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SUB- AND SUPER-ADDITIVE PROPERTIES OF THE PSI FUNCTION

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ABSTRACT. We prove the following sub- and super-additive properties of the psi function.

(i) The inequality

$$\psi\big((x+y)^\alpha\big) \le \psi(x^\alpha) + \psi(y^\alpha) \quad (\alpha \in \mathbf{R})$$

holds for all $x, y > 0$ if and only if $\alpha \le \alpha_0 = -1.0266\ldots$. Here, α_0 is given by

$$2^{\alpha_0} = \inf_{t>0} \frac{\psi^{-1}\big(2\psi(t)\big)}{t} = 0.4908\ldots.$$

(ii) The inequality

$$\psi(x^\beta) + \psi(y^\beta) \le \psi\big((x+y)^\beta\big) \quad (\beta \in \mathbf{R})$$

is valid for all $x, y > 0$ if and only if $\beta = 0$.

1. Introduction. Euler's classical gamma function is defined for positive real numbers x by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt = \frac{e^{-\gamma x}}{x} \prod_{k=1}^\infty \left\{ \left(1 + \frac{x}{k}\right)^{-1} e^{x/k} \right\}.$$

We are concerned with the logarithmic derivative of the Γ-function,

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

which is known as psi (or digamma) function. In view of its relevance in various fields, like, for example, the theory of special functions, statistics and mathematical physics, the ψ-function has been the subject of intensive work, and many interesting properties were discovered. Here are some of them:

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Series and integral representations:

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty} \frac{x}{k(k+x)} = -\gamma + \int_0^{\infty} \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}\, dt.$$

Asymptotic formula:

$$\psi(x) \sim \log(x) - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - + \cdots \quad (x \to \infty).$$

Reflection and recurrence formulas:

$$\psi(1-x) = \psi(x) + \pi \cot(\pi x), \quad \psi(x+1) = \psi(x) + \frac{1}{x}.$$

Additional information on the ψ-function can be found, for instance, in [**1,** Chapter 6].

Numerous research articles were published in the recent past providing inequalities for the gamma and psi functions and their derivatives. We refer to the detailed bibliography [**23**] as well as to [**4, 14, 18, 24, 25, 26, 27, 28, 29**], and the references given therein.

In this paper, we are interested in certain sub- and super-additive properties of ψ. We recall that a function $f : (0,\infty) \to \mathbf{R}$ is said to be sub-additive, if

$$f(x+y) \le f(x) + f(y) \quad \text{for all} \quad x, y > 0.$$

If the converse inequality holds, then f is called super-additive. These functions have applications in different mathematical branches, like, for example, semi-group theory and functional analysis, and also in economics. See [**5, 6, 7, 10, 11, 12, 16**], [**17,** Chapter 16], [**19, 20, 22**] for more information on this subject.

We ask: do there exist real parameters α and β such that $\psi(x^\alpha)$ is subadditive and that $\psi(x^\beta)$ is super-additive on $(0,\infty)$? It is our aim to answer this question. In the next section, we collect some lemmas which we need to prove our main results. In Section 3, we determine all $\alpha, \beta \in \mathbf{R}$ such that the inequalities

$$\psi\big((x+y)^\alpha\big) \le \psi(x^\alpha) + \psi(y^\alpha)$$

and

$$\psi(x^\beta) + \psi(y^\beta) \le \psi\big((x+y)^\beta\big)$$

are valid for all $x, y > 0$. We conclude our paper with a few remarks. Among others, we study convexity and concavity properties of $\psi(x^\alpha)$. These remarks are given in Section 4.

The numerical values have been calculated via the computer program `MAPLE V, Release 5.1`.

2. Lemmas. Throughout this paper, we denote by $x_0 = 1.4616\ldots$ the only positive zero of ψ. The first three lemmas provide known monotonicity properties of functions which are defined in terms of the ψ-function. Proofs are given in [**2, 3**].

Lemma 1. *The function*

$$x \longmapsto x\psi(x)$$

is strictly decreasing on $(0, r_0]$, *where* $r_0 = 0.2160\ldots$.

Lemma 2. *Let* $k \in \mathbf{N}$. *The function*

$$\tau_k(x) = x^{k+1}|\psi^{(k)}(x)| \tag{2.1}$$

is strictly increasing on $(0, \infty)$.

Lemma 3. *Let* $k \in \mathbf{N}$. *The function*

$$\Delta_k(x) = x\frac{\psi^{(k+1)}(x)}{\psi^{(k)}(x)} \tag{2.2}$$

is strictly increasing on $(0, \infty)$.

Lemma 4. *Let*

$$P_b(x) = x^b\psi'(x) \quad (b \in \mathbf{R}). \tag{2.3}$$

(i) *If* $b < 1.98$, *then* P_b *is strictly decreasing on* $(0, 0.08)$.
(ii) *If* $1.97 < b < 1.98$, *then* P_b *is strictly convex on* $(0, x_0)$.

Proof. (i) Let $b < 1.98$ and $0 < x < 0.08$. Differentiation leads to

$$x^{-b}P_b'(x) = b\frac{\psi'(x)}{x} + \psi''(x).$$

Since ψ' is positive on $(0,\infty)$, we obtain

$$x^{-b}P_b'(x) < 1.98\frac{\psi'(x)}{x} + \psi''(x) = \frac{\psi'(x)}{x}[1.98 + \Delta_1(x)], \tag{2.4}$$

where Δ_1 is defined in (2.2). Applying Lemma 3 gives

$$\Delta_1(x) < \Delta_1(0.08) = -1.982\ldots. \tag{2.5}$$

Combining (2.4) and (2.5) reveals that $P_b'(x) < 0$.

(ii) Let $1.97 < b < 1.98$. We consider two cases.

Case 1. $0 < x < 0.1$. By differentiation, we get

$$x^{4-b}P_b''(x) = (b^2 - b)\tau_1(x) - 2b\tau_2(x) + \tau_3(x),$$

where τ_1, τ_2, τ_3 are defined in (2.1). Using $\tau_1(0) = \lim_{x\to 0}\tau_1(x) = 1$, $\tau_3(0) = \lim_{x\to 0}\tau_3(x) = 6$, and Lemma 2 yields

$$x^{4-b}P_b''(x) \ge b^2 - b - 2b\tau_2(0.1) + 6 = q(b), \quad \text{say.}$$

Since q is decreasing on $[1.97,\ 1.98]$ with $q(1.98) = 0.013\ldots$, we obtain $P_b''(x) > 0$.

Case 2. $0.1 \le x \le x_0$. We have

$$\frac{x^{2-b}}{\psi'(x)}P_b''(x) = b^2 - b + 2b\Delta_1(x) + \Delta_1(x)\Delta_2(x),$$

with Δ_1 and Δ_2 as defined in (2.2). Since Δ_1 and Δ_2 are negative and increasing, we conclude that the product $\Delta_1\Delta_2$ is decreasing. Let $0.1 \le r \le x \le s \le x_0$. Then, we get

$$\begin{aligned}\frac{x^{2-b}}{\psi'(x)}P_b''(x) &\ge 1.97^2 - 1.97 + 2\cdot 1.98\Delta_1(r) + \Delta_1(s)\Delta_2(s)\\ &= K(r,s), \quad \text{say.}\end{aligned}$$

Since

$$K\Big(0.1 + \frac{k}{150}, 0.1 + \frac{k+1}{150}\Big) > 0$$

for $k = 0,1,\ldots,203$ and $K(1.46, x_0) = 0.037\ldots$, we find that P_b'' is positive on $(0, x_0)$. $\square$

Lemma 5. *Let* $c = 0.49084$. *For* $t > 0$ *we have*

$$\psi(ct) < 2\psi(t).$$

Proof. To show that $g(t) = 2\psi(t) - \psi(ct)$ is positive on $(0, \infty)$ we consider three cases.

Case 1. $0 < t \le 0.2$. Let $0 < a \le 1.08$ and $h_a(t) = t\psi(at)$. Since $0 < at \le 0.216$, we conclude from Lemma 1 that h_a is decreasing on $(0, 0.2]$. We set $0 \le r \le t \le s \le 0.2$ and obtain

$$tg(t) = 2h_1(t) - h_c(t) \ge 2h_1(s) - h_c(r) = H(r, s), \quad \text{say.}$$

By direct computation, we find

$$H(0, 0.03) = 0.0055\ldots,$$
$$H\left(0.03 + \frac{k}{100}, 0.03 + \frac{k+1}{100}\right) > 0 \quad \text{for } k = 0, 1, 2, 3, 4,$$
$$H\left(0.08 + \frac{k}{400}, 0.08 + \frac{k+1}{400}\right) > 0 \quad \text{for } k = 0, 1, \ldots, 12,$$
$$H\left(0.1125 + \frac{k}{2000}, 0.1125 + \frac{k+1}{2000}\right) > 0 \quad \text{for } k = 0, 1, \ldots, 27,$$
$$H\left(0.1265 + \frac{k}{12000}, 0.1265 + \frac{k+1}{12000}\right) > 0 \quad \text{for } k = 0, 1, \ldots, 101,$$
$$H\left(0.135 + \frac{k}{20000}, 0.135 + \frac{k+1}{20000}\right) > 0 \quad \text{for } k = 0, 1, \ldots, 99,$$
$$H\left(0.14 + \frac{k}{8000}, 0.14 + \frac{k+1}{8000}\right) > 0 \quad \text{for } k = 0, 1, \ldots, 79,$$
$$H\left(0.15 + \frac{k}{900}, 0.15 + \frac{k+1}{900}\right) > 0 \quad \text{for } k = 0, 1, \ldots, 44.$$

This leads to $g(t) > 0$ for $t \in (0, 0.2]$.

Case 2. $0.2 \le t \le x_0$. Let $0.2 \le r \le t \le s \le x_0$. Since ψ is strictly increasing on $(0, \infty)$, we obtain

$$g(t) \ge 2\psi(r) - \psi(cs) = I(r, s), \quad \text{say.}$$

We have

$$I\left(0.2 + \frac{k}{1500}, 0.2 + \frac{k+1}{1500}\right) > 0 \quad \text{for } k = 0, 1, \ldots, 149,$$
$$I\left(0.3 + \frac{k}{160}, 0.3 + \frac{k+1}{160}\right) > 0 \quad \text{for } k = 0, 1, \ldots, 15,$$

$$I\left(0.4+\frac{k}{50},0.4+\frac{k+1}{50}\right)>0 \quad \text{for } k=0,1,\ldots,29,$$
$$I(1,x_0)=0.017\ldots.$$

Thus, $g(t)>0$ for $t\in[0.2,x_0]$.

Case 3. $t\geq x_0$. Since $\psi(t)\geq 0$ and $\psi(t)>\psi(ct)$, we get $g(t)=\psi(t)+[\psi(t)-\psi(ct)]>0$. □

Lemma 6. *Let* $\alpha\in[-1.027,-1.026]$ *and*

$$F_\alpha(s,t)=\psi(s)+\psi(t)-\psi\big([s^{1/\alpha}+t^{1/\alpha}]^\alpha\big).$$

There exists a function σ_α *such that* $F_\alpha(s,t)\geq\sigma_\alpha(s)$ *for all* $s,t\in\mathbf{R}$ *with* $0<s\leq t\leq x_0$ *and* $\lim_{s\to 0}\sigma_\alpha(s)=\infty$.

Proof. We distinguish two cases.

Case 1. $t<0.08$. We have $0<s\leq t<0.08$. Partial differentiation yields

$$t^{1-1/\alpha}\frac{\partial}{\partial t}F_\alpha(s,t)=P_a(t)-P_a([s^{1/\alpha}+t^{1/\alpha}]^\alpha)$$

with $a=1-1/\alpha$ and P_b as defined in (2.3). Since

$$a<1.98 \quad\text{and}\quad 0<[s^{1/\alpha}+t^{1/\alpha}]^\alpha<t<0.08,$$

we conclude from Lemma 4 (i) that $(\partial/\partial t)F_\alpha(s,t)<0$. This implies that $t\mapsto F_\alpha(s,t)$ is strictly decreasing on $[s,0.08]$. Thus, we obtain

$$F_\alpha(s,t)\geq F_\alpha(s,0.08). \tag{2.6}$$

Let $A=[s^{1/\alpha}+0.08^{1/\alpha}]^\alpha$. We have $0<A<0.08$. Using the identity $\psi(x)=\psi(x+1)-1/x$ and the monotonicity of ψ gives

$$\begin{aligned} F_\alpha(s,0.08)&=\psi(0.08)+\psi(s+1)-\psi(A+1)+\frac{1}{A}-\frac{1}{s} \\ &\geq\psi(0.08)+\psi(1)-\psi(1.08)+\frac{1}{A}-\frac{1}{s}. \end{aligned} \tag{2.7}$$

Since $-\alpha>1$, we get

$$\frac{1}{A}-\frac{1}{s}=\frac{1}{s}\left[\left(1+\left(\frac{0.08}{s}\right)^{1/\alpha}\right)^{-\alpha}-1\right]\geq 0.08^{1/\alpha}s^{-1-1/\alpha}. \tag{2.8}$$

Combining (2.6)–(2.8) gives

$$F_\alpha(s,t) \geq c_0 + 0.08^{1/\alpha} s^{-1-1/\alpha} \tag{2.9}$$

with $c_0 = \psi(0.08) + \psi(1) - \psi(1.08) = -13.077\ldots.$

Case 2. $0.08 \leq t$. We have $0 < s \leq t \leq x_0$ and $0.08 \leq t$. Let $B = [s^{1/\alpha} + t^{1/\alpha}]^\alpha$. Then, $B \leq 2^{-1.026} x_0$. We obtain

$$\begin{aligned} F_\alpha(s,t) &\geq \psi(s) + \psi(0.08) - \psi(B) \\ &= \psi(0.08) + \psi(s+1) - \psi(B+1) + \frac{1}{B} - \frac{1}{s} \\ &\geq \psi(0.08) + \psi(1) - \psi(2^{-1.026} x_0 + 1) + \frac{1}{B} - \frac{1}{s}. \end{aligned} \tag{2.10}$$

Furthermore,

$$\begin{aligned} \frac{1}{B} - \frac{1}{s} &= \frac{1}{s}\left[\left(1 + \left(\frac{t}{s}\right)^{1/\alpha}\right)^{-\alpha} - 1\right] \\ &\geq \frac{1}{s}\left(\frac{t}{s}\right)^{1/\alpha} \\ &\geq x_0^{1/\alpha} s^{-1-1/\alpha}. \end{aligned} \tag{2.11}$$

From (2.10) and (2.11), we get

$$F_\alpha(s,t) \geq c_0^* + x_0^{1/\alpha} s^{-1-1/\alpha} \tag{2.12}$$

with $c_0^* = \psi(0.08) + \psi(1) - \psi(2^{-1.026} x_0 + 1) = -13.752\ldots.$

We have $x_0^{1/\alpha} < 0.08^{1/\alpha}$ and $c_0^* < c_0$. Thus, from (2.9) and (2.12), we obtain for $s, t \in \mathbf{R}$ with $0 < s \leq t \leq x_0$:

$$F_\alpha(s,t) \geq c_0^* + x_0^{1/\alpha} s^{-1-1/\alpha} = \sigma_\alpha(s), \quad \text{say.}$$

Since $-1 - 1/\alpha < 0$, we conclude that $\lim_{s \to 0} \sigma_\alpha(s) = \infty$. $\square$

3. Main results. We are now ready to describe completely the sub- and super-additive properties of $\psi(x^\alpha)$.

Theorem 1. *Let α be a real number. The inequality*

$$\psi\big((x+y)^\alpha\big) \leq \psi(x^\alpha) + \psi(y^\alpha) \tag{3.1}$$

holds for all positive real numbers x and y if and only if

$$\alpha \leq \alpha_0 = -1.0266\ldots. \tag{3.2}$$

Here, α_0 is given by

$$2^{\alpha_0} = \inf_{t>0} \frac{\psi^{-1}\big(2\psi(t)\big)}{t} = 0.4908\dots . \tag{3.3}$$

(As usual, ψ^{-1} denotes the inverse function of ψ.)

Proof. First, we assume that (3.1) is valid for all $x, y > 0$. If $\alpha > 0$, then we obtain

$$\lim_{x\to 0} \psi\big((x+y)^\alpha\big) = \psi(y^\alpha) \quad \text{and} \quad \lim_{x\to 0} [\psi(x^\alpha) + \psi(y^\alpha)] = -\infty,$$

a contradiction. Hence, $\alpha \le 0$. If $\alpha = 0$, then (3.1) is equivalent to $0 \le \psi(1)$. But, $\psi(1) = -\gamma < 0$. It follows that $\alpha < 0$. We set $x = y$ and $t = x^\alpha$. Then, we get for $t > 0$:

$$\psi(2^\alpha t) \le 2\psi(t).$$

Therefore,

$$2^\alpha \le \frac{\psi^{-1}\big(2\psi(t)\big)}{t} = J(t), \quad \text{say.}$$

This gives $\alpha \le \alpha_0$, where $2^{\alpha_0} = \inf_{t>0} J(t)$. From Lemma 5 we conclude that

$$0.49084 < J(t) \quad \text{for } t > 0,$$

so that $J(0.13654) = 0.49084\dots$ leads to

$$0.49084 \le 2^{\alpha_0} \le 0.49084\dots .$$

It follows that

$$2^{\alpha_0} = 0.4908\dots \quad \text{and} \quad \alpha_0 = -1.0266\dots .$$

Next, we prove: if $\alpha \le \alpha_0$ with α_0 as given in (3.2) and (3.3), respectively, then (3.1) is valid for all $x, y > 0$. We set $x = s^{1/\alpha}$ and $y = t^{1/\alpha}$. Then, (3.1) can be written as

$$\psi\big([s^{1/\alpha} + t^{1/\alpha}]^\alpha\big) \le \psi(s) + \psi(t).$$

Since ψ is increasing on $(0, \infty)$ and $a \mapsto [s^{1/a} + t^{1/a}]^a$ is increasing on $(-\infty, 0)$, see [**8,** page 18], we obtain

$$\psi\big([s^{1/\alpha} + t^{1/\alpha}]^\alpha\big) \le \psi\big([s^{1/\alpha_0} + t^{1/\alpha_0}]^{\alpha_0}\big).$$

Hence, it suffices to show that, if $0 < s \le t$, then

$$\psi\big([s^{1/\alpha_0} + t^{1/\alpha_0}]^{\alpha_0}\big) \le \psi(s) + \psi(t). \tag{3.4}$$

To prove (3.4) we consider two cases.

Case 1. $x_0 < t$. s Since $[s^{1/\alpha_0} + t^{1/\alpha_0}]^{\alpha_0} < s$, we get

$$\psi\big([s^{1/\alpha_0} + t^{1/\alpha_0}]^{\alpha_0}\big) < \psi(s) < \psi(s) + \psi(t).$$

Case 2. $t \le x_0$. Let $W = \{(s,t) \in \mathbf{R}^2 \mid 0 < s \le t \le x_0\}$ and

$$F(s,t) = \psi(s) + \psi(t) - \psi\big([s^{1/\alpha_0} + t^{1/\alpha_0}]^{\alpha_0}\big).$$

We set

$$M = \max_{0.01 \le s \le t \le x_0} F(s,t).$$

Applying Lemma 6 (with $\alpha = \alpha_0$) reveals that there exists a number $\delta > 0$ such that, for all $s, t \in \mathbf{R}$ with $0 < s < \delta$ and $s \le t \le x_0$, we have $F(s,t) \ge M$. Let $\delta^* = \min\{\delta, 0.01\}$. We show that, for all $(\widetilde{s}, \widetilde{t}) \in W$ we have

$$F(\widetilde{s}, \widetilde{t}) \ge \min_{\delta^* \le s \le t \le x_0} F(s,t). \tag{3.5}$$

Case 2.1. $\delta^* \le \widetilde{s}$. Then we have $\delta^* \le \widetilde{s} \le \widetilde{t} \le x_0$. This implies that (3.5) holds.

Case 2.2. $\widetilde{s} \le \delta^*$. Then, $0 < \widetilde{s} \le \delta$, $\widetilde{s} \le \widetilde{t} \le x_0$ and $\delta^* \le 0.01$. It follows that

$$F(\widetilde{s}, \widetilde{t}) \ge M \ge \min_{0.01 \le s \le t \le x_0} F(s,t) \ge \min_{\delta^* \le s \le t \le x_0} F(s,t).$$

Thus, there exist real numbers s_0, t_0 with $(s_0, t_0) \in W$ such that $F(s,t) \ge F(s_0, t_0)$ for all $(s,t) \in W$. We suppose that (s_0, t_0) is an interior point of W. Then we obtain

$$s_0{}^{b_0} \frac{\partial F(s,t)}{\partial s}\bigg|_{(s,t)=(s_0,t_0)} = P_{b_0}(s_0) - P_{b_0}(C) = 0$$

and

$$t_0{}^{b_0} \frac{\partial F(s,t)}{\partial t}\bigg|_{(s,t)=(s_0,t_0)} = P_{b_0}(t_0) - P_{b_0}(C) = 0$$

where $b_0 = 1 - 1/\alpha_0$, $C = [s_0^{1/\alpha_0} + t_0^{1/\alpha_0}]^{\alpha_0}$, and P_b as defined in (2.3). It follows that

$$P_{b_0}(s_0) = P_{b_0}(t_0) = P_{b_0}(C)$$

with $0 < C < s_0 < t_0 < x_0$ and $1.97 < b_0 < 1.98$. This contradicts Lemma 4 (ii). Thus, we have either $0 < s_0 = t_0 \le x_0$ or $0 < s_0 \le t_0 = x_0$. In the first case, we obtain

$$F(s_0, t_0) = 2\psi(t_0) - \psi(2^{\alpha_0} t_0).$$

Since

$$2^{\alpha_0} \le \frac{\psi^{-1}\big(2\psi(t_0)\big)}{t_0},$$

we get $F(s_0, t_0) \ge 0$. And, the second case leads to

$$F(s_0, t_0) = \psi(s_0) - \psi(C) > 0.$$

The proof of Theorem 1 is complete. $\square$

Theorem 2. *Let β be a real number. The inequality*

$$\psi(x^\beta) + \psi(y^\beta) \le \psi\big((x+y)^\beta\big) \tag{3.6}$$

is valid for all positive real numbers x and y if and only if $\beta = 0$.

Proof. Since $\psi(1) = -\gamma$, we conclude that (3.6) holds if $\beta = 0$. Next, we assume that (3.6) is valid for all $x, y > 0$. If $\beta < 0$, then the sum on the left-hand side tends to ∞ as $x \to 0$, whereas the right-hand side converges to $\psi(y^\beta)$. Hence, $\beta \ge 0$. We suppose that $\beta > 0$ and set $x = y$, $t = x^\beta$. Then, (3.6) reads

$$2\psi(t) \le \psi(2^\beta t).$$

This yields for $t > 1$:

$$2\frac{\psi(t)}{\log(t)} \le \frac{\psi(2^\beta t)}{\log(2^\beta t)}\left(1 + \frac{\beta \log(2)}{\log(t)}\right). \tag{3.7}$$

Applying $\lim_{t\to\infty} \psi(t)/\log(t) = 1$ leads to $2 \le 1$. This contradiction gives $\beta = 0$. $\square$

4. Final remarks. (I) In what follows, we set $\Phi_\alpha(x) = \psi(x^\alpha)$. The inequalities (3.1) and (3.6) are related to Jensen's inequality and its converse. Therefore, it is natural to ask for all real parameters α such that Φ_α is convex/concave on $(0, \infty)$.

Remark 1. The inequality

$$\psi\left(\left(\frac{x+y}{2}\right)^\alpha\right) < \frac{\psi(x^\alpha) + \psi(y^\alpha)}{2} \quad (\alpha \in \mathbf{R} \setminus \{0\}) \tag{4.1}$$

holds for all $x, y > 0$ with $x \neq y$ if and only if $\alpha \in [-1, 0)$. The converse of (4.1) is valid for all $x, y > 0$ with $x \neq y$ if and only if $\alpha > 0$.

Proof. Differentiation gives for $\alpha \neq 0$:

$$\frac{x^{2-\alpha}}{\alpha^2 \psi'(x^\alpha)} \Phi''_\alpha(x) = \Delta_1(x^\alpha) + 1 - \frac{1}{\alpha},$$

where Δ_1 is defined in (2.2). Using this identity as well as Lemma 3 (with $k = 1$) and the limit relations

$$\lim_{t \to 0} \Delta_1(t) = -2, \qquad \lim_{t \to \infty} \Delta_1(t) = -1,$$

we conclude that $\Phi''_\alpha(x) > 0$ for $x > 0$ if and only if $-1 \leq \alpha < 0$, and $\Phi''_\alpha(x) < 0$ for $x > 0$ if and only if $\alpha > 0$. $\square$

(II) An application of Remark 1 leads to the following functional inequality.

Remark 2. The inequality

$$\psi((x+y)^\alpha) + \psi(z^\alpha) \leq \psi(x^\alpha) + \psi((y+z)^\alpha) \quad (\alpha \in \mathbf{R} \setminus \{0\})$$

holds for all $x, y, z > 0$ with $x \leq z$ if and only if $\alpha \in [-1, 0)$.

Proof. Let $-1 \leq \alpha < 0$ and

$$Q_\alpha(x, y, z) = \Phi_\alpha(x) + \Phi_\alpha(y+z) - \Phi_\alpha(x+y) - \Phi_\alpha(z).$$

Since Φ_α is convex on $(0, \infty)$, we obtain

$$\frac{\partial}{\partial y} Q_\alpha(x, y, z) = \Phi'_\alpha(y+z) - \Phi'_\alpha(x+y) \geq 0.$$

This gives

$$Q_\alpha(x, y, z) \geq Q_\alpha(x, 0, z) = 0.$$

Let $Q_\alpha(x,y,z) \geq 0$ for all $x,y,z>0$ with $x \leq z$. If $\alpha > 0$, then $\lim_{x\to 0} Q_\alpha(x,y,z) = -\infty$. This contradiction leads to $\alpha < 0$. Then, for $z \geq x$, we get

$$\begin{aligned} Q_\alpha(x,x,z) &= \Phi_\alpha(x) + \Phi_\alpha(x+z) - \Phi_\alpha(2x) - \Phi_\alpha(z) \geq 0 \\ &= Q_\alpha(x,x,x). \end{aligned}$$

This gives

$$\begin{aligned} (4.2) \quad 0 \leq (2x)^{\alpha+1} \frac{d}{dz} Q_\alpha(x,x,z)\Big|_{z=x} \\ = \alpha\Big[(2x)^{2\alpha}\psi'((2x)^\alpha) - 2^{\alpha+1}x^{2\alpha}\psi'(x^\alpha)\Big]. \end{aligned}$$

We let x tend to ∞ and make use of the limit relation $\lim_{t\to 0} t^2\psi'(t) = 1$. Then, (4.2) leads to $0 \leq \alpha(1-2^{\alpha+1})$. Thus, $\alpha \geq -1$. $\square$

(III) The weighted power mean of order r is defined for positive real numbers $a_1,\ldots,a_n$ and $w_1,\ldots,w_n$ with $w_1+\cdots+w_n = 1$ by

$$M(r) = \left(\sum_{k=1}^n w_k a_k^r\right)^{1/r} \qquad (r \in \mathbf{R}\setminus\{0\}).$$

The main properties of this family of mean-values are collected in [**15**, Chapter 2]. In 1972, Beesack [**9**] presented a proof for the following remarkable inequality:

$$(4.3) \qquad \frac{M(t)-M(r)}{M(t)-M(s)} < \frac{s(t-r)}{r(t-s)} \quad (0<r<s<t).$$

The validity of (4.3) for the special case $w_1 = \cdots = w_n = 1/n$ was conjectured by Hsu in 1955. Here is a counterpart of (4.3) for the psi function.

Remark 3. The inequality

$$(4.4) \qquad \frac{\psi(t^\alpha)-\psi(r^\alpha)}{\psi(t^\alpha)-\psi(s^\alpha)} < \frac{s(t-r)}{r(t-s)} \quad (\alpha \in \mathbf{R}\setminus\{0\})$$

holds for all real numbers r,s,t with $0<r<s<t$ if and only if $\alpha < 0$ or $0 < \alpha \leq 1$.

Proof. Let $0<r<s<t$. To prove (4.4) we consider two cases.

Case 1. $0 < \alpha \leq 1$. Since $\Phi_{-\alpha}$ is strictly convex on $(0,\infty)$, we obtain for $x, y > 0$ with $x \neq y$ and $\lambda \in (0,1)$:

$$\Phi_{-\alpha}(\lambda x + (1-\lambda)y) < \lambda\Phi_{-\alpha}(x) + (1-\lambda)\Phi_{-\alpha}(y). \tag{4.5}$$

We set

$$x = \frac{1}{t}, \quad y = \frac{1}{r}, \quad \text{and} \quad \lambda = \frac{t(s-r)}{s(t-r)}.$$

Then, (4.5) gives

$$\Phi_\alpha(s) < \frac{t(s-r)}{s(t-r)}\Phi_\alpha(t) + \frac{r(t-s)}{s(t-r)}\Phi_\alpha(r).$$

This is equivalent to

$$r(t-s)[\Phi_\alpha(t) - \Phi_\alpha(r)] < s(t-r)[\Phi_\alpha(t) - \Phi_\alpha(s)]. \tag{4.6}$$

The function Φ_α is strictly increasing on $(0,\infty)$, so that (4.6) implies (4.4).

Case 2. $\alpha < 0$. The strict concavity of $\Phi_{-\alpha}$ reveals that (4.5) and (4.6) are valid with "$>$" instead of "$<$." Since Φ_α is strictly decreasing on $(0,\infty)$, we conclude that (4.6) leads to (4.4).

Conversely, let (4.4) be valid for all r, s, t with $0 < r < s < t$. We assume that $\alpha > 1$. Then we get

$$R_\alpha(r,t) < R_\alpha(s,t)$$

with

$$R_\alpha(x,t) = x\frac{\psi(t^\alpha) - \psi(x^\alpha)}{t-x}.$$

Let $0 < x < t$. We obtain

$$\begin{aligned} 0 &\leq (t-x)^2 x^\alpha \frac{\partial}{\partial x} R_\alpha(x,t) \\ &= -t[x^\alpha\psi(x^\alpha) - x^\alpha\psi(t^\alpha)] + \alpha(x-t)x^{2\alpha}\psi'(x^\alpha), \end{aligned} \tag{4.7}$$

and let x tend to 0. Then, the expression on the right-hand side of (4.7) converges to $t(1-\alpha)$. Hence, $\alpha \leq 1$. □

(IV) The logarithmic mean of two positive real numbers x, y with $x \neq y$ is defined by

$$L(x,y) = \frac{x-y}{\log(x) - \log(y)}.$$

This mean value plays a role not only in mathematics, but it also has applications in physics and economics. For more information on this subject we refer to **[21]** and the references given therein. In 2008, Chu et al. **[13]** proved an elegant inequality involving the psi function and the logarithmic mean:

$$(4.8) \qquad (y-x)\psi(\sqrt{xy}) < \big(L(x,y)-x\big)\psi(x) + \big(y-L(x,y)\big)\psi(y) \qquad (2 \le x < y).$$

The authors conjectured that (4.8) is valid for all $x, y > 0$ with $y > x$. However, this conjecture is not true. To show this we set $y = x_0$ and multiply both sides of (4.8) by $\sqrt{x_0 x}$. This leads to

$$\begin{aligned}(x_0-x)\sqrt{x_0 x}\psi(\sqrt{x_0 x}) &< \big(L(x,x_0)-x\big)\sqrt{x_0 x}\psi(x) \\ &= (-x\psi(x))\left[\sqrt{x_0 x} + \frac{x_0 - x}{2\sqrt{x/x_0}\log\sqrt{x/x_0}}\right].\end{aligned}$$

If x tends to 0, then the expression on the left-hand side converges to $-x_0$, whereas the right-hand side tends to $-\infty$, a contradiction.

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