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STARLIKENESS OF SECTIONS OF UNIVALENT FUNCTIONS

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ABSTRACT. Let $\mathcal{S}$ be the class of all normalized analytic and univalent functions in the unit disk $\mathbf{D}$. In this paper, we determine condition so that each section $s_n(f,z)$ of $f \in \mathcal{S}$ is starlike in the disk $|z| < r_n$. In particular, $s_n(f,z)$ is starlike in $|z| \le 1/2$ for $n \ge 47$.

1. Introduction. In this article, we consider analytic functions defined on the open unit disk $\mathbf{D} = \{z : |z| < 1\}$ of the complex plane $\mathbf{C}$. Let $\mathcal{LU}$ denote the family of all locally univalent mappings f analytic in $\mathbf{D}$ with the normalization $f(0) = 0 = f'(0) - 1$. The subfamily of univalent mappings is denoted by $\mathcal{S}$. The classes of convex, starlike, starlike of order 1/2, and close-to-convex mappings are some of the important well-known standard subclasses of $\mathcal{S}$, denoted by $\mathcal{C}$, $\mathcal{S}^*$, $\mathcal{S}^*(1/2)$, and $\mathcal{K}$, respectively. These classes are well understood and are studied extensively in the literature. We refer to the books by Duren [**3**] and Goodman [**4**] for analytic characterizations of these geometric subclasses of $\mathcal{S}$. In particular,

$$\mathcal{S}^* = \left\{ f \in \mathcal{S} : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \text{ for } z \in D \right\},$$
$$\mathcal{S}^*(1/2) = \left\{ f \in \mathcal{S} : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \frac{1}{2} \text{ for } z \in D \right\},$$

and the strict inclusions $\mathcal{C} \subsetneq \mathcal{S}^*(1/2) \subsetneq \mathcal{S}^* \subsetneq \mathcal{K} \subsetneq \mathcal{S}$ hold. For a normalized analytic function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in $\mathbf{D}$, and $n \in \mathbf{N}$, we write

$$s_n(f,z) := s_n(z) = z + \sum_{k=2}^{n} a_k z^k$$

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for the nth partial sums or sections of f. The question of finding the largest radius of *univalence* r_n of $s_n(z)$, when f belongs to $\mathcal{S}$ or some of its interesting subclasses, has been a subject of discussion by a number of researchers. For example, we refer to [**3, 4**] and the works of the present authors [**6**], Robertson [**9**], Ruscheweyh [**10**], Sheil-Small [**11**] and Szegö [**12**]. In particular, from the result of Szegö [**12**] (see also [**3**, Theorem 8.5]), we have:

Theorem A. *If $f \in \mathcal{S}$, then for each $n \geq 2$, $s_n(z)$ is univalent in the disk $|z| < 1/4$. The radius $1/4$ is best possible as the Koebe function shows for $n = 2$.*

In [**9**], Robertson proved that the nth partial sum of an arbitrary $f \in \mathcal{S}^*$ is starlike in a disk of radius $1 - 4n^{-1} \log n$ (see also [**5**]). He further showed that sections of the Koebe function $k(z)$ are univalent in the disk $|z| < 1 - 3n^{-1} \log n$ for $n \geq 5$, and that the constant 3 cannot be replaced by a smaller constant. Jenkins [**5**] observed that a modification of the proof of Szegö [**12**] shows

$$r_n \geq 1 - (4 + \varepsilon) n^{-1} \log n$$

for each $\varepsilon > 0$ and for all large n. Further, it is worth recalling that Bshouty and Hengartner [**2**, page 408] proved that the Koebe function is not extremal for the radius of univalency of the partial sums of $f \in \mathcal{S}$. However, a well-known theorem on convolution allows us to conclude immediately that if f belongs to $\mathcal{C}$, $\mathcal{S}^*$, or $\mathcal{K}$, then its nth section is respectively convex, starlike, or close-to-convex in the disk $|z| < 1 - 3n^{-1} \log n$, for $n \geq 5$. As pointed out in [**3**, subsection 8.2, page 246] (see also [**7**, subsection 6.4]), the exact (largest) radius of univalence r_n of $s_n(z)$ ($f \in \mathcal{S}$) remains an open problem. In this paper, we continue the discussion of this problem, and our primary emphasis will be to prove the starlikeness of $s_n(z)$ rather than its univalence in a disk. We now state our main result.

Lemma 1. *Let $f \in \mathcal{S}$ and $s_n(z)$ be its nth partial sum. Then, for each $r \in (0,1)$ and $n \geq 2$, we have*

$$\left|\frac{s_n'(z)}{f'(z)} - 1\right| < |z|^n\left((n+1)^2 + \left(\frac{\sqrt{(1+r)^6-(1-r)^2}}{(1-r)r^n}\right)A_n'\frac{|z|}{r-|z|}\right) \text{ for } |z|<r,$$

where

$$A_n' = \sqrt{\sum_{k=1}^{n} k^4} = \sqrt{\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}}. \tag{1}$$

The proof of Lemma 1 will be given in Section 3. As an application of Lemma 1, we have (compare with Theorem A) the following result.

Theorem 1. *Let $f \in \mathcal{S}$. Then every section $s_n(z)$ of f is starlike in the disk $|z| \le 1/2$ for all $n \ge 47$.*

From the proof of Theorem 1, it can easily be seen that the radius r_n of the disk of starlikeness of $s_n(z)$, $f \in \mathcal{S}$, can be made larger than $1/2$ for all $n \ge N$ for some $N > 47$.

2. Preliminary lemmas. In order to present the proof our main result, we need some lemmas.

Lemma 2. *For $n \ge 1$, we have*

$$S_n = \sum_{k=1}^{n} \frac{(n+1-k)^2}{k} < R_n^2$$

where $R_n = \sqrt{(n+1)(n+1/2)\log(n+1)}$.

Proof. As $(n+1-k)^2 = (n+1)^2 - 2(n+1)k + k^2$, we see that

$$S_n = (n+1)^2\sum_{k=1}^{n}\frac{1}{k} - 2n(n+1) + \frac{n(n+1)}{2}$$

$$= (n+1)^2\sum_{k=1}^{n}\frac{1}{k} - \frac{3}{2}n(n+1).$$

In view of the inequalities

$$\frac{1}{k+1} < \int_k^{k+1} \frac{dx}{x} = \log(k+1) - \log k < \frac{1}{k},$$

it follows that

$$-\frac{n}{n+1} + \sum_{k=1}^{n} \frac{1}{k} < \log(n+1).$$

This inequality together with the fact $\log(n+1) < n$ shows that

$$S_n < (n+1)\left[(n+1)\log(n+1) - \frac{n}{2}\right] < R_n^2,$$

and the proof is completed. □

Lemma 3. *Suppose that $f \in \mathcal{S}$ and $s_n(z)$ is its nth partial sum. Then, for each $n \geq 2$*

$$\left|\frac{s_n(z)}{f(z)} - 1\right| < |z|^n \left(n + 1 + R_n \frac{|z|}{1-|z|}\right), \quad |z| = r < 1,$$

where R_n is defined as in Lemma 2.

Proof. Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ so that $s_n(z) = z + a_2z^2 + a_3z^3 + \cdots + a_nz^n$ is its nth partial sum. Every $f \in \mathcal{S}$ can be written in the form

$$\frac{z}{f(z)} = 1 + b_1z + b_2z^2 + \cdots \tag{2}$$

for some complex coefficients b_n $(n \geq 1)$. In view of this observation, it follows that

$$(1 + a_2z + a_3z^2 + \cdots)(1 + b_1z + b_2z^2 + \cdots) \equiv 1.$$

From the last relation, we see that

$$\sum_{k=1}^{m-1} b_k a_{m-k} + a_m = 0 \quad (m = 2, 3, \ldots;\ a_1 = 1). \tag{3}$$

Using the representation for the partial sum $s_n(z)$ and (2), we obtain that

$$\frac{s_n(z)}{f(z)} = (1 + a_2 z + a_3 z^2 + \cdots + a_n z^{n-1})(1 + b_1 z + b_2 z^2 + \cdots)$$
$$= 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots ,$$

where

$$c_n = b_1 a_n + b_2 a_{n-1} + \cdots + b_n a_1. \tag{4}$$

By (3), we observe that the coefficients of z^k in the above expansion for $k = 2, \ldots , n-1$ vanish. Equation (3) for $m = n+1$ shows that $c_n = -a_{n+1}$. Also

$$c_m = b_{m-n+1} a_n + b_{m-n+2} a_{n-1} + \cdots + b_m a_1 \quad \text{for } m = n+1, n+2, \ldots . \tag{5}$$

By the de Branges theorem [**1**], $|a_n| \leq n$ for all $n \geq 2$, and therefore, for $m \geq n+1$, we have

$$|c_m| \leq n b_{m-n+1} + (n-1) b_{m-n+2} + \cdots + b_m$$
$$= \alpha_1 \beta_1 + \alpha_2 \beta_2 + \cdots + \alpha_n \beta_n \quad \text{(say)},$$

where

$$\alpha_k = \frac{n+1-k}{\sqrt{m-n+k-1}} \quad \text{and} \quad \beta_k = (\sqrt{m-n+k-1})\, b_{m-n+k}.$$

Using the classical Cauchy-Schwarz inequality, it follows that, for $m \geq n+1$,

$$|c_m|^2 \leq \left(\sum_{k=1}^{n} \frac{(n+1-k)^2}{m-(n+1-k)} \right) \left(\sum_{k=1}^{n} (m-n+k-1)|\, |b_{m-n+k}|^2 \right)$$
$$=: AB \quad \text{(say)}.$$

For $f \in \mathcal{S}$ of form (2), the well-known Area theorem [**4**, Vol. 2, page 193, Theorem 11] shows that

$$\sum_{n=2}^{\infty} (n-1)|b_n|^2 \leq 1,$$

and so, we have $B \leq 1$. On the other hand, for the first sum A, we observe that, for $m \geq n+1$,

$$A = \sum_{k=1}^{n} \frac{(n+1-k)^2}{m-(n+1-k)} \leq \sum_{k=1}^{n} \frac{(n+1-k)^2}{k} =: S_n$$

and so, A is less than or equal to S_n. Therefore, by Lemma 2, we deduce that

$$|c_m| < \sqrt{(n+1)(n+1/2)\log(n+1)} = R_n \quad \text{for } m \geq n+1.$$

This inequality, together with the fact that $|c_n| = |a_{n+1}| \leq n+1$, gives that, for $|z| = r < 1$

$$\begin{aligned} \left| \frac{s_n(z)}{f(z)} - 1 \right| &\leq |c_n|\,|z|^n + |c_{n+1}|\,|z|^{n+1} + \cdots \\ &< (n+1)|z|^n + R_n \left(|z|^{n+1} + |z|^{n+2} + \cdots \right) \\ &= |z|^n \left(n+1 + R_n \frac{|z|}{1-|z|} \right) \end{aligned}$$

for $n \geq 2$. This completes the proof of Lemma 3. □

Lemma 4. *Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{LU}$ such that $|a_n| \leq n$ for $n \geq 2$ and $s_n(z)$ is its nth partial sum. Assume that $|1/f'(z)| \leq M$ in $\mathbf{D}$ for some $M > 1$. Then, for each $n \geq 2$,*

$$\left| \frac{s_n'(z)}{f'(z)} - 1 \right| \leq |z|^n \left((n+1)^2 + A_n \frac{|z|}{1-|z|} \right), \quad |z| = r < 1,$$

where $A_n = A_n' \sqrt{M^2 - 1}$ and A_n' is given by (1).

Proof. For $f \in \mathcal{LU}$, we let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ so that

$$s_n(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n.$$

Since $f \in \mathcal{LU}$, $f'(z)$ is non-vanishing in $\mathbf{D}$ and hence, $1/f'(z)$ can be represented in the form

$$\frac{1}{f'(z)} = 1 + d_1 z + d_2 z^2 + \cdots$$

for some complex coefficients d_n, $n \geq 1$. Note that $2a_2 = -d_1$, and we have the identity

$$(1 + 2a_2 z + 3a_3 z^2 + \cdots)(1 + d_1 z + d_2 z^2 + \cdots) \equiv 1.$$

From the last relation, we see that

$$\sum_{k=1}^{m-1} (m-k) a_{m-k} d_k + m a_m = 0 \quad (m = 2, 3, \ldots;\ a_1 = 1). \tag{6}$$

As in Lemma 3, using the representation for the partial sum $s_n(z)$, we have

$$\begin{aligned} \frac{s_n'(z)}{f'(z)} &= (1 + 2a_2 z + 2a_3 z^2 + \cdots + n a_n z^{n-1})(1 + d_1 z + d_2 z^2 + \cdots) \\ &\equiv 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots, \end{aligned}$$

where

$$c_n = n a_n d_1 + (n-1) a_{n-1} d_2 + \cdots + a_1 b_n.$$

Equation (6) for $m = n+1$ shows that $c_n = -(n+1)a_{n+1}$. More generally,

$$c_m = n a_n d_{m-n+1} + (n-1) a_{n-1} d_{m-n+2} + \cdots + a_1 d_m \quad \text{for} \quad m = n+1, n+2, \ldots.$$

By hypothesis, $|a_n| \leq n$ for all $n \geq 2$, and therefore, we have that for $m \geq n+1$

$$|c_m| \leq \sum_{k=1}^{n} (n+1-k)^2 |d_{m-n+k}|. \tag{7}$$

By assumption, $|1/f'(z)| \leq M$ for $z \in \mathbf{D}$. Hence, for $0 < r < 1$, we have that

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1}{f'(re^{i\theta})} \right|^2 d\theta = 1 + \sum_{n=1}^{\infty} |d_n|^2 r^{2n} \leq M^2$$

which, by allowing $r \to 1^-$, shows that

$$\sum_{n=1}^{\infty} |d_n|^2 \leq M^2 - 1. \tag{8}$$

Using (8) and the Cauchy-Schwarz inequality, we see that

$$\begin{aligned}|c_m|^2 &\leq \left(\sum_{k=1}^{n}(n+1-k)^4\right)\left(\sum_{k=1}^{n}|d_{m-n+k}|^2\right)\\ &\leq (M^2-1)\sum_{k=1}^{n}k^4 = (M^2-1)A_n'^2.\end{aligned}$$

Here the precise value of A_n' is easy to compute and is given by (1). Thus, we have

$$|c_m| \leq A_n'\sqrt{M^2-1} = A_n \quad \text{for } m \geq n+1.$$

This inequality, together with the fact that $|c_n| = |(n+1)a_{n+1}| \leq (n+1)^2$, gives that, for $|z| = r < 1$,

$$\begin{aligned}\left|\frac{s_n'(z)}{f'(z)} - 1\right| &\leq |c_n|\,|z|^n + |c_{n+1}|\,|z|^{n+1} + \cdots\\ &\leq (n+1)^2|z|^n + A_n\left(|z|^{n+1} + |z|^{n+2} + \cdots\right)\\ &= |z|^n\left((n+1)^2 + A_n\frac{|z|}{1-|z|}\right)\end{aligned}$$

for $n \geq 2$. The proof is complete. □

3. Proofs of Lemma 1 and Theorem 1.

Proof of Lemma 1. We begin with $f \in \mathcal{S}$ and follow the method of proof of Lemma 4. Then a well-known sharp estimate for the derivative f' gives

$$\left|\frac{1}{f'(z)}\right| \leq \frac{(1+r)^3}{1-r} =: M(r) \quad \text{for } |z| = r < 1. \tag{9}$$

As observed in the proof of Lemma 4, it follows that

$$\sum_{k=1}^{\infty}|d_k|^2 r^{2k} \leq M(r)^2 - 1.$$

Following the notation of Lemma 4, (7) may be rewritten as

$$\begin{aligned}|c_m| &\leq \sum_{k=1}^{n}(n+1-k)^2|d_{m-n+k}| \\ &= \sum_{k=1}^{n}\left((n+1-k)^2\frac{1}{r^{m-n+k}}\right)\left(|d_{m-n+k}|r^{m-n+k}\right)\end{aligned}$$

for any $r \in (0,1)$. Thus, we have

$$\begin{aligned}|c_m|^2 &\leq \left(\sum_{k=1}^{n}(n+1-k)^4\frac{1}{r^{2(m-n+k)}}\right)\left(\sum_{k=1}^{n}|d_{m-n+k}|^2 r^{2(m-n+k)}\right) \\ &\leq \left(\frac{1}{r^{2m}}\sum_{k=1}^{n}k^4\right)\left(M(r)^2-1\right) = \frac{1}{r^{2m}}\left(M(r)^2-1\right)A_n'^2\end{aligned}$$

which is true for each $r \in (0,1)$, and so

$$|c_m| \leq \frac{1}{r^m}\left(\sqrt{M(r)^2-1}\right)A_n' \quad \text{for } m \geq n+1,$$

where A_n' is given by (1). As in the proof of Lemma 4, using the above estimate, we easily have

$$\left|\frac{s_n'(z)}{f'(z)}-1\right| < |z|^n\left((n+1)^2+\frac{1}{r^n}\left(\sqrt{M(r)^2-1}\right)A_n'\frac{|z|/r}{1-(|z|/r)}\right),$$

for $|z|<r$ and the proof of the lemma follows if we use the expression for $M(r)$ given by (9). □

Let us now demonstrate the use of Lemma 1 by fixing some values for r. For example, if we put $r=2/3$, then by (9) one has

$$M(r) = \frac{125}{9} \quad \text{and} \quad \sqrt{M(r)^2-1} = \frac{2}{9}\sqrt{3886}.$$

Thus, for $f \in \mathcal{S}$, Lemma 1 after some computation gives the estimate

$$\left|\frac{s_n'(z)}{f'(z)}-1\right| < |z|^n\left((n+1)^2+\frac{\sqrt{3886}}{3}\left(\frac{3}{2}\right)^{n-1}A_n'\frac{3|z|}{2-3|z|}\right)$$

for $|z| < 2/3$. This estimate helps us to discuss the disk of starlikeness of partial sums of functions from $\mathcal{S}$. For example, Theorem 1 is a consequence of this observation.

Proof of Theorem 1. Let $f \in \mathcal{S}$. Setting $|z| = 1/2$ in (10) yields

$$\left|\frac{s_n'(z)}{f'(z)} - 1\right| < \frac{1}{2^n}\left((n+1)^2 + \sqrt{3886}\left(\frac{3}{2}\right)^{n-1} A_n'\right) = K_1 \quad \text{for } |z| \le 1/2, \tag{11}$$

where A_n' is given by (1). We remark that we have used the maximum modulus principle for analytic functions so that the inequality (11) holds also in the closed disk $|z| \le 1/2$, as a similar observation holds for (12) and (13) below.

Next we recall a well-known result for $f \in \mathcal{S}$ (cf., [**3**, page 97, Theorem 3.6])

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \le \log \frac{1+r}{1-r}, \quad |z| = r < 1.$$

In particular, this gives

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \le \log 3 \quad \text{for } |z| \le 1/2 \tag{12}$$

and Lemma 3 implies

$$\left|\frac{s_n(z)}{f(z)} - 1\right| < \frac{1}{2^n}(n+1+R_n) = K_2 \quad \text{for } |z| \le 1/2, \tag{13}$$

where $R_n = \sqrt{(n+1)(n+1/2)\log(n+1)}$. Next, we observe that

$$\left|\arg \frac{zs_n'(z)}{s_n(z)}\right| \le \left|\arg \frac{s_n'(z)}{f'(z)}\right| + \left|\arg \frac{zf'(z)}{f(z)}\right| + \left|\arg \frac{f(z)}{s_n(z)}\right|,$$

and (11) and (13) show that

$$\max_{|z|=1/2} \left|\arg \frac{s_n'(z)}{f'(z)}\right| \le \sin^{-1}(K_1)$$

and

$$\max_{|z|=1/2}\left|\arg\frac{f(z)}{s_n(z)}\right| \le \sin^{-1}(K_2).$$

Finally, using (11), (12) and (13), we find that

$$\left|\arg\frac{zs_n'(z)}{s_n(z)}\right| < \frac{\pi}{2} \quad \text{for } |z| \le 1/2,$$

provided

$$\sin^{-1}(K_1) + \sin^{-1}(K_2) + \log 3 \le \frac{\pi}{2}.$$

The last inequality can easily be seen to be true for all $n \ge 47$ (e.g., using Mathematica). This ends the proof. ◻

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