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A STUDY ON THE INTEGRAL OF THE PRODUCT OF SEVERAL BERNOULLI POLYNOMIALS

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ABSTRACT. The purpose of this paper is to give some properties of several Bernoulli polynomials to express the integral of those polynomials from 0 to 1 in terms of beta and gamma functions. From those properties, we derive some relations on the integral of the product of Bernoulli polynomials and new identities on the Bernoulli numbers.

1. Introduction. The Bernoulli polynomials are given by the generating function as follows:

$$F(t,x) = \frac{t}{e^t - 1} e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1}$$

with the usual convention about replacing $B^n(x)$ by $B_n(x)$ (see [**1–4, 6–23**]). In the special case, $x = 0$, $B_n(0) = B_n$ are called the nth Bernoulli numbers. From (1), we have

$$B_n(x) = (B + x)^n = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l}, \quad n \in \mathbf{Z}_+. \tag{2}$$

By (1) and (2), we get the recurrence relation for the Bernoulli numbers as follows:

$$B_0 = 1, \qquad (B+1)^n - B_n = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{for } n > 1. \end{cases} \tag{3}$$

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From (2), we note that

$$\frac{d}{dx}B_n(x) = \sum_{l=0}^{n-1}\binom{n}{l}B_l(n-l)x^{n-l-1} \tag{4}$$
$$= n\sum_{l=0}^{n-1}\binom{n-1}{l}B_l x^{n-l-1} = nB_{n-1}(x).$$

By (4), we get

$$\int_0^1 B_n(x)\,dx = \frac{1}{n+1}\int_0^1 \frac{d}{dx}(B_{n+1}(x))\,dx = \frac{B_{n+1}(1)-B_{n+1}}{n+1}. \tag{5}$$

Let $n \in \mathbf{N}$. Then, by (3) and (5), we see that

$$\int_0^1 B_n(x)\,dx = 0. \tag{6}$$

Let $C([0,1])$ be the space of continuous functions on $[0,1]$. For $f \in C([0,1])$, the Bernstein operator for f is defined by

$$\mathbf{B}_n(f,x) = \sum_{k=0}^{n} f(\frac{k}{n})B_{k,n}(x), \quad \text{for } 0 \leq x \leq 1, \tag{7}$$

where

$$B_{k,n}(x) = x^k(1-x)^{n-k}, \quad n,k \in \mathbf{Z}_+ = \mathbf{N}\cup\{0\}, \quad (\text{see } [\mathbf{8}]). \tag{8}$$

Here $B_{k,n}(x)$ are called the Bernstein polynomials of degree n. By (8), we easily see that $B_{k,n}(x) = B_{n-k,n}(1-x)$. From(1), we can derive the following equation:

$$\frac{t}{e^t-1}e^{xt} = \frac{-t}{e^{-t}-1}e^{-(1-x)t}, \quad (\text{see } [\mathbf{8}]). \tag{9}$$

Thus, by (1) and (9), we get

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}(-1)^n B_n(1-x)\frac{t^n}{n!}, \quad (\text{see } [\mathbf{8}]). \tag{10}$$

By comparing the coefficients on both sides of (10), we obtain the following reflection symmetric relation for Bernoulli polynomials:

$$B_n(x) = (-1)^n B_n(1-x), \quad (n \in \mathbf{Z}_+ = \mathbf{N} \cup \{0\}). \tag{11}$$

The purpose of this paper is to give some properties of several Bernoulli polynomials to express the integral of those polynomials from 0 to 1 in terms of beta and gamma functions. From those properties, we derive some relations on the integral of the product of Bernoulli polynomials and new identities on the Bernoulli numbers.

One may compare our results with those of Carlitz in [**5**].

2. On the integral of the product of Bernoulli polynomials.

Let us take the integral for the product of Bernoulli polynomials and x^n as follows: for $n \in \mathbf{Z}_+$,

$$\int_0^1 x^n B_n(x)\,dx = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \int_0^1 x^{n+l} dx = \sum_{l=0}^{n} \binom{n}{l} \frac{B_{n-l}}{n+l+1}. \tag{12}$$

On the other hand, by (11), we get

$$\begin{aligned} \int_0^1 x^n B_n(x) dx &= (-1)^n \int_0^1 x^n B_n(1-x)\,dx \\ &= (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \int_0^1 x^n (1-x)^l dx \\ &= (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \frac{\Gamma(n+1)\Gamma(l+1)}{\Gamma(n+l+2)} \\ &= (-1)^n \sum_{l=0}^{n} \frac{B_{n-l}}{n+l+1} \frac{\binom{n}{l}}{\binom{n+l}{l}}, \end{aligned} \tag{13}$$

where $\Gamma(x)$ is the gamma function with $\Gamma(n+1) = n!(n \in \mathbf{N})$. By (12) and (13), we obtain

Proposition 2.1. *For $n \in \mathbf{Z}_+$, we have*

$$\sum_{l=0}^{n} \binom{n}{l} \frac{B_{n-l}}{n+l+1} = (-1)^n \sum_{l=0}^{n} \frac{B_{n-l}}{n+l+1} \frac{\binom{n}{l}}{\binom{n+l}{l}}.$$

Let $n \in \mathbf{N}$ with $n \geq 3$. Then we see that

$$\begin{aligned}(14)\quad \int_0^1 x^n B_n(x)\,dx &= \frac{B_n}{n+1} - \frac{n}{n+1}\int_0^1 x^{n+1}B_{n-1}(x)\,dx\\ &= \frac{B_n}{n+1} - \frac{B_{n-1}}{n+1}\frac{n}{n+2} + (-1)^2\frac{n(n-1)}{(n+1)(n+2)}\\ &\quad\times \int_0^1 x^{n+2}B_{n-2}(x)\,dx.\end{aligned}$$

Continuing this process, we obtain the following equation:

$$\begin{aligned}(15)\quad &\int_0^1 x^n B_n(x)\,dx\\ &= \frac{B_n}{n+1} + \sum_{l=2}^{n-1}\frac{n(n-1)\cdots(n-l+2)}{(n+1)(n+2)\cdots(n+l)}B_{n-l+1}(-1)^{l-1}\\ &\quad + (-1)^{n-1}\frac{n(n-1)\cdots 2}{(n+1)(n+2)\cdots(2n-1)}\int_0^1 x^{2n-1}B_1(x)\,dx\\ &= \frac{B_n}{n+1} + \sum_{l=2}^{n-1}\frac{n(n-1)\cdots(n-l+2)}{(n+1)(n+2)\cdots(n+l)}B_{n-l+1}(-1)^{l-1}\\ &\quad + (-1)^{n-1}\frac{n!}{(n+1)(n+2)\cdots(2n-1)}\left(\frac{1}{2n}B_1 + \frac{1}{2n+1}\right).\end{aligned}$$

Therefore, by (13) and (15), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbf{N}$, with $n \geq 3$, we have*

$$\begin{aligned}&\frac{B_n}{n+1} + \sum_{l=2}^{n-1}\frac{n(n-1)\cdots(n-l+2)}{(n+1)(n+2)\cdots(n+l)}(-1)^{n-l-1}B_{n-l+1}\\ &\qquad + \frac{1}{\binom{2n}{n}}\left(\frac{1}{2} - \frac{2n}{2n+1}\right) = \sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{n+l}{l}}\frac{B_{n-l}}{n+l+1}.\end{aligned}$$

Our result in (13) is related to Euler's beta function, which is defined by

$$(16)\qquad B(n,m) = \int_0^1 x^{n-1}(1-x)^{m-1}dx.$$

Thus, by (16), we know the following relation between beta and gamma function:

$$B(n,m)=\frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}. \tag{17}$$

For $n,k\in\mathbf{Z}_+$, let us consider the following integral for the product of Bernoulli polynomials and x^k:

$$\begin{aligned}\int_0^1 x^k B_n(x)\,dx &= (-1)^n\int_0^1 x^k B_n(1-x)\,dx\\ &= (-1)^n\sum_{l=0}^{n}\binom{n}{l}B_{n-l}\int_0^1 x^k(1-x)^l dx\\ &= (-1)^n\sum_{l=0}^{n}\binom{n}{l}B_{n-l}B(k+1,l+1)\\ &= (-1)^n\sum_{l=0}^{n}\binom{n}{l}B_{n-l}\frac{\Gamma(k+1)\Gamma(l+1)}{\Gamma(l+k+2)}.\end{aligned} \tag{18}$$

Therefore, by (18), we obtain the following proposition.

Proposition 2.3. *For $n,k\in\mathbf{Z}_+$, we have*

$$\begin{aligned}\int_0^1 x^k B_n(x)\,dx &= (-1)^n\sum_{l=0}^{n}\binom{n}{l}B_{n-l}\frac{\Gamma(k+1)\Gamma(l+1)}{\Gamma(l+k+2)}\\ &= (-1)^n\sum_{l=0}^{n}\frac{B_{n-l}}{l+k+1}\frac{\binom{n}{l}}{\binom{k+l}{l}}.\end{aligned}$$

Let us assume that $n,k\in\mathbf{N}$. Then

$$\int_0^1 x^k B_n(x)\,dx=\frac{B_{n+1}}{n+1}-\frac{k}{n+1}\int_0^1 x^{k-1}B_{n+1}(x)\,dx. \tag{19}$$

From (18) and (19), we can derive the following equation:

$$\begin{aligned}\int_0^1 x^k B_n(x)\,dx &= \frac{B_{n+1}}{n+1}-\frac{k}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}\\ &\quad\times B_{n+1-l}(-1)^{n+1}B(k,l+1)\end{aligned} \tag{20}$$

$$= \frac{B_{n+1}}{n+1} + \frac{k(-1)^n}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} \times B_{n+1-l} B(k, l+1).$$

By Proposition 2.3 and equation (20), we obtain the following theorem.

Theorem 2.4. *For $n, k \in \mathbf{N}$, we have*

$$\frac{B_{n+1}}{n+1} = \frac{k}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} B_{n+1-l} \frac{\Gamma(k)\Gamma(l+1)}{\Gamma(k+l+1)} - \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \frac{\Gamma(k+1)\Gamma(l+1)}{\Gamma(l+k+2)}.$$

That is,

$$\frac{B_{n+1}}{n+1} = \frac{k}{n+1} \sum_{l=0}^{n+1} \frac{\binom{n+1}{l}}{\binom{k+l-1}{l}} \frac{B_{n+1-l}}{k+l} - \sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{k+l}{l}} \frac{B_{n-l}}{l+k+1}.$$

Definite integrals for the product of two Bernoulli polynomials can be given by the following relation:

$$\begin{aligned} \int_0^1 B_n(x) B_m(x)\, dx &= \sum_{l=0}^{n} \binom{n}{l} B_l (-1)^m \times \sum_{k=0}^{m} \binom{m}{k} B_k \int_0^1 x^{n-l} (1-x)^{m-k} dx \\ &= \sum_{l=0}^{n} \binom{n}{l} B_l (-1)^m \times \sum_{k=0}^{m} \binom{m}{k} B_k B(n-l+1, m-k+1) \\ &= \sum_{l=0}^{n} \sum_{k=0}^{m} \binom{n}{l} \binom{m}{k} (-1)^m \times B_l B_k B(n-l+1, m-k+1). \end{aligned} \tag{21}$$

Let $m, n \in \mathbf{N}$ with $m \geq 2$. Then we get

$$\int_0^1 B_n(x)B_m(x)\,dx = -\frac{m}{n+1}\int_0^1 B_{n+1}(x)B_{m-1}(x)\,dx.$$

Continuing this process, we get

$$\begin{aligned}(22)\quad &\int_0^1 B_n(x)B_m(x)\,dx \\ &= (-1)^{m-1}\frac{m(m-1)\cdots 1}{(n+1)(n+2)\cdots(n+m-1)}\int_0^1 B_{n+m-1}(x)B_1(x)\,dx.\end{aligned}$$

It is easy to show that

$$\begin{aligned}(23)\qquad \int_0^1 B_{n+m-1}(x)B_1(x)\,dx &= \left[\frac{B_{n+m}(x)}{m+n}B_1(x)\right]_0^1 \\ &\quad - \frac{1}{m+n}\int_0^1 B_{m+n}(x)\,dx \\ &= \frac{B_{m+n}}{m+n}B_1(1) - \frac{B_{n+m}}{m+n}B_1 \\ &= \frac{B_{m+n}}{m+n} + \frac{B_{m+n}B_1}{m+n} - \frac{B_{m+n}B_1}{m+n}.\end{aligned}$$

From (22) and (23), we note that

$$\begin{aligned}(24)\quad \int_0^1 B_n(x)B_m(x)\,dx &= (-1)^{m-1}\frac{m!B_{n+m}}{(n+1)(n+2)\cdots(n+m)} \\ &= (-1)^{m-1}\frac{m!n!B_{n+m}}{(n+m)!} = (-1)^{m-1}\frac{B_{m+n}}{\binom{m+n}{n}}.\end{aligned}$$

Therefore, by (21) and (23), we obtain the following theorem.

Theorem 2.5. *For $m, n \in \mathbf{N}$ with $m \geq 2$, we have*

$$-\frac{B_{n+m}}{\binom{m+n}{n}} = \sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}B_l B_k \frac{\Gamma(n-l+1)\Gamma(m-k+1)}{\Gamma(n+m-l-k+2)}.$$

That is,

$$-\frac{B_{n+m}}{\binom{m+n}{n}} = \sum_{l=0}^{n}\sum_{k=0}^{m}\frac{\binom{n}{l}\binom{m}{k}}{\binom{n+m-l-k}{n-l}}\frac{B_l B_k}{n+m-l-k+1}.$$

From (21), we note that

$$\begin{aligned}(25)\quad &\int_0^1 B_n(x)B_m(x)\,dx\\ &=\sum_{l=0}^{n}\sum_{k=0}^{m}\sum_{j=0}^{m-k}\binom{n}{l}\binom{m}{k}\binom{m-k}{j}(-1)^{k-j}\frac{B_lB_k}{n+m-l-k-j+1}.\end{aligned}$$

Therefore, by (25), we obtain the following corollary.

Corollary 2.6. *For $m,n\in\mathbf{N}$ with $m\geq 2$, we have*

$$\begin{aligned}&(-1)^{m-1}\frac{B_{n+m}}{\binom{m+n}{n}}\\ &=\sum_{l=0}^{n}\sum_{k=0}^{m}\sum_{j=0}^{m-k}\binom{n}{l}\binom{m}{k}\binom{m-k}{j}(-1)^{k-j}\frac{B_lB_k}{n+m-l-k-j+1}.\end{aligned}$$

From (1), we can derive the following equation:

$$\begin{aligned}(26)\quad &\sum_{m,n=0}^{\infty}\left(mB_{m-1}(x)B_n(x)+nB_{n-1}(x)B_m(x)\right)\frac{t^m}{m!}\frac{s^n}{n!}\\ &=\frac{d}{dx}\left(\frac{t}{e^t-1}e^{tx}\right)\left(\frac{s}{e^s-1}e^{sx}\right)=\frac{d}{dx}\left(\frac{tse^{(s+t)x}}{(e^t-1)(e^s-1)}\right)\\ &=(t+s)\frac{ts}{(e^t-1)(e^s-1)}e^{(s+t)x}.\end{aligned}$$

It is easy to show that

$$\begin{aligned}(27)\quad &\\ &(t+s)\frac{ts}{(e^t-1)(e^s-1)}e^{(s+t)x}\\ &=\frac{(s+t)e^{x(s+t)}}{e^{s+t}-1}\left(st+\frac{st}{e^t-1}+\frac{st}{e^s-1}\right)\\ &=\left(\sum_{l=0}^{\infty}B_l(x)\frac{(t+s)^l}{l!}\right)\left(st+s\sum_{r=0}^{\infty}B_r\frac{t^r}{r!}+t\sum_{r=0}^{\infty}B_r\frac{s^r}{r!}\right)\end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{m,n=0}^{\infty} \frac{B_{m+n}(x) t^m s^n}{m!n!} \right) \left(\sum_{r=0}^{\infty} \frac{B_{2r}}{(2r)!} (t^{2r} s + t s^{2r}) \right) \\
&= \sum_{m,n=0}^{\infty} \left\{ \sum_{r=0}^{\infty} \frac{B_{2r}}{(2r)!} \frac{B_{m+n}(x)}{m!n!} \left(t^{2r+m} s^{n+1} + t^{m+1} s^{2r+n} \right) \right\} \\
&= \sum_{m,n=0}^{\infty} \sum_{r=0}^{\infty} \frac{B_{2r}}{(2r)!} \left(\frac{B_{m-2r+n-1}(x)}{(m-2r)!(n-1)!} \frac{t^m s^n m!}{m!} \right. \\
&\quad \left. + \frac{t^m s^n n! B_{m-2r+n-1}(x)}{(m-1)!(n-2r)!} \frac{1}{n!} \right) \\
&= \sum_{m,n=0}^{\infty} \left\{ \sum_{r=0}^{\infty} B_{2r} \left(\binom{m}{2r} n B_{m-2r+n-1}(x) \right. \right. \\
&\quad \left. \left. + \binom{n}{2r} m B_{m-2r+n-1}(x) \right) \right\} \frac{t^m}{m!} \frac{s^n}{n!}.
\end{aligned}
$$

By (26) and (27), we get

$$
\begin{aligned}
(28) \quad & m B_{m-1}(x) B_n(x) + n B_m(x) B_{n-1}(x) \\
&= \sum_{r=0}^{\infty} B_{2r} B_{m+n-2r-1}(x) \left(\binom{m}{2r} n + \binom{n}{2r} m \right).
\end{aligned}
$$

Thus, we have

(29)

$$
\begin{aligned}
\frac{d}{dx} (B_m(x) B_n(x)) &= m B_{m-1}(x) B_n(x) + n B_m(x) B_{n-1}(x) \\
&= \sum_{r=0}^{\infty} B_{2r} B_{m+n-2r-1}(x) \left\{ \binom{m}{2r} n + \binom{n}{2r} m \right\}.
\end{aligned}
$$

Thus, by (29), we get

$$
\begin{aligned}
(30) \quad B_m(x) B_n(x) &= \int \frac{d}{dx} (B_m(x) B_n(x)) \, dx \\
&= \sum_{r=0}^{\infty} \frac{B_{2r} B_{m+n-2r}(x)}{m+n-2r} \left(\binom{m}{2r} n + \binom{n}{2r} m \right) + C,
\end{aligned}
$$

where C is some constant. Let $m, n \in \mathbf{N}$. Then we see that

$$\int_0^1 B_m(x)B_n(x)\,dx = C\int_0^1 dx = C. \tag{31}$$

Hence,

$$C = \int_0^1 B_m(x)B_n(x)\,dx = (-1)^{m-1}\frac{n!m!}{(n+m)!}B_{n+m}, \quad \text{(cf. (24))}.$$

From (30), and (31), we have

$$\begin{aligned} &B_m(x)B_n(x) \\ &= \sum_{r=0}^{\infty}\left(\binom{m}{2r}n+\binom{n}{2r}m\right)B_{2r}\frac{B_{m+n-2r}(x)}{m+n-2r}+(-1)^{m+1}\frac{m!n!}{(m+n)!}B_{m+n}, \end{aligned} \tag{32}$$

where $m, n \in \mathbf{N}$. For $m, n, p \in \mathbf{N}$, by (24) and (32), we get

$$\begin{aligned} &\int_0^1 B_m(x)B_n(x)B_p(x)\,dx \\ &\quad = \int_0^1 B_p(x)(B_m(x)B_n(x))\,dx \\ &\quad = \sum_{r=0}^{\infty}\left(\binom{m}{2r}n+\binom{n}{2r}m\right)\frac{B_{2r}}{m+n-2r} \\ &\qquad \times \int_0^1 B_{m+n-2r}(x)B_p(x)\,dx \\ &\qquad + \frac{(-1)^{m+1}B_{m+n}}{\binom{m+n}{n}}\int_0^1 B_p(x)\,dx \\ &\quad = \sum_{r=0}^{\infty}\left(\binom{m}{2r}n+\binom{n}{2r}m\right)B_{2r}\frac{p!(m+n-2r)!(-1)^{p+1}B_{m+n+p-2r}}{(m+n-2r)(m+n+p-2r)!} \\ &\quad = (-1)^{p+1}\sum_{r=0}^{\infty}\left(\binom{m}{2r}n+\binom{n}{2r}m\right)\frac{B_{2r}B_{m+n+p-2r}}{\binom{m+n+p-2r}{p}(m+n-2r)}. \end{aligned} \tag{33}$$

On the other hand,

(34)

$$\int_0^1 B_m(x)B_n(x)B_p(x)\,dx = (-1)^p \sum_{l=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p}\binom{m}{l}\binom{n}{j}\binom{p}{k}$$
$$\times B_{m-l}B_{n-j}B_k \int_0^1 x^{l+j}(1-x)^{p-k}dx$$
$$= (-1)^p \sum_{l=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p}\binom{m}{l}\binom{n}{j}\binom{p}{k}$$
$$\times B_{m-l}B_{n-j}B_k B(l+j+1, p-k+1)$$
$$= (-1)^p \sum_{l=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p}\binom{m}{l}\binom{n}{j}\binom{p}{k}$$
$$\times B_{m-l}B_{n-j}B_k \frac{\Gamma(l+j+1)\Gamma(p-k+1)}{\Gamma(l+j+p-k+2)}$$

Therefore, by (33) and (34), we obtain the following theorem.

Theorem 2.7. *For $m, n, p \in \mathbf{N}$, we have*

$$\sum_{r=0}^{\infty}\left(\binom{m}{2r}n + \binom{n}{2r}m\right)\frac{B_{2r}B_{m+n+p-2r}}{\binom{m+n+p-2r}{p}(m+n-2r)}$$
$$= -\sum_{l=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p}\binom{m}{l}\binom{n}{j}\binom{p}{k}\frac{B_{m-l}B_{n-j}B_k}{(l+j+p-k+1)\binom{l+j+p-k}{l+j}}.$$

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