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THE HARMONIC INDEX OF A GRAPH

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ABSTRACT. The harmonic index of a graph G is defined as the sum of weights $\frac{2}{d(v_i)+d(v_j)}$ of all edges v_iv_j of G, where $d(v_i)$ denotes the degree of the vertex v_i in G. In this paper, we study how the harmonic index behaves when the graph is under perturbations. These results are used to provide a simpler method for determining the unicyclic graphs with maximum and minimum harmonic index among all unicyclic graphs, respectively. Moreover, a lower bound for harmonic index is also obtained.

1. Introduction. Let G be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Its *order* is $|V(G)|$, denoted by n, and its *size* is $|E(G)|$, denoted by m. For $v_i \in V(G)$, let $N_G(v_i)$ (or $N(v_i)$ for short) be the set of vertices which are adjacent to v_i in G, and let $d_G(v_i)$ (or $d(v_i)$ for short) be the degree of v_i. Clearly, $d(v_i) = |N(v_i)|$. The maximum and minimum degrees of G are denoted by Δ and δ, respectively.

The Randić index is one of the most successful molecular descriptors in structure-property and structure-activity relationships studies. The Randić index of a graph G is defined as the sum of the weights $(d(v_i)d(v_j))^{-1/2}$ over all edges v_iv_j of G. The mathematical properties of this graph invariant have been studied extensively (see the recent book **[4]** and survey **[6]**). Motivated by the success of the Randić index, various generalizations and modifications were introduced, such as the sum-connectivity index **[7, 10]** and the general sum-connectivity index **[1, 2]**.

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In this paper, we consider another variant of the Randić index, called the *harmonic index* $H(G)$, which is defined as

$$H(G) = \sum_{v_i v_j \in E(G)} \frac{2}{d(v_i) + d(v_j)},$$

where the summation goes over all edges of G.

Favaron et al. [**3**] considered the relationship between the harmonic index and the eigenvalues of graphs; Zhong [**8**] determined the minimum and maximum values of harmonic index on simple connected graphs and trees, and characterized the corresponding extremal graphs. Moreover, some of results in [**8**] are generalized by Ilić [**5**]. Zhong [**9**] determined the minimum and maximum values of harmonic index on unicyclic graphs and characterized the corresponding extremal graphs. In this paper, we present a lower bound for harmonic index and characterize graphs for which this bound is attained. Moreover, we study how the harmonic index behaves when the graph is perturbed by separating, grafting or deleting an edge. These results are used to provide a simpler method of determining the unicyclic graphs with maximum and minimum harmonic index among all unicyclic graphs of order n, respectively.

2. Lower bounds for harmonic index. In this section, we establish a lower bound on $H(G)$ in terms of its structural parameters, such as the number of edges, the number of pendent edges and maximum vertex degree.

Theorem 2.1. *Let G be a simple connected graph of order n with m edges, maximum degree Δ and p pendent edges. Then*

$$H(G) \geq \frac{2p}{\Delta + 1} + \frac{m - p}{\Delta}. \tag{2.1}$$

The equality holds if and only if $G \cong K_{1,n-1}$, or G is a regular graph or G is a $(\Delta, 1)$-semiregular graph.

Proof. Note that there are p pendent edges in G. Then we have

$$\begin{aligned}
H(G) &= \sum_{v_iv_j \in E(G)} \frac{2}{d(v_i)+d(v_j)} \\
&= \sum_{v_iv_j \in E(G) d(v_j)=1} \frac{2}{d(v_i)} \\
&\quad + d(v_j) + \sum_{v_iv_j \in E(G) d(v_i)d(v_j)>1} \frac{2}{d(v_i)+d(v_j)} \\
&\geq \frac{2p}{\Delta+1} + \sum_{v_iv_j \in E(G) d(v_i)d(v_j)>1} \frac{2}{d(v_i)+d(v_j)}, \\
&\qquad \text{as } d(v_i) \leq \Delta \\
&\geq \frac{2p}{\Delta+1} + \frac{m-p}{\Delta}, \quad \text{as } d(v_i), d(v_j) \leq \Delta.
\end{aligned}$$

Now suppose that the equality holds in (2.1). Then all inequalities in the above argument must be equalities. Therefore, we have $d(v_i) = \Delta$ and $d(v_j) = 1$ for each pendent edge $v_iv_j \in E(G)$, and $d(v_i) = \Delta$ for each non-pendent vertex $v_i \in V(G)$. Suppose that $m = p$, i.e., all edges are pendent. Hence, G is the star S_n since G is connected; suppose that $m > p$. If $p = 0$, i.e., there is no pendent vertex in G, then we have $d(v_i) = \Delta$ for each $v_i \in V(G)$. Hence, G is a regular graph. If $p > 0$, in this case we have $d(v_i) = \Delta$ for each non-pendent vertex $v_i \in V(G)$. Hence, G is a $(\Delta, 1)$-semiregular graph.

Conversely, one can easily check that the equality holds in (2.1) for the star S_n or a regular graph or a $(\Delta, 1)$-semiregular graph. This completes the proof. □

In particular, there is no pendent edge in G. Then we have:

Corollary 2.2. *Let G be a simple connected graph of order n with m edges and $\delta > 1$. Then*

$$H(G) \geq \frac{m}{\Delta}.$$

The equality holds if and only if G is a regular graph.

3. Effects on harmonic index under graph perturbations. In this section, we consider how the harmonic index behaves when the graph is perturbed by separating, grafting or deleting an edge.

Let $e = uv$ be an edge of a graph G. Let G' be the graph obtained from G by contracting the edge e into a new vertex u_e and adding a new pendent edge $u_e v_e$, where v_e is a new pendent vertex. We say that G' is obtained from G by separating an edge uv (see Figure 1).

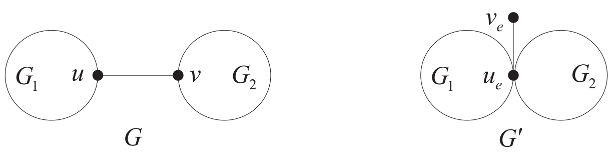

FIGURE 1. Separating an edge uv.

Theorem 3.1. *Let $e = uv$ be a cut edge of a connected graph G, and suppose that $G - uv = G_1 \cup G_2$ ($|V(G_1)|, |V(G_2)| \geq 2$), where G_1 and G_2 are two components of $G - uv$, $u \in V(G_1)$ and $v \in V(G_2)$. Let G' be the graph obtained from G by separating the edge uv. Then $H(G) > H(G')$.*

Proof. Let $N_G(u) = \{x_1, x_2, \ldots, x_p, v\}$ and $N_G(v) = \{y_1, y_2, \ldots, y_q, u\}$. Then $d_{G'}(u_e) = d(u) + d(v) - 1$ and $N_{G'}(u_e) = \{x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q, v_e\}$. Therefore, we have

$$\begin{aligned}
H(G) - H(G') = &\left(\sum_{i=1}^{p} \frac{2}{d(u) + d(x_i)} + \frac{2}{d(u) + d(v)} + \sum_{i=1}^{q} \frac{2}{d(v) + d(y_i)}\right)\\
&- \left(\sum_{i=1}^{p} \frac{2}{d(u)+d(v)-1+d(x_i)} + \frac{2}{d(u)+ d(v)-1+1}\right.\\
&\left. + \sum_{i=1}^{q} \frac{2}{d(u) + d(v) - 1 + d(y_i)}\right)\\
= &\sum_{i=1}^{p} \left(\frac{2}{d(u) + d(x_i)} - \frac{2}{d(u) + d(v) - 1 + d(x_i)}\right)
\end{aligned}$$

$$+\sum_{i=1}^{q}\left(\frac{2}{d(v)+d(y_i)}-\frac{2}{d(u)+d(v)-1+d(y_i)}\right)$$
$$>0, \quad \text{as } d(u), d(v) \geq 2,$$

that is, $H(G) > H(G')$, which completes the proof. □

Note that $H(S_n) = [2(n-1)]/n$. Let T ($T \neq S_n$) be a tree of order n. By repetitive separating of the non-pendent edges of T, the resulting tree is S_n. Then Theorem 3.1 implies that:

Corollary 3.2 ([8]). *Let T be a tree of order $n \geq 3$. Then since $H(T) \geq H(S_n) = [2(n-1)]/n$, the equality holds if and only if $T \cong S_n$.*

Let $S_{n_1,n-n_1}$ be a double star obtained by connecting the centers S_{n_1} and $S(n-n_1)$ with an edge, where $2 \leq n_1 \leq \lfloor n/2 \rfloor$. Then the harmonic index of $S_{n_1,n-n_1}$ is $H(S_{n_1,n-n_1}) = [2(n_1-1)/(n_1+1)] + [2(n-n_1-1)/(n-n_1+1)] + 2/n$.

Similarly, for $n \geq 5$, using Theorem 3.1, we can conclude that the tree with the second minimum value of harmonic index is $S_{2,n-2}$.

Corollary 3.3. *Let T ($T \neq S_n$) be a tree of order $n \geq 5$. Then $H(T) \geq H(S_{2,n-2}) = 2/3 + [2(n-3)/n-1] + 2/n$, the equality holds if and only if $T \cong S_{2,n-2}$.*

Proof. Let e be an non-pendent edge of T, since $T \neq S_n$. Then by Theorem 3.1, we may construct a new tree $S_{n_1,n-n_1}$ such that $H(T) \geq H(S_{n_1,n-n_1})$, where $2 \leq n_1 \leq \lfloor n/2 \rfloor$ and $S_{n_1,n-n_1}$ is obtained from T by separating all non-pendent edges except for e. Note that

$$H(S_{n_1,n-n_1}) = \frac{2(n_1-1)}{n_1+1} + \frac{2(n-n_1-1)}{n-n_1+1} + \frac{2}{n}.$$

Let

$$f(x) = \frac{2(x-1)}{x+1} + \frac{2(n-1-x)}{n-x+1} + \frac{2}{n}$$

for $2 \leq x \leq \lfloor n/2 \rfloor$. Then

$$f'(x) = \frac{4}{(x+1)^2} - \frac{4}{(n-x+1)^2} > 0$$

for $2 \le x \le \lfloor n/2 \rfloor$. Therefore, $f(x)$ is an increasing function for $2 \le x \le \lfloor n/2 \rfloor$. Hence, we have $H(S_{n_1,n-n_1}) = f(n_1) \ge f(2) = 2/3 + [2(n-3)/n-1] + 2/n$. This completes the proof. $\square$

Let u and v be two vertices of a graph G. Suppose that two new paths $P = uu_l \cdots u_2u_1$ and $Q = vv_k \cdots v_2v_1$ of lengths l and k ($l \ge k \ge 1$), respectively, are attached to G at u and v to form a new graph $G^2_{l,k}$ (shown in Figure 2), where $u_1, u_2, \ldots, u_l$ and $v_1, v_2, \ldots, v_k$ are distinct. Let $G^2_{l+1,k-1} = G^2_{l,k} - v_2v_1 + u_1v_1$. We say that $G^2_{l+1,k-1}$ is obtained from $G^2_{l,k}$ by grafting an edge (see Figure 2).

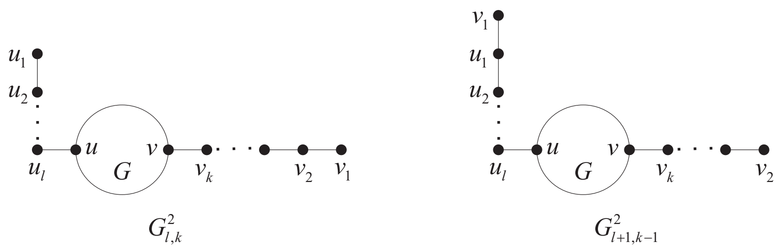

FIGURE 2. Grafting an edge.

Theorem 3.4. *Let $G^2_{l,k}$ and $G^2_{l+1,k-1}$ ($l \ge k \ge 1$) be the graphs as defined above, and let $d_{G^2_{l,k}}(u)$ and $d_{G^2_{l,k}}(v) \ge 3$. We have:*

(i) *if $l \ge k \ge 3$, then $H(G^2_{l,k}) = H(G^2_{l+1,k-1})$;*
(ii) *if $l \ge k = 2$, then $H(G^2_{l,2}) > H(G^2_{l+1,1})$. Moreover, when $d_{G^2_{l,k}}(v) \ge 4$, $H(G^2_{l,2}) < H(G^2_{l+2,0})$; when $d_{G^2_{l,k}}(v) = 3$, let $N_{G^2_{l,k}}(v) = \{v_2, y_1, y_2\}$. If*

$$\frac{2}{(2+d(y_1))(3+d(y_1))} + \frac{2}{(2+d(y_2))(3+d(y_2))} > \frac{1}{15},$$

then $H(G^2_{l,2}) < H(G^2_{l+2,0})$;
(iii) *if $l \ge k = 1$, then $H(G^2_{l,1}) < H(G^2_{l+1,0})$.*

Combining (i)–(iii), *we have that* $H(G^2_{l+k,0}) > H(G^2_{l,k})$ *holds for* $l \geq k \geq 1$ *when* $d_{G^2_{l,k}}(v) \geq 4$; *when* $d_{G^2_{l,k}}(v) = 3$, *if*

$$\frac{2}{(2+d(y_1))(3+d(y_1))} + \frac{2}{(2+d(y_2))(3+d(y_2))} > \frac{1}{15},$$

then $H(G^2_{l+k,0}) > H(G^2_{l,k})$ *holds for* $l \geq k \geq 1$, *where* $N_{G^2_{l,k}}(v) = \{v_k, y_1, y_2\}$.

Proof. Let $N_{G^2_{l,k}}(u) = \{x_1, x_2, \ldots, x_p, u_l\}$, $N_{G^2_{l,k}}(v) = \{y_1, y_2, \ldots, y_q, v_k\}$, $d_{G^2_{l,k}}(u) = x$ and $d_{G^2_{l,k}}(v) = y$.

(i) $l \geq k \geq 3$. It is easy to see that $H(G^2_{l,k}) = H(G^2_{l+1,k-1})$.
(ii) $l \geq k = 2$. Then we have

$$\begin{aligned}
&H(G^2_{l,2}) - H(G^2_{l+1,1})\\
&\quad= \Bigg(\sum_{i=1}^{p}\frac{2}{x+d(x_i)} + \frac{2}{x+2} + \underbrace{\frac{1}{2}+\cdots+\frac{1}{2}}_{l-2} + \frac{2}{3} + \sum_{i=1}^{q}\frac{2}{y+d(y_i)} + \frac{2}{y+2} + \frac{2}{3}\Bigg)\\
&\qquad - \Bigg(\sum_{i=1}^{p}\frac{2}{x+d(x_i)} + \frac{2}{x+2} + \underbrace{\frac{1}{2}+\cdots+\frac{1}{2}}_{l-1} + \frac{2}{3} + \sum_{i=1}^{q}\frac{2}{y+d(y_i)} + \frac{2}{y+1}\Bigg)\\
&\quad= \left(\frac{2}{3} - \frac{1}{2}\right) - \left(\frac{2}{y+1} - \frac{2}{y+2}\right)\\
&>0, \quad \text{as } y \geq 3,
\end{aligned}$$

which implies that $H(G^2_{l,2}) > H(G^2_{l+1,1})$. Moreover,

$$\begin{aligned}
&H(G^2_{l+2,0}) - H(G^2_{l,2})\\
&\quad= \Bigg(\sum_{i=1}^{p}\frac{2}{x+d(x_i)} + \frac{2}{x+2} + \underbrace{\frac{1}{2}+\cdots+\frac{1}{2}}_{l} + \frac{2}{3} + \sum_{i=1}^{q}\frac{2}{y-1+d(y_i)}\Bigg)\\
&\qquad - \Bigg(\sum_{i=1}^{p}\frac{2}{x+d(x_i)} + \frac{2}{x+2} + \underbrace{\frac{1}{2}+\cdots+\frac{1}{2}}_{l-2}\\
&\qquad + \frac{2}{3} + \sum_{i=1}^{q}\frac{2}{y+d(y_i)} + \frac{2}{y+2} + \frac{2}{3}\Bigg)
\end{aligned}$$

$$= \frac{1}{3} - \frac{2}{y+2} + \sum_{i=1}^{q} \frac{2}{(y-1+d(y_i))(y+d(y_i))}.$$

When $y \geq 4$, we have $1/3 - (2/y+2) \geq 0$. Therefore, $H(G^2_{l+2,0}) > H(G^2_{l,2})$.

When $y = 3$, in this case $q = 2$, if $2/[(2+d(y_1))(3+d(y_1))] + 2/[(2+d(y_2))(3+d(y_2))] > 1/15$, then $H(G^2_{l+2,0}) > H(G^2_{l,2})$.

(iii) $l \geq k = 1$. If $l = 1$, then

$$\begin{aligned} &H(G^2_{1,1}) - H(G^2_{2,0}) \\ &\quad = \left(\sum_{i=1}^{p} \frac{2}{x+d(x_i)} + \frac{2}{x+1} + \sum_{i=1}^{q} \frac{2}{y+d(y_i)} + \frac{2}{y+1} \right) \\ &\qquad - \left(\sum_{i=1}^{p} \frac{2}{x+d(x_i)} + \frac{2}{x+2} + \frac{2}{2+1} + \sum_{i=1}^{q} \frac{2}{y-1+d(y_i)} \right) \\ &\quad = \frac{2}{x+1} - \frac{2}{x+2} + \frac{2}{y+1} - \frac{2}{3} \\ &\qquad + \sum_{i=1}^{q} \left(\frac{2}{y+d(y_i)} - \frac{2}{y-1+d(y_i)} \right) \\ &\quad \leq \frac{1}{10} + \frac{1}{2} - \frac{2}{3} \\ &\qquad + \sum_{i=1}^{q} \left(\frac{2}{y+d(y_i)} - \frac{2}{y-1+d(y_i)} \right), \quad \text{as } x, y \geq 3 \\ &\quad \leq -\frac{1}{15} < 0, \end{aligned}$$

that is, $H(G^2_{1,1}) < H(G^2_{2,0})$.

If $l > 1$, then

$$\begin{aligned} &H(G^2_{l,1}) - H(G^2_{l+1,0}) \\ &= \left(\sum_{i=1}^{p} \frac{2}{x+d(x_i)} + \frac{2}{x+2} + \underbrace{\frac{1}{2} + \cdots + \frac{1}{2}}_{l-2} + \frac{2}{3} + \sum_{i=1}^{q} \frac{2}{y+d(y_i)} + \frac{2}{y+1} \right) \\ &\quad - \left(\sum_{i=1}^{p} \frac{2}{x+d(x_i)} + \frac{2}{x+2} + \underbrace{\frac{1}{2} + \cdots + \frac{1}{2}}_{l-1} + \frac{2}{3} + \sum_{i=1}^{q} \frac{2}{y-1+d(y_i)} \right) \end{aligned}$$

$$= \frac{2}{y+1} - \frac{1}{2} + \sum_{i=1}^{q} \left(\frac{2}{y+d(y_i)} - \frac{2}{y-1+d(y_i)} \right)$$
$$< 0, \quad \text{as } y \geq 3,$$

that is, $H(G^2_{l,1}) < H(G^2_{l+1,0})$, which completes the proof. $\square$

A special case of Theorem 3.4 is that $u = v$ in a graph G, that is, two new paths $P = uu_l \cdots u_2u_1$ and $Q = uv_k \cdots v_2v_1$ of lengths l and k ($l \geq k \geq 1$), respectively, are attached to G at u to form a new graph $G_{l,k}$, where $u_1, u_2, \ldots, u_l$ and $v_1, v_2, \ldots, v_k$ are distinct. Let $G_{l+1,k-1} = G_{l,k} - v_2v_1 + u_1v_1$. We say that $G_{l+1,k-1}$ is obtained from $G_{l,k}$ by grafting an edge.

Ilić [**5**] proved that $H(G_{l+k,0}) > H(G_{l,k})$ for $l \geq k \geq 1$. Similar to the proof of Theorem 3.4, the general result will be obtained.

Theorem 3.5. *Let $G_{l,k}$ and $G_{l+1,k-1}$ ($l \geq k \geq 1$) be the graphs as defined above. We have*

(i) *if $l \geq k \geq 3$, then $H(G_{l,k}) = H(G_{l+1,k-1})$;*
(ii) *if $l \geq k = 2$, then $H(G_{l+1,1}) < H(G_{l,2}) < H(G_{l+2,0})$;*
(iii) *if $l \geq k = 1$, then $H(G_{l,1}) < H(G_{l+1,0})$.*

Combining (1)–(3), *we have that $H(G_{l+k,0}) > H(G_{l,k})$ holds for $l \geq k \geq 1$.*

Note that $H(P_n) = (n-3)/4 + 4/3$. If T ($T \neq P_n$) is a tree of order n, then by Theorem 3.5, the following corollary is immediate.

Corollary 3.6 ([**8**]). *Let T be a tree of order $n \geq 3$. Then $H(T) \leq H(P_n) = (n-3)/4 + 4/3$ with equality if and only if $T \cong P_n$.*

Let $2/[d(u) + d(v)]$ be the weight of an edge $e = uv$. Assume that $e = uv$ is an edge with minimal weight among all edges of G. Ilić [**5**] proved that $H(G) < H(G - uv)$. In what follows, we will show that the harmonic index of a graph strictly decreases by removing a pendent vertex.

Theorem 3.7. *Let G be a connected graph with a pendent vertex v. Then $H(G) > H(G - v)$.*

Proof. Let uv be a pendent edge of G, and let $N_G(u) = \{x_1, x_2, \ldots, x_p, v\}$. Clearly, $p = d(u) - 1$. Then we have

$$\begin{aligned}
H(G) - H(G - v) &= \left(\sum_{i=1}^{p} \frac{2}{d(u) + d(x_i)} + \frac{2}{d(u) + 1} \right) \\
&\quad - \sum_{i=1}^{p} \frac{2}{d(u) - 1 + d(x_i)} \\
&= \frac{2}{d(u) + 1} - \sum_{i=1}^{p} \frac{2}{(d(u) + d(x_i))\,(d(u) - 1 + d(x_i))} \\
&\geq \frac{2}{d(u) + 1} - \frac{2(d(u) - 1)}{d(u)(d(u) + 1)}, \\
&\quad \text{as } d(x_i) \geq 1 \text{ for } i = 1, 2, \ldots, p \\
&= \frac{2}{d(u)(d(u) + 1)} > 0,
\end{aligned}$$

that is, $H(G) > H(G - v)$. $\square$

Remark 3.8. Similarly, we have

(i) Let $e = uv$ be an edge of G such that uv does not belong to any triangle. Let G^0 be the graph obtained from G by contracting the edge e into a new vertex u_e. Then $H(G) > H(G^0)$.
(ii) Let G^+ be a graph obtained from a graph G by inserting a vertex of degree 2 in an edge $e = uv$, where $e \in E(G)$. Then $H(G^+) > H(G)$.

4. Applications. The unicyclic graphs with maximum and minimum harmonic index among all unicyclic graphs of order n were determined by Zhong **[9]**. In this section, using Theorems 3.1, 3.4 and 3.5, we provide a simpler method for determining the unicyclic graphs with maximum and minimum harmonic index among all unicyclic graphs of order n, respectively. To begin, some notation is needed.

Let $\mathscr{U}_n$ be the set of unicyclic graphs of order n, and let $\mathscr{U}_n^g$ be the set of unicyclic graphs of order n with girth g $(3 \leq g \leq n)$. Obviously, if $U \in \mathscr{U}_n^n$, then U is a cycle C_n. Note that, for each $U \in \mathscr{U}_n^g$, U consists of the (unique) cycle (say C_g) of length g and a certain number of trees attached at vertices of C_g having (in total) $n - g$

edges. We assume that the vertices of C_g are $v_1, v_2, \dots, v_g$ (ordered in a natural way around C_g, say in the clockwise direction). Then U can be written as $C(T_1, T_2, \dots, T_g)$, which is obtained from a cycle C_g on vertices $v_1, v_2, \dots, v_g$ by identifying v_i with the root of a tree T_i of order n_i for each $i = 1, 2, \dots, g$, where $n_i \geq 1$ and $\sum_{i=1}^{g} n_i = n$. If T_i, for each i, is a path of order n_i, whose root is a vertex of minimum degree, then we write $U = P(n_1, n_2, \dots, n_g)$. If T_i, for each i, is a star of order n_i, whose root is a vertex of maximum degree, then we write $U = S(n_1, n_2, \dots, n_g)$.

From Theorems 3.1 and 3.5, the following result is immediate.

Theorem 4.1. *Let $U \in C(T_1, T_2, \dots, T_g)$, where $|V(T_i)| = n_i$ for $i = 1, 2, \dots, g$, and $\sum_{i=1}^{g} n_i = n$. Then*

$$H(P(n_1, n_2, \dots, n_g)) \geq H(U) \geq H(S(n_1, n_2, \dots, n_g)),$$

where the degree of the root in P_{n_i} (S_{n_i}) is 1 (respectively, $n_i - 1$). Moreover, both extremal graphs are unique.

Theorem 4.2. *For any $U \in \mathscr{U}_n$, we have $H(U) \leq H(C_n) = n/2$, and the equality holds if and only if $U \cong C_n$.*

Proof. Assume that the girth of U is g. If $g = n$, then $U = C_n$ and the result holds. Now assume that $g < n$. Then U can be rewritten as $C(T_1, T_2, \dots, T_g)$, where $|V(T_i)| = n_i$ for $i = 1, 2, \dots, g$ and $\sum_{i=1}^{g} n_i = n$. Thus, Theorem 4.1 implies that $H(U) \leq H(P(n_1, n_2, \dots, n_g))$. Moreover, for $P(n_1, n_2, \dots, n_g)$, since each vertex belongs to the cycle C_g with degree 2 or 3, it is easy to check that it satisfies the conditions of Theorem 3.4. Then by Theorem 3.4, we have $H(P(n_1, n_2, \dots, n_g)) \leq H(P(n - g + 1, 1, \dots, 1))$. Note that $H(P(n - g + 1, 1, \dots, 1)) = (n - 4/2) + 6/5 + 2/3 = (n - 4/2) + 28/15 < n/2 = H(C_n)$. This completes the proof. $\square$

Lemma 4.3. *For $n_1 \geq n_2 \geq n_3 \geq 1$ and $n_1 + n_2 + n_3 = n$, we have*

$$H(S(n_1, n_2, n_3)) \geq H(S(n - 2, 1, 1)) = \frac{2(n-3)}{n} + \frac{4}{n+1} + \frac{1}{2}.$$

The equality holds if and only if $n_2 = 1$ and $n_3 = 1$.

Proof. Note that

$$H(S(n_1,n_2,n_3)) = \frac{2(n_1-1)}{n_1+2} + \frac{2(n_2-1)}{n_2+2} + \frac{2(n_3-1)}{n_3+2} + \frac{2}{n_1+n_2+2} + \frac{2}{n_1+n_3+2} + \frac{2}{n_2+n_3+2}.$$

Since $n_1 \geq n_2 \geq n_3 \geq 1$ and $n_1+n_2+n_3 = n$, $n_1 = n-n_2-n_3$ and $\lfloor n/3 \rfloor \geq n_2 \geq n_3 \geq 1$, that is,

$$H(S(n-n_2-n_3,n_2,n_3)) = \frac{2(n-n_2-n_3-1)}{n-n_2-n_3+2} + \frac{2(n_2-1)}{n_2+2} + \frac{2(n_3-1)}{n_3+2} + \frac{2}{n-n_3+2} + \frac{2}{n-n_2+2} + \frac{2}{n_2+n_3+2}.$$

Let

$$f(x,y) = \frac{2(n-x-y-1)}{n-x-y+2} + \frac{2(x-1)}{x+2} + \frac{2(y-1)}{y+2} + \frac{2}{n-y+2} + \frac{2}{n-x+2} + \frac{2}{x+y+2}$$

for $\lfloor n/3 \rfloor \geq x \geq y \geq 1$.

Then

$$f_x = -\left[\frac{6}{a^2} - \frac{2}{(a+y)^2}\right] + \left[\frac{6}{b^2} - \frac{2}{(b+y)^2}\right],$$

where $a = n-x+2-y$ and $b = x+2$. Note that $a = n-x-y+2 \geq y+2 \geq x+2 = b$. For $y \geq 1$, it is easy to check that

$$g(t) = \frac{6}{t^2} - \frac{2}{(t+y)^2}$$

is a decreasing function for $t \geq b$. Then $f_x = -g(a) + g(b) \leq 0$. Similarly, we have

$$f_y = -\left[\frac{6}{(n-y+2-x)^2} - \frac{2}{(n-y+2)^2}\right] + \left[\frac{6}{(y+2)^2} - \frac{2}{(y+2+x)^2}\right] \leq 0.$$

Therefore, $f(x,y)$ is a decreasing function for $\lfloor n/3 \rfloor \geq x \geq y \geq 1$, that is,

$$\begin{aligned} H(S(n-n_2-n_3,n_2,n_3)) &= f(n_2,n_3) \geq f(1,1) = H(S(n-2,1,1)) \\ &= \frac{2(n-3)}{n} + \frac{4}{n+1} + \frac{1}{2}. \end{aligned}$$

This completes the proof. $\square$

Theorem 4.4. *For any $U \in \mathscr{U}_n$, we have $H(U) \geq H(S(n-2,1,1)) = [2(n-3)/n] + 4/n+1+1/2$. The equality holds if and only if $U \cong S(n-2,1,1)$.*

Proof. Assume that the girth of U is g. We need to distinguish between two cases: (a) $g = 3$, and (b) $g \geq 4$.

Case (a). $g = 3$. If $U \neq S(n-2,1,1)$, then Theorem 3.1 implies that $H(U) > H(S(n_1,n_2,n_3))$, where $n_1 \geq n_2 \geq n_3 \geq 1$ and $n_1+n_2+n_3 = n$. Thus, the result follows from Lemma 4.3.

Case (b). $g \geq 4$. Then U can be rewritten as $C(T_1,T_2,\ldots,T_g)$, where $|V(T_i)| = n_i$ for $i = 1,2,\ldots,g$ and $\sum_{i=1}^{g} n_i = n$. Theorem 4.1 implies that $H(U) \geq H(S(n_1,n_2,\ldots,n_g))$. Moreover, by Theorem 3.1, we have $H(S(n_1,n_2,\ldots,n_g)) > H(S(n_1',n_2',n_3'))$, where $n_1'+n_2'+n_3' = n$. Then the result follows from Lemma 4.3. $\square$

5. Concluding remarks. In this paper, we mainly study how the harmonic index behaves when the graph is perturbed by separating, grafting or deleting an edge. It would be interesting to consider more graph perturbations, such as adding or rotating an edge.

Moreover, in Theorems 3.4 and 3.5, when $l \geq k \geq 3$, we find some graphs with the same harmonic index. Therefore, the problem of constructing graphs with the same harmonic index (or determining graphs with a given harmonic index) is also interesting.

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