ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 44, Number 4, 2014

CARLITZ INVERSIONS AND IDENTITIES OF THE ROGERS-RAMANUJAN TYPE

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ABSTRACT. By means of the inverse series relations due to Carlitz **[11]**, we establish several transformation formulae for nonterminating q-series, which will systematically be employed to review identities of the Rogers-Ramanujan type moduli 5, 7, 8, 10, 14 and 27.

1. Introduction and notation. For two indeterminate x and q, the shifted factorial of x with base q is defined by

$$(x;q)_0 = 1$$

and

$$(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}) \text{ for } n \in \mathbf{N}.$$

When $|q|<1$, we have two well-defined infinite products

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^k x) \quad \text{and} \quad (x;q)_n = \frac{(x;q)_\infty}{(q^n x;q)_\infty}.$$

The product and fraction of shifted factorials are abbreviated, respectively, as

$$\begin{aligned}[\,\alpha,\ \beta,\ \cdots,\ \gamma;q\,]_n &= (\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n,\\ \left[\begin{matrix}\alpha,\ \beta,\ \cdots,\ \gamma\\ A,\ B,\ \cdots,\ C\end{matrix}\,\middle|\,q\right]_n &= \frac{(\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n}{(A;q)_n(B;q)_n\cdots(C;q)_n}.\end{aligned}$$

2010 AMS *Mathematics subject classification.* Primary 33D15, Secondary 05A30.

Keywords and phrases. Basic hypergeometric series, Rogers-Ramanujan identities.

Supported by National Science Foundation of China (Youth grant No. 11201484).

Received by the editors on June 16, 2008, and in revised form on April 10, 2012.

DOI:10.1216/RMJ-2014-44-4-1125

Following Gasper and Rahman [**18,** page 4], the basic hypergeometric series is defined by

$$
{}_{1+r}\phi_s\left[\begin{matrix} a_0, a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \,\Big|\, q; z\right] = \sum_{n=0}^{\infty} \left\{(-1)^n q^{\binom{n}{2}}\right\}^{s-r} \left[\begin{matrix} a_0, a_1, \ldots, a_r \\ q, b_1, \ldots, b_s \end{matrix} \,\Big|\, q\right]_n z^n
$$

where the base q will be restricted to $|q| < 1$ for nonterminating q-series.

In 1973, Gould and Hsu [**20**] discovered a very general pair of inverse series relations. Its q-analogue was established by Carlitz [**11**] in the same year. Subsequently, Chu [**14, 15, 16**] found its important applications to the evaluation of terminating q-series. Specializing Carlitz's inversions, Chu [**12**] in 1990 derived the following transformation formula.

Theorem 1.1. *Let $f(n)$ and $g(n)$ be two sequences tied by one of the equations*

$$
f(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (q^k a; q)_n \, g(k), \tag{1a}
$$

$$
g(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1 - q^{2k} a}{(q^n a; q)_{k+1}} f(k). \tag{1b}
$$

Then the transformation formula holds:

$$
\sum_{n=0}^{\infty} \frac{q^{n^2} a^n}{(q;q)_n (a;q)_n} g(n) = \sum_{k=0}^{\infty} \frac{1 - q^{2k} a}{(a;q)_\infty} \frac{q^{k^2} (-a)^k}{(q;q)_k} f(k). \tag{2}
$$

This theorem has been utilized in the same paper to review the celebrated Rogers-Ramanujan identities (cf., Bailey [**7,** subsection 8.6], Slater [**27,** subsection 3.5] and Watson [**30**])

$$
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{\left[q^5, q^2, q^3; q^5\right]_\infty}{(q;q)_\infty} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{\left[q^5, q, q^4; q^5\right]_\infty}{(q;q)_\infty}.
$$

There exist numerous identities of this type expressing infinite sums in terms of infinite products. Their proofs require, in general, deep understanding of q-series theory. Among different approaches, the

Bailey lemma has been shown powerful to deal with identities of the Rogers-Ramanujan type (RR-identities) in [**8, 9, 28, 29, 6**]. The purpose of this paper is to explore further applications of Theorem 1.1 to RR-identities. Several transformation formulae will be established. As consequences, numerous identities of the Rogers-Ramanujan type moduli 5, 7, 8, 10, 14 and 27 will systematically be reviewed.

2. Three identities modulo 5. By combining Theorem 1.1 with the following q-analog of Bailey's ${}_2F_1(1/2)$-sum due to Andrews [**2,** equation (1.9)] (cf., Gasper-Rahman [**18,** II-10])

$$(3)\qquad {}_2\phi_2\left[\begin{matrix} e, & q/e \\ -q, & c\end{matrix}\,\Big|\, q; -c\right] = \left[\begin{matrix} ce, \ qc/e \\ c, \ qc\end{matrix}\,\Big|\, q^2\right]_\infty,$$

we first prove the following transformation formula.

Proposition 2.1.

$$\sum_{n=0}^{\infty}\frac{q^{3n^2-n}c^n}{(q^4;q^4)_n(c;q^2)_n} = \sum_{k=0}^{\infty}(-1)^k\frac{1-q^{4k+2}}{(q^2;q^2)_\infty}\frac{(q^{-2k}c;q^4)_k}{(c;q^2)_k}q^{3k^2+k}.$$

Proof. Define the sequence $g(k)$ by

$$g(k) = \frac{(q;q)_k(c/q)^k}{(-q;q)_k(c;q)_k}q^{\binom{k}{2}}.$$

Then, for $a = q$, we can determine, by means of (1a) and (3), the dual sequence $f(n)$ as follows:

$$\begin{aligned} f(n) &= q^{\binom{n}{2}}(q;q)_n\sum_{k=0}^{n}c^k\frac{(q^{-n};q)_k(q^{n+1};q)_k}{(q;q)_k(-q;q)_k(c;q)_k}q^{\binom{k}{2}} \\ &= (q;q)_n\frac{(q^{-n}c;q^2)_n}{(c;q)_n}q^{\binom{n}{2}}. \end{aligned}$$

Substituting them into (2) and replacing q by q^2, we get the transformation displayed in Proposition 2.1. $\square$

We are going to show three RR-identities modulo 5 by means of Proposition 2.1.

Corollary 2.2. ([**23**], [**29,** equation 19]).

$$\sum_{n=0}^{\infty}(-1)^n\frac{(q;q^2)_n}{(q^2;q^2)_{2n}}q^{3n^2}=\frac{\left[q^5,q^2,q^3;q^5\right]_\infty}{(q^2;q^2)_\infty}.$$

Proof. Letting $c=-q$ in Proposition 2.1 and then observing that

$$(-q^{1-2k};q^4)_k=q^{-\binom{k+1}{2}}(-q;q^2)_k, \tag{4}$$

we may reformulate the sum on the right hand side as follows:

$$\sum_{k=0}^{\infty}(-1)^k q^{5\binom{k}{2}+3k}(1-q^{4k+2})=\sum_{k=-\infty}^{\infty}(-1)^k q^{5\binom{k}{2}+3k}.$$

Recalling Jacobi's triple product identity [**22**] (see [**13**] and [**18,** subsection 1.6] also)

$$\sum_{n=-\infty}^{+\infty}(-1)^n\,q^{\binom{n}{2}}\,x^n=[q,x,q/x;q]_\infty\,, \tag{5}$$

we find that the last bilateral sum with respect to k factorizes into the infinite product $\left[q^5,q^2,q^3;q^5\right]_\infty$. This proves the identity stated in the corollary. □

Instead, taking $c=-q^3$ in Proposition 2.1 and then observing that

$$(-q^{3-2k};q^4)_k=q^{-\binom{k}{2}}(-q;q^2)_k, \tag{6}$$

we may recover another identity of the Rogers-Ramanujan type.

Corollary 2.3. ([**21, 24**]).

$$\sum_{n=0}^{\infty}(-1)^n\frac{(q;q^2)_{n+1}}{(q^2;q^2)_{2n+1}}q^{3n^2+2n}=\frac{\left[q^5,q,q^4;q^5\right]_\infty}{(q^2;q^2)_\infty}.$$

In addition, for the U_n-sequence defined by

$$U_n=\frac{(-1)^n q^{3n^2-2n}}{(-q;q^2)_n(q^4;q^4)_{n-1}},$$

it is trivial to check the difference

$$U_n - U_{n+1} = (-1)^n \frac{(q;q^2)_n}{(q^2;q^2)_{2n}} q^{3n^2-2n} - (-1)^n \frac{(q;q^2)_{n+1}}{(q^2;q^2)_{2n+1}} q^{3n^2+2n}.$$

According to the telescoping method, Corollary 2.3 implies the following identity.

Corollary 2.4. ([**8**], [**19,** equation (7.11)]).

$$\sum_{n=0}^{\infty} (-1)^n \frac{(q;q^2)_n}{(q^2;q^2)_{2n}} q^{3n^2-2n} = \frac{\left[q^5, q, q^4; q^5\right]_\infty}{(q^2;q^2)_\infty}.$$

3. Three identities modulo 7. Recall the terminating q-analogue of Whipple's theorem on ${}_3F_2$-series due to Andrews [**3,** Theorem 2] (see also [**18,** II-19])

$$(7) \quad {}_4\phi_3 \left[\begin{matrix} q^{-n}, \, q^{1+n}, \, \sqrt{c}, \, -\sqrt{c} \\ -q, \, e, \, qc/e \end{matrix} \,\middle|\, q; q \right] = q^{\binom{n+1}{2}} \frac{[q^{-n}e, q^{1-n}c/e; q^2]_n}{[e, qc/e; q]_n}.$$

According to Theorem 1.1, we are going to utilize this formula to derive two general transformations and review the Rogers-Selberg identities modulo 7.

Consider the case $a = q$ of Theorem 1.1. For the sequence $g(k)$ defined by

$$g(k) = \frac{(q;q)_k (c;q^2)_k}{(-q;q)_k (e;q)_k (qc/e;q)_k},$$

we can compute, according to (1a) and (7), the dual sequence $f(n)$ as follows:

$$\begin{aligned} f(n) &= q^{\binom{n}{2}} (q;q)_n \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{1+n};q)_k (c;q^2)_k}{(q^2;q^2)_k (e;q)_k (qc/e;q)_k} q^k \\ &= q^{n^2} \frac{[q^{-n}e, q^{1-n}c/e; q^2]_n}{[e, qc/e; q]_n} (q;q)_n. \end{aligned}$$

Substituting them into (2) and then replacing q by q^2, we derive the following transformation formula.

Proposition 3.1.

$$\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}(c;q^4)_n}{(q^4;q^4)_n[e,q^2c/e;q^2]_n}$$
$$=\sum_{k=0}^{\infty}(-1)^k\frac{1-q^{4k+2}}{(q^2;q^2)_\infty}\frac{[q^{-2k}e,q^{2-2k}c/e;q^4]_k}{[e,q^2c/e;q^2]_k}q^{4k^2+2k}.$$

Now we examine the limiting case $c \to 0$ of this proposition. For $e = -q$ and $e = -q^3$, taking into account (4) and (6) and then factorizing the corresponding right members through (5), we recover the following two Rogers-Selberg identities, respectively.

Corollary 3.2 ([**23, 25**]).

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n}{(q^2;q^2)_{2n}}q^{2n^2+2n}=\frac{[q^7,q^2,q^5;q^7]_\infty}{(q^2;q^2)_\infty}.$$

Corollary 3.3 ([**24, 25**]).

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_{n+1}}{(q^2;q^2)_{2n+1}}q^{2n^2+2n}=\frac{[q^7,q,q^6;q^7]_\infty}{(q^2;q^2)_\infty}.$$

There is a third Rogers-Selberg identity which reads as follows.

Corollary 3.4 ([**23, 25**]).

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n}{(q^2;q^2)_{2n}}q^{2n^2}=\frac{[q^7,q^3,q^4;q^7]_\infty}{(q^2;q^2)_\infty}.$$

It follows by specifying $a \to 1$ and $c \to \infty$ in the next transformation formula.

Proposition 3.5.

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}a^n(-\sqrt{a}/c;q)_{2n}}{[q^2,a/c^2;q^2]_n(-q\sqrt{a};q)_{2n}}$$
$$=\sum_{k=0}^{\infty}\frac{1-q^{2k}\sqrt{a}}{1-q^k\sqrt{a}}\begin{bmatrix}q\sqrt{a},qc\\q,\sqrt{a}/c\end{bmatrix}q\Big]_k\frac{(a^{3/2}/c)^k}{(q^2a;q^2)_\infty}q^{3k^2-k}.$$

Proof. Define the sequence $f(k)$ by

$$f(k)=q^{\binom{k}{2}}\frac{[-q^{1/2},\sqrt{a},q^{1/2}c;q^{1/2}]_k}{(1+q^k\sqrt{a})\,(\sqrt{a}/c;q^{1/2})_k}\Big(-\frac{\sqrt{a}}{c}\Big)^k.$$

Then the dual sequence $g(n)$ corresponding to (1b) reads as follows

$$g(n)=\frac{1-\sqrt{a}}{1-q^na}\,{}_6\phi_5\left[\begin{matrix}\sqrt{a},\ \pm q^{\frac{1}{2}}\sqrt[4]{a},\ \pm q^{-\frac{n}{2}},\ q^{\frac{1}{2}}c\\ \pm\sqrt[4]{a},\ \pm q^{\frac{1+n}{2}}\sqrt{a},\sqrt{a}/c\end{matrix}\ \Big|\ q^{\frac{1}{2}};-q^n\frac{\sqrt{a}}{c}\right].$$

Evaluating the last ${}_6\phi_5$-series by the q-Dougall sum (11a)–(11b) (see page 1138), we find that $g(n)$ admits the closed expression below

$$g(n)=\frac{(a;q)_n}{(a/c^2;q)_n}\frac{(-\sqrt{a}/c;q^{1/2})_{2n}}{(-\sqrt{a};q^{1/2})_{2n+1}}.$$

Substituting $f(k)$ and $g(n)$ into (2) and then replacing q by q^2, we get the transformation stated in Proposition 3.5. □

In addition, we point out that when $c\to\infty$, Proposition 3.1 leads to alternative proofs of Corollaries 2.2 and 2.3, respectively, under the specifications $e=-q$ and $e=-q^3$. Instead, we can derive two RR-identities modulo 6 from Proposition 3.5. The first one follows from the case $a=1$ and $c=-q^{-1}$.

Corollary 3.6. ([**4,** equation (3.2)]).

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n\,q^{2n^2}}{(-q;q^2)_n(q^4;q^4)_n}=\frac{[q^6,q^3,q^3;q^6]_\infty}{(q^2;q^2)_\infty}.$$

The second one is done by letting $a=q^2$ and $c=-1$ in Proposition 3.5.

Corollary 3.7. ([**29,** equation (27)]).

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n\,q^{2n^2+2n}}{(-q;q^2)_{n+1}(q^4;q^4)_n}=\frac{\left[q^6,q,\,q^5;q^6\right]_\infty}{(q^2;q^2)_\infty}.$$

This can also be proved by putting $c=q^2$ and $e=-q$ in Proposition 3.1.

4. Two identities modulo 8. Recall the q-Chu-Vandermonde-Gauss summation formula (cf., [**7,** Section 8] and [**27,** subsection 3.3]):

$$ {}_2\phi_1\left[\begin{matrix}q^{-n},\,a\\ c\end{matrix}\,\middle|\,q;\,q\right]=\frac{(c/a;q)_n}{(c;q)_n}a^n. \tag{8}$$

For the sequence $g(k)$ defined by

$$g(k)=\frac{(a;q)_k}{(q^{1/2}a;q)_k},$$

we can determine, according to (1a) and (8), the dual sequence $f(n)$

$$\begin{aligned}f(n)&=q^{\binom{n}{2}}(a;q)_n\sum_{k=0}^{n}\frac{(q^{-n};q)_k(q^na;q)_k}{(q;q)_k(q^{1/2}a;q)_k}q^k\\&=(-a)^n\frac{(a;q)_n(q^{1/2};q)_n}{(q^{1/2}a;q)_n}q^{n^2-(n/2)}.\end{aligned}$$

Then the transformation corresponding to (2) reads as follows.

Proposition 4.1.

$$\sum_{n=0}^{\infty}\frac{q^{n^2}a^n}{(q;q)_n(q^{1/2}a;q)_n}=\sum_{k=0}^{\infty}\{1-q^{2k}a\}\frac{a^{2k}}{(a;q)_\infty}\frac{(a;q)_k(q^{1/2};q)_k}{(q;q)_k(q^{12}a;q)_k}q^{2k^2-(k/2)}.$$

Letting $a\to q$ in Proposition 4.1 and then evaluating the right member by means of Jacobi's triple product identity (5), we obtain, under the base change $q\to q^2$, the following RR-identity.

Corollary 4.2. ([**29,** equation (38)], [**17,** equation (1.6)]).

$$\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(q;q)_{2n+1}}=\frac{\left[q^8,-q,-q^7;q^8\right]_\infty}{(q^2;q^2)_\infty}.$$

We also can derive from Proposition 4.1 another identity given below.

Corollary 4.3. ([**10,** equation (3.2)], [**29,** equation (39)]).

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q)_{2n}}=\frac{\left[q^8,-q^3,-q^5;q^8\right]_\infty}{(q^2;q^2)_\infty}.$$

Proof. Letting $a\to 1$ in Proposition 4.1 and then separating the initial term from the others, we can reformulate the corresponding right member as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n(q^{1/2};q)_n}&=\frac{1}{(q;q)_\infty}\left\{1+\sum_{k=1}^{\infty}q^{2k^2-(k/2)}(1+q^k)\right\}\\&=\frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{\infty}q^{2k^2-(k/2)}\end{aligned}$$

where the replacement $k\to -k$ has been made for the sum corresponding to q^k in the factor $1+q^k$. Applying again (5) and replacing q by q^2 in the resulting equation, we get the identity stated in the corollary. □

5. Three identities modulo 10. This section will review three RR-identities. Recall the q-analogue of Gauss's ${}_2F_1(1/2)$-sum due to Andrews [**2,** equation (1.8)] (cf., [**18,** II-11]):

$$(9)\qquad {}_2\phi_2\left[\begin{matrix}a,\ b\\ \sqrt{qab},\ -\sqrt{qab}\end{matrix}\ \middle|\ q;-q\right]=\left[\begin{matrix}qa,\ qb\\ q,\ qab\end{matrix}\ \middle|\ q^2\right]_\infty.$$

We can establish the infinite series transformation formula.

Proposition 5.1.

$$\sum_{n=0}^{\infty}\frac{q^{n^2+\binom{n}{2}}a^n}{(q;q)_n(qa;q^2)_n}=\sum_{k=0}^{\infty}\{1-q^{4k}a\}\frac{(-a^2)^k}{(a;q)_\infty}\frac{(a;q^2)_k}{(q^2;q^2)_k}q^{5k^2-k}.$$

Proof. For the sequence $g(k)$ defined by

$$g(k) = \frac{(a;q)_k}{(qa;q^2)_k} q^{\binom{k}{2}},$$

the dual sequence $f(n)$ in (1a) can be determined, by means of (9), as follows:

$$\begin{aligned}
f(n) &= \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (q^k a;q)_n \, g(k) \\
&= q^{\binom{n}{2}} (a;q)_n \sum_{k=0}^{n} \left[\begin{matrix} q^{-n}, \, q^n a \\ q, \, \pm\sqrt{qa} \end{matrix} \,\middle|\, q \right]_k q^{\binom{k+1}{2}} \\
&= \begin{cases} 0, & n\text{-odd}; \\ (-1)^\ell \left[q, a; q^2\right]_\ell q^{\ell^2-\ell}, & n = 2\ell. \end{cases}
\end{aligned}$$

Then Proposition 5.1 follows immediately from (2). □

For $a \to 1$ and $a = q^2$, the transformation displayed in Proposition 5.1 leads, respectively, to the following two RR-identities modulo 10.

Corollary 5.2. ([**8,** equation (10.4)], [**29,** equation (46)]).

$$\sum_{n=0}^{\infty} \frac{q^{n^2+\binom{n}{2}}}{(q;\,q)_n (q;q^2)_n} = \frac{\left[q^{10}, q^4, q^6; q^{10}\right]_\infty}{(q;\,q)_\infty}.$$

Corollary 5.3. ([**24**], [**8,** equation (10.5)]).

$$\sum_{n=0}^{\infty} \frac{q^{3\binom{n+1}{2}}}{(q;\,q)_n (q;q^2)_{n+1}} = \frac{\left[q^{10}, q^2, q^8; q^{10}\right]_\infty}{(q;\,q)_\infty}.$$

In the same manner as the derivation from Corollary 2.3 to Corollary 2.4, we can deduce another identity from Corollary 5.2. In fact, define the V_n-sequence by

$$V_n = \frac{q^{n^2+\binom{n}{2}}}{(q;q)_{n-1}(q;q^2)_n}.$$

It is not hard to verify the difference equation

$$V_n - V_{n+1} = \frac{q^{n^2+\binom{n}{2}}}{(q;q)_n(q;q^2)_n} - \frac{q^{n^2+\binom{n+1}{2}}}{(q;q)_n(q;q^2)_{n+1}}.$$

This yields the following identity of Rogers-Ramanujan type.

Corollary 5.4. ([**5**], [**1,** equation (2.3)]).

$$\sum_{n=0}^{\infty} \frac{q^{n^2+\binom{n+1}{2}}}{(q;\,q)_n(q;\,q^2)_{n+1}} = \frac{\left[q^{10}, q^4, q^6; q^{10}\right]_\infty}{(q;\,q)_\infty}.$$

6. Three identities modulo 14. Recall the terminating q-analogue of Watson's theorem on ${}_3F_2$-series due to Andrews [**3,** Theorem 1] (see also [**18,** II-17])

(10)

$${}_4\phi_3\left[\begin{matrix} q^{-n},\ q^n a,\ \sqrt{c},\ -\sqrt{c} \\ c,\ \sqrt{qa},\ -\sqrt{qa}\end{matrix}\,\Big|\, q; q\right] = \begin{cases} c^{n/2}\left[\begin{matrix} q, qa/c \\ qa, qc\end{matrix}\,\Big|\, q^2\right]_{n/2} & n\text{-even;} \\ 0 & n\text{-odd.}\end{cases}$$

For the sequence $g(k)$ defined by

$$g(k) = \frac{(a;q)_k(c;q^2)_k}{(c;q)_k(qa;q^2)_k}$$

we can compute, according to (1a) and (10), the dual sequence $f(n)$ as follows:

$$\begin{aligned} f(n) &= q^{\binom{n}{2}}(a;q)_n \sum_{k=0}^{n} \frac{(q^{-n};q)_k(q^n a;q)_k(c;q^2)_k}{(q;q)_k(c;q)_k(qa;q^2)_k} q^k \\ &= \begin{cases} q^{2\ell^2-\ell}\left[\begin{matrix} q, a, qa/c \\ qc\end{matrix}\,\Big|\, q^2\right]_\ell c^\ell, & n = 2\ell; \\ 0, & n\text{-odd.}\end{cases}\end{aligned}$$

Substituting them into (2), we derive the following transformation.

Proposition 6.1.

$$\sum_{n=0}^{\infty}\frac{q^{n^2}a^n(c;q^2)_n}{(q;q)_n(c;q)_n(qa;q^2)_n}=\sum_{k=0}^{\infty}\{1-q^{4k}a\}\frac{(a^2c)^k}{(a;q)_\infty}\left[\begin{matrix}a,qa/c\\q^2,qc\end{matrix}\,\middle|\,q^2\right]_k q^{6k^2-k}.$$

Letting $a=q^2$ and $c\to 0$ in the last equation and then factorizing the right member through the Jacobi triple product identity (5), we get the following RR-identity.

Corollary 6.2. ([**24**], [**8,** equation (10.2)]).

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q;q)_n(q;q^2)_{n+1}}=\frac{\left[q^{14},q^2,q^{12};q^{14}\right]_\infty}{(q;q)_\infty}.$$

Similarly letting $a\to 1$ and $c\to 0$ in Proposition 6.1 leads us to another identity.

Corollary 6.3. ([**23**], [**8,** equation (10.3)]).

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n(q;q^2)_n}=\frac{\left[q^{14},q^6,q^8;q^{14}\right]_\infty}{(q;q)_\infty}.$$

Define the sequence $g(k)$ by

$$g(k)=\frac{(a;q)_k(c;q^2)_k}{(c;q)_k(a;q^2)_{k+1}}.$$

The dual sequence $f(n)$ corresponding to (1a) reads as

$$f(n)=\frac{(a;q)_n}{1-a}q^{\binom{n}{2}}\sum_{k=0}^{n}\frac{(q^{-n};q)_k(q^na;q)_k(c;q^2)_k}{(q;q)_k(c;q)_k(q^2a;q^2)_k}q^k.$$

By inserting the factor

$$1=\frac{1-q^{k+n}a}{1-q^{2n}a}-\frac{1-q^{k-n}}{1-q^{2n}a}q^{2n}a$$

in the last sum, we can evaluate it through (10) as follows:

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^n a;q)_k (c;q^2)_k}{(q;q)_k (c;q)_k (q^2 a;q^2)_k} q^k \\
&= \frac{1-q^n a}{1-q^{2n}a} {}_4\phi_3 \left[\begin{matrix} q^{-n},\ q^{n+1}a,\ \sqrt{c},\ -\sqrt{c} \\ c, q\sqrt{a}, -q\sqrt{a} \end{matrix} \,\Big|\, q;q \right] \\
&\quad + \frac{q^n a - q^{2n} a}{1-q^{2n}a} {}_4\phi_3 \left[\begin{matrix} q^{1-n},\ q^n a,\ \sqrt{c},\ -\sqrt{c} \\ c, q\sqrt{a}, -q\sqrt{a} \end{matrix} \,\Big|\, q;q \right] \\
&= \begin{cases} \dfrac{1-a}{1-q^{4\ell}a} \left[\begin{matrix} q, q^2 a/c \\ a, qc \end{matrix} \,\Big|\, q^2 \right]_\ell c^\ell, & n = 2\ell; \\ \dfrac{q^{2\ell+1} a(1-q)}{1-q^{4\ell+2}a} \left[\begin{matrix} q^3, q^2 a/c \\ q^2 a, qc \end{matrix} \,\Big|\, q^2 \right]_\ell c^\ell, & n = 2\ell+1. \end{cases}
\end{aligned}$$

Therefore, we have the following expression

$$f(n) = \begin{cases} \dfrac{q^{\binom{2\ell}{2}}}{1-q^{4\ell}a} \left[\begin{matrix} q, qa, q^2 a/c \\ qc \end{matrix} \,\Big|\, q^2 \right]_\ell c^\ell, & n = 2\ell; \\ \dfrac{q^{\binom{2\ell+2}{2}} a(1-q)}{1-q^{4\ell+2}a} \left[\begin{matrix} q^3, qa, q^2 a/c \\ qc \end{matrix} \,\Big|\, q^2 \right]_\ell c^\ell, & n = 2\ell+1. \end{cases}$$

Substituting $f(n)$ and $g(k)$ into (2) and then simplifying the result, we derive the following transformation formula.

Proposition 6.4.

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{n^2} a^n (c;q^2)_n}{(q;q)_n (c;q)_n (a;q^2)_{n+1}} \\
&= \sum_{k=0}^{\infty} \{1 - q^{8k+2} a^2\} \frac{(a^2 c)^k}{(a;q)_\infty} \left[\begin{matrix} qa, q^2 a/c \\ q^2, qc \end{matrix} \,\Big|\, q^2 \right]_k q^{6k^2-k}.
\end{aligned}$$

Letting $a = q$ and $c \to 0$ in this equation gives rise to the following RR-identity.

Corollary 6.5. ([**24**], [**29,** equation (60)]).

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n(q;q^2)_{n+1}}=\frac{\left[q^{14},q^4,q^{10};q^{14}\right]_\infty}{(q;q)_\infty}.$$

Furthermore, we can also derive three RR-identities modulo 12.

For the case $c=-q$ of Proposition 6.1, specifying further $a\to 1$ and $a\to q^2$, we recover the following two identities.

Corollary 6.6. ([**26,** equation 5.4]).

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q;q)_{2n}}q^{n^2}=\frac{\left[q^{12},q^6,q^6;q^{12}\right]_\infty}{(q;q)_\infty}.$$

Corollary 6.7. ([**29,** equation (50)]).

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q;q)_{2n+1}}q^{n^2+2n}=\frac{\left[q^{12},q^2,q^{10};q^{12}\right]_\infty}{(q;q)_\infty}.$$

Similarly, letting $a=-c=q$ in Proposition 6.4 leads us to another RR-identity.

Corollary 6.8. ([**29,** equation (51)]).

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q;q)_{2n+1}}q^{n^2+n}=\frac{\left[q^{12},q^4,q^8;q^{12}\right]_\infty}{(q;q)_\infty}.$$

7. Four identities modulo 27. This section will review four RR-identities modulo 27 by combining Theorem 1.1 with the following identity of the q-Dougall sum [**18,** II-20]:

$$\text{(11a)}\qquad {}_6\phi_5\left[\begin{matrix}a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d\\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d\end{matrix}\,\middle|\, q;\frac{qa}{bcd}\right]$$

$$\text{(11b)}\qquad =\left[\begin{matrix}qa,\ qa/bc,\ qa/bd,\ qa/cd\\ qa/b,\ qa/c,\ qa/d,\ qa/bcd\end{matrix}\,\middle|\, q\right]_\infty$$

provided $|qa/bcd|<1$.

For the sequence $f(m)$ defined by

$$f(m) = \begin{cases} q^{\binom{3k}{2}} \frac{(a;q^3)_k}{(q^3;q^3)_k}(q;q)_{3k}a^k, & m = 3k \\ 0, & m = 3k+1; \\ 0, & m = 3k+2, \end{cases}$$

we can evaluate the sequence $g(n)$ corresponding to (1b) through (11a)–(11b) as

$$\begin{aligned} g(n) &= \frac{1-a}{1-q^n a} {}_6\phi_5 \left[\begin{matrix} a,\ \pm q^3\sqrt{a},\ q^{-n},\ q^{1-n},\ q^{2-n} \\ \pm\sqrt{a},\ q^{3+n}a,\ q^{2+n}a,\ q^{1+n}a \end{matrix} \,\middle|\, q^3; q^{3n}a \right] \\ &= \frac{(a;q^3)_n (a;q)_n}{(a;q)_{2n}}. \end{aligned}$$

Substituting them into (2), we find the transformation formula.

Proposition 7.1.

$$\sum_{n=0}^{\infty} \frac{q^{n^2} a^n (a;q^3)_n}{(q;q)_n (a;q)_{2n}} = \sum_{k=0}^{\infty} (-a^4)^k \frac{1-q^{6k}a}{(a;q)_\infty} \frac{(a;q^3)_k}{(q^3;q^3)_k} q^{27\binom{k}{2}+12k}.$$

When $a \to 1$, this transformation results in the following identity.

Corollary 7.2. ([**8,** equation (10.7)], [**29,** equation (93)]).

$$1 + \sum_{n=1}^{\infty} \frac{(q^3;q^3)_{n-1} q^{n^2}}{(q;q)_n (q;q)_{2n-1}} = \frac{\left[q^{27}, q^{12}, q^{15}; q^{27}\right]_\infty}{(q;q)_\infty}.$$

Alternatively, when $a = q^3$, we get another identity from Proposition 7.1.

Corollary 7.3. ([**8,** equation (10.8)], [**29,** equation (90)]).

$$\sum_{n=0}^{\infty} \frac{(q^3;q^3)_n q^{n^2+3n}}{(q;q)_n (q;q)_{2n+2}} = \frac{\left[q^{27}, q^{3}, q^{24}; q^{27}\right]_\infty}{(q;q)_\infty}.$$

Similarly in Theorem 1.1, define the sequence $f(m)$ by

$$f(m)=\begin{cases} q^{\binom{3k}{2}+k}\frac{(qa;q^3)_k}{(q^3;q^3)_k}\frac{(q;q)_{3k}}{1-q^{6k}a}a^k, & m=3k;\\ q^{\binom{3k+2}{2}+k}\frac{(qa;q^3)_k}{(q^3;q^3)_k}\frac{(q;q)_{3k+1}}{1-q^{6k+2}a}a^{k+1}, & m=3k+1;\\ 0, & m=3k+2.\end{cases}$$

According to (1b), we may evaluate

$$\begin{aligned} g(n) &= \frac{1-qa}{(q^na;q)_2}\sum_{k\geq 0}\frac{1-q^{6k+1}a}{1-qa}\frac{(qa;q^3)_k(q^{-n};q)_{3k}}{(q^3;q^3)_k(q^{n+2}a;q)_{3k}}q^{3nk+k}a^k\\ &= \frac{(a;q)_n(qa;q^3)_n}{(a;q)_{2n+1}}.\end{aligned}$$

Substituting them into (2) and simplifying the result, we get the transformation.

Proposition 7.4.

$$\sum_{n=0}^{\infty}\frac{q^{n^2}a^n(qa;q^3)_n}{(q;q)_n(a;q)_{2n+1}}=\sum_{k=0}^{\infty}\left\{1-q^{12k+2}a^2\right\}\frac{(-a^4)^k}{(a;q)_\infty}\frac{(qa;q^3)_k}{(q^3;q^3)_k}q^{27\binom{k}{2}+13k}.$$

For $a=q^2$, this proposition recovers the following identity.

Corollary 7.5. ([**8,** B2], [**29,** equation (91)]).

$$\sum_{n=0}^{\infty}\frac{(q^3;q^3)_nq^{n^2+2n}}{(q;q)_n(q;q)_{2n+2}}=\frac{\left[q^{27},q^6,q^{21};q^{27}\right]_\infty}{(q;q)_\infty}.$$

Finally, in Theorem 1.1, let $f(m)$ be the sequence given by

$$f(m)=\begin{cases} q^{\binom{3k}{2}+2k}\frac{(q^2a;q^3)_k}{(q^3;q^3)_k}\frac{(q;q)_{3k}}{1-q^{6k}a}a^k, & m=3k;\\ 0, & m=3k+1;\\ -q^{\binom{3k+3}{2}}\frac{(q^2a;q^3)_k}{(q^3;q^3)_k}\frac{(q;q)_{3k+2}}{1-q^{6k+4}a}\left(\frac{a}{q}\right)^{k+1}, & m=3k+2.\end{cases}$$

The dual sequence $g(n)$ corresponding to (1b) may be evaluated as

$$g(n) = \frac{1-q^{2n+1}a}{(q^n a;q)_3} \sum_{k\geq 0} \{1-q^{6k+2}a\} \frac{(q^2a;q^3)_k (q^{-n};q)_{3k}}{(q^3;q^3)_k (q^{n+3}a;q)_{3k}} q^{3nk+2k} a^k$$
$$= \frac{(a;q)_n (q^2a;q^3)_n}{(a;q)_{2n+1}}.$$

Substituting them into (2) and then unifying the two sums with respect to k, we derive the transformation formula.

Proposition 7.6.

$$\sum_{n=0}^{\infty} \frac{q^{n^2} a^n (q^2a;q^3)_n}{(q;q)_n (a;q)_{2n+1}} = \sum_{k=0}^{\infty} \{1-q^{18k+6}a^3\} \frac{(-a^4)^k}{(a;q)_\infty} \frac{(q^2a;q^3)_k}{(q^3;q^3)_k} q^{27\binom{k}{2}+14k}.$$

For $a=q$, this proposition yields the following identity.

Corollary 7.7. ([**8,** B3], [**29,** equation (92)]).

$$\sum_{n=0}^{\infty} \frac{(q^3;q^3)_n q^{n^2+n}}{(q;q)_n (q;q)_{2n+1}} = \frac{[q^{27},q^9,q^{18};q^{27}]_\infty}{(q;q)_\infty}.$$

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