xi + (\xi\mu)W + \mu\left(\frac{3c}{4\alpha}+\alpha\right)\phi W. \tag{3.10}$$

By a similar argument as the above, we can verify that the gradient vector fields of the smooth functions $\gamma = g(AW, W)$ and $-c/(4\alpha) = g(A\phi W, \phi W)$ are given respectively by

$$\nabla\gamma = -(A-\gamma I)\nabla_W W + (W\mu)\xi + (W\gamma)W + \mu\left(\gamma - \frac{c}{2\alpha}\right)\phi W, \tag{3.11}$$

$$\frac{c}{4\alpha}\nabla\alpha = -\alpha\left(A + \frac{c}{4\alpha}I\right)\phi\nabla_{\phi W}W + \frac{c}{4\alpha}((\phi W)\alpha)\phi W. \tag{3.12}$$

Taking the inner product of (3.12) with ξ and W respectively and using (2.6) and (3.7), and comparing the resultant equations, we have

$$\alpha\mu W\alpha = \left(\frac{c}{4}+\alpha\gamma\right)\xi\alpha. \tag{3.13}$$

By means of (2.1), (2.6), (3.2) and (3.7), we can verify that

$$\begin{aligned}(\nabla_{\phi W}A)\xi &= \mu\nabla_{\phi W}W \\ &\quad + \Big\{(\phi W)\alpha - \frac{c}{4\alpha}\mu\Big\}\xi + \Big\{(\phi W)\mu + \frac{c}{4} - \frac{c}{4\alpha}\gamma\Big\}W, \\ (\nabla_\xi A)\phi W &= -\Big(A + \frac{c}{4\alpha}I\Big)\phi\nabla_\xi W + \mu\Big(\frac{c}{4\alpha} + \alpha\Big)\xi \\ &\quad + \mu^2 W + \frac{c}{4\alpha^2}(\xi\alpha)\phi W.\end{aligned}$$

Therefore, it follows from equation (2.3) of Codazzi that

$$\begin{aligned}\mu\nabla_{\phi W}W + \Big(A + \frac{c}{4\alpha}I\Big)\phi\nabla_\xi W &= -\Big\{(\phi W)\alpha - \mu\Big(\frac{c}{2\alpha} + \alpha\Big)\Big\}\xi \\ &\quad - \Big\{(\phi W)\mu - \mu^2 - \frac{c}{4\alpha}\gamma\Big\}W \\ &\quad + \frac{c}{4\alpha^2}(\xi\alpha)\phi W.\end{aligned} \tag{3.14}$$

By a similar argument as the above, we can also verify from $(\nabla_\xi A)W - (\nabla_W A)\xi$ that

$$\begin{aligned}&\mu\nabla_W W + (A - \gamma I)\nabla_\xi W \\ &\quad = (\xi\mu - W\alpha)\xi + (\xi\gamma - W\mu)W + \Big(\mu^2 - \frac{c}{4} - \alpha\gamma - \frac{c}{4\alpha}\gamma\Big)\phi W.\end{aligned}$$

If we take the inner product of this equation with ξ and W, respectively, then we have

$$\xi\mu = W\alpha \quad \text{and} \quad \xi\gamma = W\mu \tag{3.15}$$

on Ω, and hence the initial equation is reduced to

$$\mu\nabla_W W + (A - \gamma I)\nabla_\xi W = \Big(\mu^2 - \frac{c}{4} - \alpha\gamma - \frac{c}{4\alpha}\gamma\Big)\phi W. \tag{3.16}$$

Let D be the distribution spanned by the unit vector fields ξ, W and ϕW on Ω, that is, $D_p = \mathrm{span}\,\{\xi, W, \phi W\}_p$ for any point p of Ω. Then we see from (2.6), (3.2) and (3.7) that D is invariant under the shape operator A and the structure tensor field ϕ.

If we eliminate the term $(A-\gamma I)\nabla_\xi W$ from (3.9) and (3.16), then we obtain

$$\nabla\mu - \mu\nabla_W W \equiv 0 (\operatorname{mod} D). \tag{3.17}$$

Applying the operators μI to (3.11) and $A-\gamma I$ to (3.17) and eliminating the term $\nabla_W W$ from them, we have

$$(A-\gamma I)\nabla\mu + \mu\nabla\gamma \equiv 0 \ (\operatorname{mod} D). \tag{3.18}$$

By a similar argument as the above, we see from (3.9) and (3.10) that

$$(A-\gamma I)\nabla\alpha + \mu\nabla\mu \equiv 0 \ (\operatorname{mod} D). \tag{3.19}$$

Since D is invariant under the operator $A\phi+(c/4)\phi$, substituting (3.14) into (3.12) yields

$$\frac{c}{4}\mu\nabla\alpha - \left\{\alpha^2 A\phi A\phi + \frac{c}{4}\alpha\phi A\phi - \frac{c}{4}\left(\alpha A + \frac{c}{4}I\right)\right\}\nabla_\xi W \equiv 0 \ (\operatorname{mod} D) \tag{3.20}$$

by the use of $\eta(\nabla_\xi W)=0$. If we apply the operator $A-\gamma I$ to (3.14), then we can find

$$\mu(A-\gamma I)\nabla_{\phi W} W + (A-\gamma I)\left(A\phi + \frac{c}{4\alpha}\phi\right)\nabla_\xi W \equiv 0 \ (\operatorname{mod} D). \tag{3.21}$$

By virtue of (2.1), (2.3), (3.2) and (3.7), it follows from $(\nabla_W A)\phi W - (\nabla_{\phi W} A)W$ that $(A-\gamma I)\nabla_{\phi W} W - (A\phi + c/(4\alpha)\phi)\nabla_W W \equiv 0 (\operatorname{mod} D)$, from which together with (3.16), we find

$$\mu(A-\gamma I)\nabla_{\phi W} W + \left(A\phi + \frac{c}{4\alpha}\phi\right)(A-\gamma I)\nabla_\xi W \equiv 0 \ (\operatorname{mod} D). \tag{3.22}$$

Comparing (3.21) with (3.22), we can verify that

$$\left\{A^2\phi - A\phi A - \frac{c}{4\alpha}(\phi A - A\phi)\right\}\nabla_\xi W \equiv 0 \quad (\operatorname{mod} D). \tag{3.23}$$

It follows from (3.1) that $(\phi A^2 + A^2\phi - 2A\phi A)X \equiv 0 \ (\operatorname{mod} D)$ for any vector field X on Ω, since by use of (3.2) and (3.7) the vector field on the right side of (3.1) belongs to D. Putting $X=\phi\nabla_\xi W$ into this relation above yields

$$(A^2 - \phi A^2\phi + 2A\phi A\phi)\nabla_\xi W \equiv 0 \ (\operatorname{mod} D).$$

Since we have $\phi(\phi A^2+A^2\phi-2A\phi A)\nabla_\xi W \equiv 0 \ (\mathrm{mod}\, D)$ and $\eta(A^2\nabla_\xi W) = 0$, we obtain

$$(A^2 - \phi A^2\phi + 2\phi A\phi A)\nabla_\xi W \equiv 0 \ (\mathrm{mod}\, D).$$

From the above two relations, we have

$$(A\phi A\phi - \phi A\phi A)\nabla_\xi W \equiv 0 \ (\mathrm{mod}\, D). \tag{3.24}$$

Combining (3.23) with the relation $(\phi A^2 + A^2\phi - 2A\phi A)\nabla_\xi W \equiv 0 \ (\mathrm{mod}\, D)$, we get

$$\left\{\phi A^2 - A\phi A + \frac{c}{4\alpha}(\phi A - A\phi)\right\}\nabla_\xi W \equiv 0 \ (\mathrm{mod}\, D).$$

If we apply ϕ to this relation and take account of $\eta(A\nabla_\xi W) = 0$ and $\eta(A^2\nabla_\xi W) = 0$, then we have

$$\left(A^2 + \phi A\phi A + \frac{c}{4\alpha}\phi A\phi + \frac{c}{4\alpha}A\right)\nabla_\xi W \equiv 0 \ (\mathrm{mod}\, D). \tag{3.25}$$

By virtue of (3.24), it follows from (3.20) and (3.25) that

$$\frac{c}{4}\mu\nabla\alpha + \left\{\alpha A^2 + \frac{c}{2}\alpha A + \left(\frac{c}{4}\right)^2 I\right\}\nabla_\xi W \equiv 0 \ (\mathrm{mod}\, D).$$

Since we have $\nabla\alpha - \mu\nabla_\xi W \equiv 0 \ (\mathrm{mod}\, D)$ from (3.10), the above relation is rewritten as

$$\left\{\alpha^2 A^2 + \frac{c}{2}\alpha A + \frac{c}{4}\left(\mu^2 + \frac{c}{4}\right)I\right\}\nabla\alpha \equiv 0 \ (\mathrm{mod}\, D). \tag{3.26}$$

4. Some lemmas. In this section we assume that M is a real hypersurface satisfying $R_\xi L_\xi = L_\xi R_\xi$ in a complex space form $M_n(c)$, $c \neq 0$, and the open subset Ω given in (2.7) is not empty. Then we may consider from (3.8) that we have either $\alpha\gamma + c/4 = 0$ or $\alpha\gamma + c/4 = \mu^2$ on Ω. The distribution D is given by $D = \mathrm{span}\,\{\xi, W, \phi W\}$ as before. We shall prove some Lemmas, which will be used later.

Lemma 4.1. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, satisfying $R_\xi L_\xi = L_\xi R_\xi$. If the open subset Ω is not empty, then the vector fields $\nabla\alpha$, $\nabla\mu$, $\nabla\gamma$, $\nabla_\xi W$, $\nabla_W W$ and $\nabla_{\phi W} W$ are expressed in terms of the vector fields ξ, W and ϕW only on Ω.*

Proof. In the case where $\alpha\gamma + c/4 = \mu^2$ on Ω, we have

$$\alpha\nabla\gamma + \gamma\nabla\alpha = 2\mu\nabla\mu. \tag{4.1}$$

If we substitute (4.1) into (3.18) and make use of $\alpha\gamma + c/4 = \mu^2$, then we obtain

$$\left\{\alpha A + \left(\mu^2 + \frac{c}{4}\right)I\right\}\nabla\gamma + \gamma(A - \gamma I)\nabla\alpha \equiv 0 \ (\mathrm{mod}\, D). \tag{4.2}$$

Substituting (4.1) into (3.19), we also obtain

$$(2A - \gamma I)\nabla\alpha + \alpha\nabla\gamma \equiv 0(\mathrm{mod}\, D). \tag{4.3}$$

If we apply the operators αI to (4.2) and $\alpha A + (\mu^2 + c/4)I$ to (4.3), and eliminate the term $\nabla\gamma$ from them, then we have

$$\left\{\alpha^2 A^2 + \frac{c}{2}\alpha A - \frac{c}{4}\left(\mu^2 - \frac{c}{4}\right)I\right\}\nabla\alpha \equiv 0 \ (\mathrm{mod}\, D). \tag{4.4}$$

By subtracting (4.4) from (3.26), we see that $\nabla\alpha$ is expressed in terms of ξ, W and ϕW only. Since D is invariant under the operator $2A - \gamma I$, similar expressions as that of $\nabla\alpha$ hold for $\nabla\mu$ and $\nabla\gamma$ by (4.1) and (4.3).

In the case where $\alpha\gamma + c/4 = 0$ on Ω, we have

$$\gamma\nabla\alpha + \alpha\nabla\gamma = 0. \tag{4.5}$$

It is easily seen from (3.18) and (4.5) that

$$\left(\alpha A + \frac{c}{4}I\right)\nabla\mu - \mu\gamma\nabla\alpha \equiv 0 \ (\mathrm{mod}\, D). \tag{4.6}$$

Applying $\alpha A + (c/4)I$ to (3.19) and μ to (4.6), and eliminating the term $\nabla\mu$ from them, we can verify that

$$\left\{\alpha^2 A^2 + \frac{c}{2}\alpha A - \frac{c}{4}\left(\mu^2 - \frac{c}{4}\right)I\right\}\nabla\alpha \equiv 0 \ (\mathrm{mod}\, D). \tag{4.7}$$

Therefore, it follows from (3.26) and (4.7) that $\nabla\alpha$ is given by a linear combination of ξ, W and ϕW only, and hence from (3.19) and (4.5) that $\nabla\mu$ and $\nabla\gamma$ have similar expressions as that of $\nabla\alpha$.

Since $\nabla\alpha$ is expressed in terms of ξ, W and ϕW only, so is $\nabla_\xi W$ by (3.10). For the vector fields $\nabla_{\phi W}W$ and $\nabla_W W$, the similar expressions as that of $\nabla_\xi W$ are given by (3.14) and (3.16), respectively. $\square$

Lemma 4.2. *Under the assumptions of Lemma* 4.1*, if* $c/4+\alpha\gamma=\mu^2$ *holds on* Ω*, then we have*

$$\begin{aligned}
&\alpha\nabla\alpha = (\xi\alpha)A\xi - 3\mu\left(\alpha^2-\frac{c}{4}\right)\phi W,\\
&\alpha^2\nabla\mu = \alpha\mu(\xi\alpha)\xi + \left(\mu^2+\frac{c}{4}\right)(\xi\alpha)W\\
&\qquad + \left\{\mu^2\left(\frac{c}{4}-3\alpha^2\right)-\frac{c^2}{16}\right\}\phi W,\\
&\nabla_\xi W = -4\alpha\phi W, \qquad \mu\nabla_{\phi W}W = -\frac{c}{4\alpha}\mu\xi + \frac{c}{4\alpha^2}(\xi\alpha)\phi W.
\end{aligned} \tag{4.8}$$

Proof. At first, we see from (3.13) and (3.15) that

$$\xi\mu = W\alpha = \frac{\mu}{\alpha}\xi\alpha, \qquad \xi\gamma = W\mu. \tag{4.9}$$

If we substitute (3.9), (3.11) and (3.12) into (4.1), then we have

$$\begin{aligned}
&2\mu(A-\gamma I)\nabla_\xi W - \alpha(A-\gamma I)\nabla_W W - \frac{4\alpha^2}{c}\gamma\left(A+\frac{c}{4\alpha}I\right)\phi\nabla_{\phi W}W\\
&= (2\mu\xi\mu - \alpha W\mu)\xi + (2\mu\xi\gamma - \alpha W\gamma)W\\
&\quad - \left\{\gamma((\phi W)\alpha) - 2\mu\left(\mu^2+\frac{c}{4\alpha}\gamma\right) + \alpha\mu\left(\gamma-\frac{c}{2\alpha}\right)\right\}\phi W.
\end{aligned}$$

Taking the inner product of this equation with ξ and W, respectively, we obtain

$$\begin{aligned}
4\alpha\mu\left(\mu^2-\frac{c}{4}\right)g(\nabla_{\phi W}W,\phi W) &= c(2\mu\xi\mu - \alpha W\mu),\\
4\mu^2\left(\mu^2-\frac{c}{4}\right)g(\nabla_{\phi W}W,\phi W) &= c(2\mu\xi\gamma - \alpha W\gamma)
\end{aligned}$$

by making use of (2.6), (3.7) and the relations $\eta(\nabla_\xi W)=\eta(\nabla_W W)=0$ and $c/4+\alpha\gamma=\mu^2$. From the above two equations, we have

$$\xi\gamma = W\mu = \frac{1}{\alpha^2}\left(\mu^2+\frac{c}{4}\right)\xi\alpha \tag{4.10}$$

by virtue of (4.9) and the equation $\alpha W\gamma = 2\mu W\mu - \gamma W\alpha$.

Substituting (3.9), (3.10) and (3.11) into (4.1) and using (4.1) and (4.9), we have

$$\mu(2A - \gamma I)\nabla_\xi W - \alpha(A - \gamma I)\nabla_W W = \frac{c}{4\alpha^2}\mu\left(4\alpha^2 - \mu^2 + \frac{c}{4}\right)\phi W.$$

If we take the inner product of the above equation with ϕW and use (3.2), then we get

$$\begin{aligned} -(\mu^2 + \frac{c}{4})g(\nabla_\xi W, \phi W) + \alpha\mu g(\nabla_W W, \phi W) \\ = \frac{c}{4\alpha}\left(4\alpha^2 - \mu^2 + \frac{c}{4}\right). \end{aligned} \tag{4.11}$$

On the other hand, taking the inner product of (3.16) with ϕW, we can easily find that

$$\mu^2 g(\nabla_\xi W, \phi W) - \alpha\mu g(\nabla_W W, \phi W) = \frac{c}{4\alpha}\left(\mu^2 - \frac{c}{4}\right). \tag{4.12}$$

It follows from (4.11) and (4.12) that $g(\nabla_\xi W, \phi W) = -4\alpha$, and hence from Lemma 4.1 that the third equation of (4.8) holds on Ω. Substituting the third of (4.8) into (3.14) and taking the inner product of it with ϕW, we obtain $\mu g(\nabla_{\phi W} W, \phi W) = c/(4\alpha^2)\xi\alpha$. Since we have $g(\nabla_{\phi W} W, \xi) = -c/(4\alpha)$, we get the fourth equation of (4.8) by Lemma 4.1.

Using (4.9), we can verify from (3.10) and the third of (4.8) that the first equation of (4.8) holds on Ω. By substituting the third of (4.8) into (3.9) and making use of (4.9) and (4.10), we obtain the second equation of (4.8). $\square$

Lemma 4.3. *Under the assumptions of Lemma* 4.1, *if* $c/4+\alpha\gamma=0$ *holds on* Ω, *then we have*

$$\begin{aligned}\nabla\alpha &= (\xi\alpha)\xi - 3\alpha\mu\phi W,\\ \nabla_\xi W &= -\left(4\alpha+\frac{3c}{4\alpha}\right)\phi W,\\ \nabla_{\phi W}W &= -\frac{c}{4\alpha}\xi + \frac{c}{4\alpha^2\mu}(\xi\alpha)\phi W.\end{aligned} \tag{4.13}$$

Proof. It follows from (3.13) and (3.15) that

$$W\alpha = \xi\mu = 0, \qquad \xi\gamma = W\mu \tag{4.14}$$

on Ω. Using (4.5) and (4.14), we have

$$\alpha\xi\gamma + \gamma\xi\alpha = 0, \qquad W\gamma = 0. \tag{4.15}$$

If we substitute (3.10) and (3.11) into (4.5) and use (4.14) and (4.15), then we obtain

$$\frac{c}{4}\mu\nabla_\xi W + \alpha\left(\alpha A + \frac{c}{4}I\right)\nabla_W W = -c\mu\left(\frac{3c}{16\alpha}+\alpha\right)\phi W. \tag{4.16}$$

Taking the inner product of (4.16) with ϕW and using (3.2), we get $g(\nabla_\xi W, \phi W) = -(4\alpha + (3c)/(4\alpha))$. Therefore, we see from Lemma 4.1 that the second equation of (4.13) holds on Ω. Moreover, if we substitute the second of (4.13) into (3.14) and take the inner product of it with ϕW, then we obtain $\mu g(\nabla_{\phi W}W, \phi W) = c/(4\alpha^2)(\xi\alpha)$. Thus, the third equation of (4.13) follows immediately from Lemma 4.1. By comparing (3.10) with the second equation of (4.13) and using (4.14) and Lemma 4.1, we have the first equation of (4.13). □

5. Characterizations of real hypersurfaces. At first, we shall prove Theorem 1 given in the Introduction.

Proof of Theorem 1. Assume that the open set $\Omega = \{p \in M \mid \mu(p) \neq 0\}$ is not empty. Then we can consider from (3.8) that either $c/4+\alpha\gamma = \mu^2$ or $c/4+\alpha\gamma = 0$ hold on Ω.

In the case where $c/4+\alpha\gamma=\mu^2$ holds on Ω, it follows from the first equation of (4.8) in Lemma 4.2 that $X\alpha^2 = 2(\xi\alpha)\eta(AX) - 6\mu(\alpha^2 - c/4)v(X)$, where v is the dual 1-form of the unit vector field ϕW, that is, $v(X) = g(\phi W, X)$. From the identity $[X,Y]\alpha^2 = XY\alpha^2 - YX\alpha^2$, we can verify that

$$
\begin{aligned}
&(X\xi\alpha)\eta(AY) - (Y\xi\alpha)\eta(AX)\\
&+ 2(\xi\alpha)g(A\phi AX, Y) - \frac{c}{2}(\xi\alpha)g(\phi X, Y)\\
&- 3\left(\alpha^2 - \frac{c}{4}\right)\{(X\mu)v(Y) - (Y\mu)v(X)\}\\
&- 6\alpha\mu\{(X\alpha)v(Y) - (Y\alpha)v(X)\}\\
&- 6\mu\left(\alpha^2 - \frac{c}{4}\right) dv(X, Y) = 0
\end{aligned}
\tag{5.1}
$$

by virtue of equations (2.1) and (2.3), where

$$2dv(X, Y) = Xv(Y) - Yv(X) - v([X, Y])$$

for any vector fields X and Y on Ω. Since we have $dv(\xi, \phi W) = 0$ by (2.1) and (3.2), putting $X = \phi W$ and $Y = \xi$ into (5.1) and using (3.2), (3.7) and (4.9) yields

$$(\phi W)\xi\alpha = \mu\left(\frac{c}{4\alpha^2} - 9\right)\xi\alpha. \tag{5.2}$$

By virtue of the fourth equation of (4.8), we obtain $dv(\phi W, W) = -c/(8\alpha^2\mu)\xi\alpha$. If we put $X = \phi W$ and $Y = W$ into (5.1) and make use of (3.2), (3.7), (4.9), (4.10) and (5.2), then we have $(\alpha^2 - c/4)\xi\alpha = 0$, from which $\xi\alpha = 0$ on Ω.

By use of (3.7) and the third equation of (4.8), we obtain

$$2dv(\xi, W) = 4\alpha + \gamma = \frac{1}{\alpha}\left(4\alpha^2 + \mu^2 - \frac{c}{4}\right) \tag{5.3}$$

on Ω. If we put $X = \xi$ and $Y = W$ into (5.1) and use (5.3) and $\xi\alpha = 0$, then we have

$$\left(\alpha^2 - \frac{c}{4}\right)\left(4\alpha^2 + \mu^2 - \frac{c}{4}\right) = 0. \tag{5.4}$$

The second equation of (4.8) is rewritten as $X\mu = (1/\alpha^2)\{\mu^2(c/4 - 3\alpha^2) - (c^2/16)\}v(X)$. From the identity $[X, Y]\mu = XY\mu - YX\mu$, we have

$$\begin{aligned}&c\left(\mu^2-\frac{c}{4}\right)\{(X\alpha)v(Y)-(Y\alpha)v(X)\}\\&-4\alpha\mu\left(\frac{c}{4}-3\alpha^2\right)\{(X\mu)v(Y)-(Y\mu)v(X)\}\\&-4\alpha\left\{\mu^2\left(\frac{c}{4}-3\alpha^2\right)-\frac{c^2}{16}\right\}dv(X,Y)=0\end{aligned}$$

for any vector fields X and Y on Ω. Putting $X=\xi$ and $Y=W$ into the above equation and using (5.3), we obtain

$$\left\{\mu^2\left(\frac{c}{4}-3\alpha^2\right)-\frac{c^2}{16}\right\}\left(4\alpha^2+\mu^2-\frac{c}{4}\right)=0. \tag{5.5}$$

It follows from (5.4) and (5.5) that $4\alpha^2+\mu^2=c/4$ holds on Ω. Substituting the first and second equations into $4\alpha\nabla\alpha+\mu\nabla\mu=0$ and using $4\alpha^2+\mu^2=c/4$, we have $c=0$, and hence a contradiction.

Therefore, $c/4+\alpha\gamma=0$ holds on Ω. It follows from $\gamma\xi\alpha+\alpha\xi\gamma=0$ and (4.14) that

$$W\alpha=0, \qquad \alpha^2W\mu=\frac{c}{4}\xi\alpha. \tag{5.6}$$

Since we have $[X,Y]\alpha=XY\alpha-YX\alpha$, we can verify from the first equation of (4.13) in Lemma 4.3 that

$$\begin{aligned}&(X\xi\alpha)\eta(Y)-(Y\xi\alpha)\eta(X)+(\xi\alpha)g((\phi A+A\phi)X,Y)\\&-3\mu\{(X\alpha)v(Y)-(Y\alpha)v(X)\}-3\alpha\{(X\mu)v(Y)-(Y\mu)v(X)\}\\&-6\alpha\mu\, dv(X,Y)=0\end{aligned} \tag{5.7}$$

for any vector fields X and Y on Ω. Making use of the third equation of (4.13), we get $dv(W,\phi W)=c/(8\alpha^2\mu)\xi\alpha$. If we put $X=W$ and $Y=\phi W$ into (5.7) and use (5.6), then we can find $\xi\alpha=0$ on Ω. Putting $X=\xi$ and $Y=W$ into (5.7) and using $\xi\alpha=0$ yields $dv(\xi,W)=0$, from which together with the second equation of (4.13), we get $4\alpha^2+c/2=0$ by use of (2.1) and (3.7). Since we have $\nabla\alpha=0$, the first equation of (4.13) turns to $\alpha\mu=0$ on Ω, and it is a contradiction.

Thus, the set Ω is empty, and hence M is a Hopf hypersurface. $\square$

As a characterization of the Hopf hypersurface, we can state:

Theorem 2. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. Then it satisfies $R_\xi L_\xi = L_\xi R_\xi$ on M if and only if M is locally congruent to*:

(1) *a Hopf hypersurface with $A\xi = 0$, or*

(2) *one of the model spaces of type A.*

Proof. By Theorem 1, the real hypersurface M satisfying $R_\xi L_\xi = L_\xi R_\xi$ is a Hopf hypersurface in $M_n(c)$, that is, $A\xi = \alpha\xi$.

In the case where the constant α does not vanish, the assumption $R_\xi L_\xi = L_\xi R_\xi$ is equivalent to

$$A^2\phi + \phi A^2 - 2A\phi A = 0 \tag{5.8}$$

by use of (2.1) and (3.7). On the other hand, if we differentiate $A\xi = \alpha\xi$ covariantly and make use of equation (2.3) of Codazzi, then we have

$$A\phi A - \frac{\alpha}{2}(\phi A + A\phi) - \frac{c}{4}\phi = 0. \tag{5.9}$$

For any vector field X on M such that $AX = \lambda X$, it follows from (5.9) that

$$\left(\lambda - \frac{\alpha}{2}\right)A\phi X = \frac{1}{2}\left(\alpha\lambda + \frac{c}{2}\right)\phi X. \tag{5.10}$$

We can choose an orthonormal frame field $\{X_0 = \xi, X_1, X_2, \ldots, X_{2(n-1)}\}$ on M such that $AX_i = \lambda_i X_i$ for $1 \leq i \leq 2(n-1)$. If $\lambda_i \neq \alpha/2$ for $1 \leq i \leq p \leq 2(n-1)$, then we see from (5.10) that ϕX_i is also a principal direction, say $A\phi X_i = \mu_i \phi X_i$. From (5.8), we have $\mu_i = \lambda_i$, and hence $A\phi X_i = \phi A X_i$ for $1 \leq i \leq p$. If $\lambda_i \neq \alpha/2$ and $\lambda_j = \alpha/2$ for $1 \leq i \leq p$ and $p+1 \leq j \leq 2(n-1)$, respectively, then it follows from (5.8) that

$$A^2\phi X_j - \alpha A\phi X_j + \frac{\alpha^2}{4}\phi X_j = 0. \tag{5.11}$$

Taking the inner product of (5.11) with X_i, we obtain $g(\phi X_j, X_i) = 0$ for $1 \leq i \leq p$. Thus, the vector field ϕX_j is expressed by a linear combination of X_j's only, which implies $A\phi X_j = (\alpha/2)\phi X_j = \phi A X_j$. If $\lambda_j = \alpha/2$ for $1 \leq j \leq 2(n-1)$, then it is easily seen that $A\phi X_j = \phi A X_j$ for all j. Therefore, we have $L_\xi = \phi A - A\phi = 0$ on M. Statement (2) of Theorem 5.2 follows immediately from Theorem A or Theorem B.

In the case where $\alpha = 0$, we see from (2.4) that $R_\xi = (c/4)\{X - \eta(X)\xi\}$. Since we have $R_\xi \xi = 0$, it is easily seen that $R_\xi L_\xi = L_\xi R_\xi$ holds on M. $\square$

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Department of Mathematics, Hankuk University of Foreign Studies, Seoul 130-791, Republic of Korea
Email address: **ibkim@hufs.ac.kr**

Department of Mathematics, Hankuk University of Foreign Studies, Seoul 130-791, Republic of Korea
Email address: **dhlnys@hufs.ac.kr**

Department of Mathematics, Hankuk University of Foreign Studies, Seoul 130-791, Republic of Korea
Email address: **hsong@hufs.ac.kr**