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A NEW CLASS OF INEQUALITIES FOR POLYNOMIALS

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ABSTRACT. We extend a recent inequality due to Fournier, Letac and Ruscheweyh to a class of inequalities involving a bound-preserving operator as a parameter.

1. Introduction. Let $\mathbf{D}$ be the unit disc in the complex plane $\mathbf{C}$. $\mathcal{P}_n$ denotes the set of complex polynomials of degree at most n and $|p|_{\mathbf{D}}$ stands for the uniform norm of $p \in \mathcal{P}_n$. The following result has been obtained recently [**4**]:

Theorem A. *For $p \in \mathcal{P}_n$ and $n \geq 2$,*

$$|p - p(0)|_{\mathbf{D}} \leq n(|p|_{\mathbf{D}} - |p(0)|). \tag{1}$$

The constant n is the best possible and equality holds only for constant polynomials $p \equiv p(0)$.

Ruscheweyh and Woloszkiewicz [**8**] have extended (1) by determining the "best" function M_n such that

$$\frac{1}{n} \leq M_n\left(\frac{|p(0)|}{|p - p(0)|_{\mathbf{D}}}\right) \leq \frac{|p|_{\mathbf{D}} - |p(0)|}{|p - p(0)|_{\mathbf{D}}}, \quad p \in \mathcal{P}_n. \tag{2}$$

They also studied some cases of equality for (2).

Of course, one may think of (1) and (2) as generalizations of the classical triangle inequality to a special finite-dimensional vector space. In the present note, we shall further extend (1) from the point of view of bound-preserving operators over $\mathcal{P}_n$. A polynomial $P \in \mathcal{P}_n$ is called a bound-preserving operator over $\mathcal{P}_n$ if

$$|P \star p|_{\mathbf{D}} \leq |p|_{\mathbf{D}}, \quad \text{for all } p \in \mathcal{P}_n.$$

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Here $\star$ denotes the convolution (sometimes called Hadamard product) of two functions in $\mathcal{H}(\mathbf{D})$, the class of functions analytic in $\mathbf{D}$. We refer the reader to [**7,** Chapter 4] and [**9,** Chapter 4] concerning the class of bound-preserving operators; we shall be interested here in the subclass $\mathcal{B}_n$ of those operators Q such that $Q(0) = 1$.

It is well known that

$$Q \in \mathcal{B}_n \iff Q(z) + o(z^n) \in \mathfrak{P}_{1/2}$$

where $\mathfrak{P}_{1/2} = \{f \in \mathcal{H}(\mathbf{D}) \mid f(0) = 1 \text{ and } \operatorname{Re} f(z) > 1/2, z \in \mathbf{D}\}$. We associate to each $Q(z) := 1 + \sum_{k=1}^{n} A_k z^k \in \mathcal{B}_n$ a sequence of Toeplitz matrices T_k, $1 \le k \le n$, whose first row is $(1, A_1, A_2, \dots, A_k)$. Crucial classical information due to Carathéodory, Fejér and Toeplitz is available in the following:

Lemma 1.1. *If $Q \in \mathcal{P}_n$ and $\det T_k(Q) > 0$ for all $1 \le k \le n$, then $Q \in \mathcal{B}_n$. Conversely, for each $Q \in \mathcal{B}_n$, we have $\det T_k(Q) > 0$ for all $1 \le k \le n$ or else there exists a smallest positive integer K, $1 \le K \le n$, such that $\det T_k = 0$ if $K \le k \le n$. In that case,*

$$Q(z) = \sum_{j=1}^{K} \frac{\ell_j}{1 - \zeta_j z} + o(z^n),$$

where $0 < \ell_j$ and $\{\zeta_j\}_{j=1}^{K}$ is a set of distinct nodes in $\partial\mathbf{D}$.

A good reference concerning Lemma 1.1 is the book of Tsuji [**10,** pages 153–159].

Let $\mathcal{B}_n^0 = \{Q \in \mathcal{B}_n \mid \det T_n > 0\}$. Our main result is:

Theorem 1.2. *For any $Q \in \mathcal{B}_n^0$, $n \ge 2$, there exists an optimal constant $0 < d_n = d(Q, n) < 1$ such that*

$$|Q \star p|_{\mathbf{D}} + d_n |p - Q \star p|_{\mathbf{D}} \le |p|_{\mathbf{D}}, \quad p \in \mathcal{P}_n. \tag{3}$$

Clearly, this is an extension of Theorem 1 which is the case $Q \equiv 1$ with $d_n = d(1, n) = 1/n$. In the next section we shall prove Theorem 1.2 and establish cases of equality in (3). We shall also discuss Theorem 1.2, assuming that $Q \in \mathcal{B}_n \backslash \mathcal{B}_n^0$. Finally, inspired by an inequality of

Ruscheweyh ([**5**], [**7**]) we shall introduce alternate versions of our theorem.

2. Proof of Theorem 1.2. Let $I(z) = \sum_{j=0}^{n} z^j$. The inequality (3) is clearly equivalent with

$$\widetilde{Q}(z) := Q(z) + \delta u\big(I(vz) - Q(vz)\big) \in \mathcal{B}_n$$

for any $0 \le \delta \le d_n$ and $u, v \in \partial\mathbf{D}$. If $Q(z) := 1 + \sum_{k=1}^{n} A_k z^k$, the first row of the Toeplitz matrix $T_k(\widetilde{Q})$ is

$$(1, A_1 + \delta u(1 - A)v, \dots, A_k + \delta u(1 - A_k)v^k).$$

We define, for $1 \le k \le n$,

$$d_k = \sup_{\delta \ge 0}\{\delta \mid \det T_j(\widetilde{Q}) > 0, j = 1, 2, \dots, k, u, v \in \partial\mathbf{D}\}.$$

The Taylor coefficients of Q are bounded and $\det T_k(Q) > 0$ by hypothesis. This is sufficient to conclude that

$$0 < d_n \le d_{n-1} \le d_{n-2} \cdots \le d_1.$$

By Lemma 1.1, we obtain that $\widetilde{Q} \in \mathcal{B}_n$ when $u, v \in \partial\mathbf{D}$ and $\delta < d_n$. By continuity, this must also hold for $\delta \le d_n$ and, by definition, there must exist, given $\delta > d_n$, numbers $u, v \in \partial\mathbf{D}$ such that $\det T_n(\widetilde{Q}) < 0$ for the corresponding $\widetilde{Q}$. It follows that $d_n = d(Q, n) > 0$.

When $n = 1$, it is rather trivial that $d_n = (1 - |A_n|)/|1 - A_n|$ and, surely, $0 < d_n \le 1$, where equality is possible if A_n is positive. It should be noted that $|A_k| < 1$ for $1 \le k \le n$ when $\det T_n(Q) > 0$. We shall now prove that $d_n < 1$ when $n \ge 2$; assume for now that $d_n = 1$, and let $u \in \partial\mathbf{D}$, $1 \le k \le n/2$ and $v = \overline{u}^{1/k}$. Then, if

$$\begin{aligned}\widetilde{Q}(z) &:= Q(z) + u d_n\big(I(vz) - Q(vz)\big)\\ &= 1 + \sum_{j=1}^{n}(A_j + d_n u(1 - A_j)v^j)z^j,\end{aligned}$$

where $A_k + d_n u(1 - A_k)v^k = A_k + (1 - A_k) = 1$, it follows that $\widetilde{Q}(z) + o(z^n)$ is a support point of $\mathfrak{P}_{1/2}$ (see [**6**] for details) since it maximizes $\operatorname{Re} f^{(k)}(0)$ within this class, and therefore

$$\widetilde{Q}(z) = \sum_{j=1}^{k} \frac{\ell_j(u)}{1 - w_j z} + o(z^n),$$

where $\ell_j(u) \geq 0$ and $\{w_j\}_{j=1}^{k}$ is the set of distinct k-roots of unity. We have, in particular,

$$\begin{aligned} 1 &= \sum_{j=1}^{k} \ell_j(u) w_j^{2k} = A_{2k} + d_n u(1 - A_{2k}) v^{2k} \\ &= A_{2k} + \overline{u}(1 - A_{2k}), \end{aligned}$$

which is impossible because u is arbitrary in $\partial\mathbf{D}$ and $|A_{2k}| < 1$.

Concerning the cases of equality in (3), we shall rely on two more hypotheses:

$$d_n < d_{n-1} \leq d_{n-2} \cdots \leq d_1 \tag{4}$$

and

$$d_n < \frac{1 - |A_n|}{|1 - A_n|}. \tag{5}$$

These hypotheses may look artificial, but we remark that they were verified in the case of Theorem 1. We shall also need the following easy consequence of Theorem 1.2:

Corollary 2.1. *Let $Q \in \mathcal{B}_n^0$ with $n \geq 2$. Then the constant polynomials are the only polynomials $p \in \mathcal{P}_n$ such that $|Q \star p|_{\mathbf{D}} = |p|_{\mathbf{D}}$.*

Let us now assume that $n \geq 2$ and that equality holds for some polynomial $p \in \mathcal{P}_n$ in (3). There must exist $Z, u, v \in \partial\mathbf{D}$ such that

$$\begin{aligned} &\Big| \big(Q(z) + d_n u\big(I(vz) - Q(vz)\big)\big) \star p(z) \Big|_{z=Z} \\ &\quad = |Q \star p(Z)| + d_n |(I - Q) \star p(vZ)| \\ &\quad = |Q \star p|_{\mathbf{D}} + d_n |p - Q \star p|_{\mathbf{D}} \\ &\quad = |p|_{\mathbf{D}}. \end{aligned} \tag{6}$$

Then, either $\det T_n(Q + d_n u(I(v\cdot) - Q(v\cdot))) > 0$ or else the same determinant vanishes. It follows, in the first case and by Corollary 2.1, that the polynomial p is constant. In the second case, it shall follow from hypothesis (4) and Lemma 1.1 that

$$Q(z) + d_n u\big(I(vz) - Q(vz)\big) = \sum_{j=1}^{n} \frac{\ell_j}{1-\zeta_j z} + o(z^n), \tag{7}$$

where $\ell_j > 0$ and the set of distinct nodes $\{\zeta_j\}_{j=1}^n$ lies in $\partial\mathbf{D}$. We obtain, in particular, from (6) and (7) that

$$\left|\sum_{j=1}^{n} \ell_j p(\zeta_j Z)\right| = |p|_{\mathbf{D}},$$

and there must exist some real number ρ such that

$$p(\zeta_j Z) = |p|_{\mathbf{D}} e^{i\rho}, \quad j = 1, 2, \dots, n.$$

It is known **[3]** that such polynomials must be of the type $p(z) = \beta + \alpha z^n$ for some $\alpha, \beta \in \mathbf{C}$. We now have from (6) that

$$|\beta + \alpha A_n z^n|_{\mathbf{D}} + d_n|\alpha|\,|1 - A_n| = |\beta| + |\alpha|,$$

and, with $\alpha \neq 0$, this amounts to

$$d_n = \frac{1 - |A_n|}{|1 - A_n|},$$

which is ruled out by hypothesis (5) We conclude that, for $n \geq 2$ and under (4) and (5), equality holds in Theorem 1.2 if and only if the polynomial p is constant.

It seems at first sight difficult to exhibit functions $Q \in \mathcal{B}_n$ with given A_n and d_n satisfying (5). We remark, however, that any one of the two statements

$$0 \leq A_n < 1$$

or

$$\min_{1 \leq j \leq n} \frac{1 - |A_j|}{|1 - A_j|} < \frac{1 - |A_n|}{|1 - A_n|}$$

admits (5) as a consequence.

3. What about $Q \in \mathcal{B}_n \backslash \mathcal{B}_n^0$? It is a natural question to ask if Theorem 1.2 remains valid for some polynomials $Q \in \mathcal{B}_n$ with

$\det T_n(Q) = 0$ since our proof depends heavily on the fact that $Q \in \mathcal{B}_n^0$. We only have partial answers concerning this question.

Let $F(z) = \sum_{j=1}^{k} \ell_j/(1 - w_j z) \in \mathfrak{P}_{1/2}$, $\ell_j > 0$, and $\{w_j\}_{j=1}^{k}$ is the set of distinct k roots of unity with $2 \le k$. There exists [**2,** Lemma 2.2] a non-constant polynomial $P \in \mathcal{P}_{\lceil k/2 \rceil}$ and $0 < a < 1$ such that, if

$$p(z) = 1 - a(1 - z^k)P(z) \in \mathcal{P}_{k+\lceil k/2 \rceil}, \tag{8}$$

then $|p|_{\mathbf{D}} = 1$. We set

$$Q(z) := 1 + \sum_{t=1}^{k+\lceil k/2 \rceil} \left(\sum_{j=1}^{k} \ell_j w_j^t \right) z^t = F(z) + o(z^{k+\lceil k/2 \rceil}).$$

Clearly, $Q \in \mathcal{B}_{k+\lceil k/2 \rceil}$ and, by Lemma 1.1,

$$\det T_j(Q) > 0 \quad \text{if } 1 \le j < k$$

and

$$\det T_j(Q) = 0 \quad \text{if } k \le j \le k + \left\lceil \frac{k}{2} \right\rceil.$$

Let us define $n = k + \lceil k/2 \rceil$; clearly, $Q \in \mathcal{B}_n \backslash \mathcal{B}_n^o$, and we claim that Theorem 1.2 is not valid for Q and that choice of n; otherwise, there would exist a constant $d > 0$ such that

$$|Q \star p|_{\mathbf{D}} + d|p - Q \star p|_{\mathbf{D}} \le |p|_{\mathbf{D}}, \quad p \in \mathcal{P}_n, \tag{9}$$

and, for p defined by (8), we obtain

$$\begin{aligned} 1 = |p|_{\mathbf{D}} &= |Q \star p(1)| \\ &\le |Q \star p|_{\mathbf{D}} + d|p - Q \star p|_{\mathbf{D}} \le |p|_{\mathbf{D}} = 1, \end{aligned}$$

i.e., $p(z) \equiv Q \star p(z)$ and $p(z) \equiv \sum_{j=1}^{k} \ell_j p(w_j z)$. Since p is non-constant, one of its Taylor coefficients (say $a_t(p)$ with $0 < t \le \lceil k/2 \rceil$) does not vanish with

$$a_t(p) = \left(\sum_{j=1}^{k} \ell_j w_j^t \right) a_t(p),$$

i.e., $1 = \sum_{j=1}^{k} \ell_j w_j^t = w_\ell^t$ for $1 \le \ell \le k$, and there would exist k distinct t-roots of unity with $0 < t \le \lceil k/2 \rceil < k$. We are, however, unable to decide if we can choose $k \le n < k + \lceil k/2 \rceil$.

We may also consider $F(z) = \sum_{j=1}^{k} \ell_j/(1 - e^{i\theta_j} z)$ where $\ell_j > 0$ and the $k+1$ nodes $\{e^{i\theta_j}\}_{j=1}^{k+1}$ satisfy

$$0 \le \theta_1 < \theta_2 \cdots < \theta_k < \theta_{k+1} < 2\pi,$$

but are otherwise arbitrary. Also let $0 < \psi < \varphi < 2\pi$; according to a result of Clunie, Hallenbeck and MacGregor **[1]** there exists, for each n, a polynomial p_n univalent in $\mathbf{D}$ such that

$$p_n(e^{i\theta_j}) = e^{i(\psi - (1/jn))}, \quad 1 \le j \le k, \text{ and } p_n(e^{i\theta_{k+1}}) = e^{i\varphi},$$

and $|p_n|_{\mathbf{D}} = 1$ where, for each n, $p_n \in \mathcal{P}_M$ where M depends only on k, $\{\theta_j\}_{j=1}^{k+1}$, ψ and φ but does not depend on n.

Due to the finiteness of M, the family $\{p_n\}$ has a subsequence $\{p_{n_j}\}$ converging uniformly over $\overline{\mathbf{D}}$ to a polynomial p which is univalent since $p(e^{i\theta_1}) = e^{i\psi} \ne e^{i\varphi} = p(e^{i\theta_{k+1}})$. We now let

$$Q(z) = 1 + \sum_{t=1}^{M} \left(\sum_{j=1}^{k} \ell_j e^{it\theta_j} \right) z^t = F(z) + o(z^M). \tag{10}$$

We may assume $k < M$ and $Q \in \mathcal{B}_M \backslash \mathcal{B}_M^0$. If there exists a constant $d > 0$ such that (9) holds with $n = M$, then for p as above,

$$1 = |Q \star p(1)| \le |Q \star p|_{\mathbf{D}} + d|p - Q \star p|_{\mathbf{D}} \le 1$$

and again $p \equiv Q \star p$. Since p is univalent, we have $p'(0) \ne 0$ and, by (10),

$$\sum_{j=1}^{k} \ell_j e^{i\theta_j} = 1,$$

which is impossible for $k > 1$ since, for any j, $\ell_j > 0$ and $\{e^{i\theta_j}\}_{j=1}^{k}$ contains k different nodes!

4. Another extension of (1)**.** Given $Q \in \mathcal{B}_n^0$, we may look for a slightly different extension of (1), namely, statements of the type

$$|Q \star p(z)| + c|p(z) - Q \star p(z)| \le |p|_{\mathbf{D}}, \quad p \in \mathcal{P}_n, \ |z| \le 1,$$

where $c > 0$. As in the proof of Theorem 1.2, we define, for $\delta \ge 0$, $|u| = 1$,

$$\widetilde{Q}(z) := Q(z) + \delta u\big(I(z) - Q(z)\big)$$

and $T_k(\widetilde{Q})$ the Toeplitz matrix whose first row equals $1 \le k \le n$,

$$\big(1, A_1 + \delta u(1 - A_1), A_2 + \delta u(1 - A_2), \ldots, A_k + \delta u(1 - A_k)\big).$$

Also, as in Theorem 1.2, we set

$$c_k = \sup_{\delta \ge 0} \left\{ \delta \mid \det T_j(\widetilde{Q}) > 0,\ j = 1, 2, \ldots, k, u \in \partial \mathbf{D} \right\}.$$

We obtain the following result (the omitted proof runs as the proof of Theorem 1.2):

Theorem 4.1. *For each $Q \in \mathcal{B}_n^0$, we have*

$$0 < c_n \le c_{n-1} \le \cdots \le c_1$$

and

$$|Q \star p(z)| + c_n |p(z) - Q \star p(z)| \le |p|_{\mathbf{D}}, \quad p \in \mathcal{P}_n,\ z \in \overline{\mathbf{D}}. \tag{11}$$

The constant c_n is the best possible. Further, if $n \ge 2$ and $c_n < \min(c_{n-1}, (1 - |A_n|)/|1 - A_n|)$, equality holds in (11) *only for constant polynomials.*

There are, however, striking differences between the inequalities (3) and (11). It is clear that $d_n \le c_n \le 1$. The polynomial $Q_n(z) := \sum_{k=0}^n (1 - k/n) z^k$ belongs to $\mathcal{P}_{n-1} \cap \mathfrak{P}_{1/2} \subset \mathcal{B}_n^0$ and, in that context, the inequality (11) is nothing but the classical

$$\left| p(z) - \frac{zp'(z)}{n} \right| + \left| \frac{zp'(z)}{n} \right| \le |p|_{\mathbf{D}}, \quad z \in \mathbf{D},\ p \in \mathcal{P}_n, \tag{12}$$

for which we have $c_j = c_n = (1 - |A_n|)/|1 - A_n| = 1$, for any $1 \le j \le n$. Indeed, the cases of inequality in (12) are numerous, and they were studied in [**3**]; we also can prove (unpublished) that in (11) we may have $c_n = 1$ if and only if $Q(z) = \sum_{k=0}^n (1 - tk/n) z^k$ for some t in $[0, 1]$.

The inequality (11) is reminiscent of a result of Ruscheweyh (see [**5**] or [**7,** Chapter 4]), claiming that

$$|q \star p(z)| + |q^s \star p(z)| \le |p|_{\mathbf{D}}, \quad p \in \mathcal{P}_n,\ z \in \overline{\mathbf{D}}, \tag{13}$$

for any $q \in \mathcal{P}_{n-1} \cap \mathfrak{P}_{1/2}$ and $q^s(z) := z^n \overline{q}(1/\overline{z})$. It is easily seen that both of (11) and (13) reduce to (12) when both q and Q equal Q_n.

We remark, however, that this is their only point of "intersection" by showing that, for $q \in \mathcal{P}_{n-1} \cap \mathfrak{P}_{1/2}$,

$$I - q \equiv q^s \iff q \equiv Q_n.$$

As a matter of fact, it was shown in **[3]** that, for $q \in \mathcal{P}_{n-1} \cap \mathfrak{P}_{1/2}$ and θ real,

$$q(z) + e^{i\theta} q^s(z) \equiv \sum_{j=0}^{n-1} \frac{\ell_j}{1 - w_j e^{i\theta/n} z} + o(z^n), \tag{14}$$

where $\{w_j\}_{j=0}^{n-1}$ is the set of nth roots of unity and $\ell_j = (2/n)(\operatorname{Re} q(\overline{w}_j e^{-i\theta/n}) - 1/2)$. In particular, if $q(z) = 1 + \sum_{k=1}^{n-1} a_k z^k$, we obtain

$$|a_k + e^{i\theta} a_{n-k}| \le 1, \quad 1 \le k \le n-1,$$

and, because θ is arbitrary, it follows that

$$|a_k| + |a_{n-k}| \le 1, \quad 1 \le k \le n-1.$$

Assume now that $q + q^s \equiv I$. Then, for $1 \le k \le n-1$,

$$1 = |a_k + \overline{a}_{n-k}| \le |a_k| + |a_{n-k}| = 1,$$

i.e. $\overline{a}_{n-k} = t_k a_k$ with $t_k \ge 0$ and

$$a_k = \frac{1}{1 + t_k}, \quad 1 \le k \le n-1. \tag{15}$$

The condition $q + q^s \equiv I$ is equivalent with $q(z) + q^s(z) = 1/(1-z) + o(z^n)$, and a comparison with (14) yields

$$\operatorname{Re} q(e^{-2ij\pi/n}) = \begin{cases} \dfrac{1}{2} & \text{if } 1 \le j \le n-1 \\[2ex] \dfrac{n+1}{2} & \text{if } j = 0. \end{cases} \tag{16}$$

By (15), the Taylor coefficients of q are real, and therefore

$$\operatorname{Re} q(e^{i\theta}) = 1 + \sum_{k=1}^{n-1} a_k \cos(k\theta) = 1 + \sum_{k=1}^{n-1} a_k T_k(\cos\theta),$$

where T_k is the kth Chebyshev polynomial. There exists at most one polynomial of this form satisfying the interpolation conditions (16), and it is now a routine calculation to show that $q \equiv Q_n$.

We finally obtain an analogue of Theorem 1.2 for Ruscheweyh's inequality (13):

Theorem 4.2. *Let $q \in \mathcal{P}_{n-1} \cap \mathfrak{P}_{1/2}$ be non-constant. There shall exist an optimal constant $b_n \in (0,1)$ such that*

$$|q \star p|_{\mathbf{D}} + b_n |q^s \star p|_{\mathbf{D}} \leq |p|_{\mathbf{D}}, \quad p \in \mathcal{P}_n.$$

Cases of equality could be discussed as above. We omit the details.

5. Conclusion.

We shall end this paper with the following problem.

Problem 5.1. Let $1 \leq k \leq n$ and $\{z_j\}_{j=1}^k \subset \partial\mathbf{D}$ be a set of distinct nodes. What are the polynomials $p \in \mathcal{P}_n$ such that

$$p(z_j) = |p|_{\mathbf{D}}, \quad j = 1, 2, \ldots, k?$$

Problem 5.1 is trivial when $k = 1$ and relatively easy when $k = n$, (see **[3]**). Not much seems to be known about the existence of such polynomials when $1 < k < n$; as an example, we remark **[3]** that, for $|b| \leq 1$, $b \neq -1$ and $0 < a$, the polynomial $p(z) := 1 - a[(1-z^n)/(1-z)](1+bz)$ always satisfies $|p|_{\mathbf{D}} > 1 = p(e^{2ij\pi/n})$, $j = 1, 2, \ldots, n-1$.

Such polynomials are related to problems considered in the present paper; if

$$F(z) = \sum_{j=1}^{k} \frac{\ell_j}{1 - z_j z}, \quad \ell_j > 0,$$

the solutions to the extremal problem

$$|F \star p|_{\mathbf{D}} = |p|_{\mathbf{D}}, \quad p \in \mathcal{P}_n$$

are precisely, up to a multiplicative constant of modulus 1, the polynomials $p \in \mathcal{P}_n$ such that $p(z_j Z) = |p|_{\mathbf{D}}$ for some $Z \in \partial\mathbf{D}$. Further, in the case where such non-constant polynomials do exist, there cannot be $d > 0$ such that

$$|F \star q|_{\mathbf{D}} + d|q - F \star q|_{\mathbf{D}} \leq |q|_{\mathbf{D}}$$

for all $q \in \mathcal{P}_n$.

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