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ON ROOTS OF DEHN TWISTS

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ABSTRACT. Margalit and Schleimer **[4]** discovered a nontrivial root of the Dehn twist about a nonseparating curve on a closed oriented connected surface. We give a complete set of conjugacy invariants for such a root by using a classification theorem of Matsumoto and Montesinos **[5, 6]** for pseudo-periodic maps of negative twists. As an application, we determine the range of degree for roots of a Dehn twist.

1. Introduction. Let Σ_{g+1} be a closed, oriented connected surface of genus $g+1 \geq 2$ and $\mathcal{M}_{g+1}$ the mapping class group of Σ_{g+1}, the group of isotopy classes of orientation-preserving homeomorphisms of Σ_{g+1}. We denote by $[h] \in \mathcal{M}_{g+1}$ the isotopy class of an orientation-preserving homeomorphism h of Σ_{g+1}. It seems natural to ask whether the Dehn twist about a curve C on Σ_{g+1} has a root in $\mathcal{M}_{g+1}$. In other words, given an integer degree $n > 1$, does there exist $[h] \in \mathcal{M}_{g+1}$ such that $[t_C] = [h]^n$? If C is separating, it is well-known that the Dehn twist has a root of degree two (a "half twist") derived from a chain relation in $\mathcal{M}_{g+1}$. Margalit and Schleimer **[4]** discovered a nontrivial root of the Dehn twist about a nonseparating curve C on Σ_{g+1}. They constructed a root of degree $2g-1$ by using a relation coming from the Artin group of type B_n.

In this paper we clarify several properties of roots of Dehn twists. We first apply a classification theorem of Matsumoto and Montesinos **[5, 6]** for pseudo-periodic maps of negative twists to roots of Dehn twists and obtain a complete set of conjugacy invariants for a root of the Dehn twist about a nonseparating curve C on Σ_{g+1} (see Theorem 3.4). Making use of these invariants, we then prove that the degree n of a root of the Dehn twist about C must be odd, and it satisfies the condition $3 \leq n \leq 2g-1$ (see Corollary 3.5).

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McCullough and Rajeevsarathy [**7**] have recently obtained the same results as Theorem 3.4 and Corollary 3.5 without using the theorem of Matsumoto and Montesinos [**5, 6**] (see [**7,** Theorem 2.1 and Corollary 2.2]). They have found more constraints on degree of roots of Dehn twists (see [**7,** Corollaries 3.1, 3.2 and Theorem 4.2]).

In Section 2 we review definitions and basic properties of pseudo-periodic maps and their conjugacy invariants. We apply Matsumoto-Montesinos' theorem to roots of Dehn twists and determine the range of degree for roots of a Dehn twist in Section 3. Sections 4 and 5 are devoted to an explicit enumerate of the root of a Dehn twist of degree three and an alternative proof of the latter part of Corollary 3.5, respectively. We end with a discussion about roots of Dehn twists for surfaces with boundary and punctures in Section 6.

2. Preliminaries.

Matsumoto and Montesinos [**5, 6**] established the theory of pseudo-periodic maps, which renewed Nielsen's work [**9**] on "surface transformation classes of algebraically finite type" from the viewpoint of degenerations of Riemann surfaces. In this section we review a part of their theory which is applied to roots of Dehn twists in the next section.

Hereafter, all surfaces will be oriented, and all homeomorphisms between them will be orientation-preserving. For us, a Dehn twist means a left-handed Dehn twist. Let Σ_{g+1} be a closed, connected oriented surface of genus $g+1 \geq 2$.

2.1. Pseudo-periodic map and screw number.

We begin with a precise definition of pseudo-periodic maps.

Definition (Matsumoto-Montesions [**5, 6**], cf., Nielsen [**9**])**.** Let $f : \Sigma_{g+1} \to \Sigma_{g+1}$ be a homeomorphism. f is called a *pseudo-periodic map* if f is isotopic to a homeomorphism $f' : \Sigma_{g+1} \to \Sigma_{g+1}$ which satisfies the following conditions:

(i) there exists a disjoint union $\mathcal{C}$ of simple closed curves $C_1, C_2, \ldots, C_r$ on Σ_{g+1} such that $f'(\mathcal{C}) = \mathcal{C}$,

(ii) the restriction $f'|_{\Sigma_{g+1}-\mathcal{C}}$ of f' to the complement $\Sigma_{g+1} - \mathcal{C}$ of $\mathcal{C}$ is isotopic to a periodic map of $\Sigma_{g+1} - \mathcal{C}$.

Note that if $\mathcal{C}$ is empty, f is isotopic to a periodic map of Σ_{g+1}. The set $\{C_i\}_{i=1}^r$ of curves is called a *system of cut curves* subordinate to f. If every connected component of $\Sigma_{g+1} - \mathcal{C}$ has a negative Euler characteristic, the system $\{C_i\}_{i=1}^r$ of cut curves is called *admissible.*

Remark 1 ([**5**], Lemma 2.1)**.** For any pseudo-periodic map $f : \Sigma_{g+1} \to \Sigma_{g+1}$, there exists an admissible system of cut curves subordinate to f.

Let $f : \Sigma_{g+1} \to \Sigma_{g+1}$ be a pseudo-periodic map and $\{C_i\}_{i=1}^r$ an admissible system of cut curves subordinate to f. We fix an orientation of each curve C_i arbitrarily. Deforming f by an isotopy, if necessary, we assume that f keeps $\mathcal{C} = C_1 \cup \cdots \cup C_r$ invariant: $f(\mathcal{C}) = \mathcal{C}$.

Choose and fix a curve C_i. Let α be the smallest positive integer such that $f^\alpha(C_i) = C_i$ and f^α preserves the orientation of C_i. If we take a point on C_i and its small disk neighborhood D in Σ_{g+1}, $D - C_i$ is a disjoint union of two connected components Δ and Δ'. Let b (respectively b') be the connected component of $\Sigma_{g+1} - \mathcal{C}$ which includes Δ (respectively Δ'), and β (respectively β') the smallest positive integer such that $f^\beta(b) = b$ (respectively $f^{\beta'}(b') = b'$). Note that α is a common multiple of β and β'. Since $f|_{\Sigma_{g+1}-\mathcal{C}}$ is isotopic to a periodic map of $\Sigma_{g+1} - \mathcal{C}$, there exists a positive integer n such that $(f^\beta|_b)^n$ is isotopic to the identity map id_b of b. Let n_b be the smallest one among such integers n. We choose a positive integer $n_{b'}$ for b' in a similar way. Let L be the least common multiple of $n_b\beta$ and $n_{b'}\beta'$. Since $f^L|_b$ (respectively $f^L|_{b'}$) is isotopic to the identity id_b (respectively $\mathrm{id}_{b'}$), the restriction $f^L|_{b\cup C_i\cup b'}$ of f^L to the union $b \cup C_i \cup b'$ is isotopic to a power of a left-handed Dehn twist map t_{C_i} about C_i. Let e be a unique integer such that $f^L|_{b\cup C_i\cup b'}$ is isotopic to $t_{C_i}^e$ on $b \cup C_i \cup b'$.

Definition (Nielsen [**9**], Matsumoto and Montesinos [**5, 6**])**.** For a pseudo-periodic map f and a fixed curve C_i above, we define the *screw number* $s(C_i)$ of f about C_i to be the rational number $e\alpha/L$.

A system $\{C_i\}_{i=1}^r$ of cut curves subordinate to f is called *precise* if it is admissible and the screw number $s(C_i)$ for each curve C_i is not zero. For an admissible system of cut curves subordinate to f, one can make it precise by removing all curves with screw number zero from the system.

If every curve C_i in a precise system $\{C_i\}_{i=1}^r$ subordinate to f has negative screw number, f is called a pseudo-periodic map *of negative twist.*

The next technical term was introduced by Nielsen.

Definition (Nielsen [**9**], Matsumoto and Montesinos [**5, 6**]). For a pseudo-periodic map f and a fixed curve C_i above, C_i is called *amphidrome* with respect to f if there exists an integer γ such that $f^\gamma(C_i)$ is equal to C_i with the opposite orientation. Here we assume that $f(\mathcal{C}) = \mathcal{C}$. It is easily seen that γ can be taken to be $\alpha/2$.

We now recall a theorem of Matsumoto and Montesinos.

Theorem 2.1 (Matsumoto and Montesinos [**5, 6**]). *Let $f : \Sigma_{g+1} \to \Sigma_{g+1}$ be a pseudo-periodic map of negative twist. The conjugacy class of $[f]$ in $\mathcal{M}_{g+1}$ is completely determined by following data*:

(i) *A precise system of cut curves $\mathcal{C} = \bigcup_i^r C_i$ on Σ_{g+1},*
(ii) *for cut curve $C_i \in \mathcal{C}$, α and the screw number $s(C_i)$ of f,*
(iii) *C_i's character of being amphidrome or not with respect to f,*
(iv) *for each connected component b of $\Sigma_{g+1} - \mathcal{C}$, β and n_b,*
(v) *for each connected component b of $\Sigma_{g+1} - \mathcal{C}$, the conjugacy class of the periodic map $f^\beta|_b$, and*
(vi) *the action of f on the oriented graph $G_{\mathcal{C}}$ whose vertices and edges correspond to connected components of $\Sigma_{g+1} - \mathcal{C}$ and $\{C_i\}_{i=1}^r$.*

2.2. Valency. Let Σ be an oriented connected surface, and let $f : \Sigma \to \Sigma$ be a smooth periodic map of order $n > 1$. Let p be a point on Σ. There is a positive integer $\alpha(p)$ such that the points $p, f(p), \ldots, f^{\alpha(p)-1}(p)$ are mutually distinct and $f^{\alpha(p)}(p) = p$. If $\alpha(p) < n$, we call p a *multiple point* of f. Note that a multiple point is an isolated and interior point of Σ.

Let $\overrightarrow{\mathcal{C}} = \{\overrightarrow{C_1}, \overrightarrow{C_2}, \ldots, \overrightarrow{C_s}\}$ be a set of oriented and disjoint simple closed curves in the surface Σ, and g be a map $g : \Sigma \to \Sigma$ such that $g(\overrightarrow{\mathcal{C}}) = \overrightarrow{\mathcal{C}}$ and $g|_{\overrightarrow{\mathcal{C}}}$ is periodic. Let m_j be the smallest positive integer such that $g^{m_j}(\overrightarrow{C_j}) = \overrightarrow{C_j}$. The restriction $g^{m_j}|_{\overrightarrow{C_j}}$ is a periodic map of $\overrightarrow{C_j}$. Let $\lambda_j > 0$ be the order of this map, the $(g|_{\overrightarrow{C_j}})^{m_j\lambda_j}$ is the identity map on $\overrightarrow{C_j}$. Let q be a point on C_j, and suppose that the images of q under the iteration of g^{m_j} are ordered as $(q, g^{m_j\sigma_j}(q), g^{2m_j\sigma_j}(q), \ldots, g^{(\lambda_j-1)m_j\sigma_j}(q))$ viewed in the direction of $\overrightarrow{C_j}$, where σ_j is an integer with $0 \le \sigma_j \le \lambda_j - 1$ such that $\gcd(\sigma_j, \lambda_j) = 1$ when $\lambda_j > 1$, and $\sigma_j = 0$ when $\lambda_j = 1$. Let δ_j be the integer with $0 \le \delta_j \le \lambda_j - 1$ which satisfies $\sigma_j\delta_j \equiv 1 \pmod{\lambda_j}$ when $\lambda_j > 0$, and $\delta_j = 0$ when $\lambda_j = 1$. Then the action of g^{m_j} on $\overrightarrow{C_j}$ is the rotation of angle $2\pi\delta_j/\lambda_j$ with a suitable parametrization of $\overrightarrow{C_j}$ as an oriented circle.

Definition. [**8**] The triple $(m_i, \lambda_i, \sigma_i)$ and $(m_i, \lambda_i, \delta_i)$ are called the *valency* and the *second valency* of $\overrightarrow{C_j} \in \overrightarrow{\mathcal{C}}$ *with respect to g.*

Nielsen also defined the *valency of a boundary curve* as its valency with respect to f assuming it has the orientation induced by that of the surface Σ. The *valency of a multiple point p* is defined to be the valency of the boundary curve ∂D_p, oriented from the outside of a disk neighborhood D_p of p.

The quotient space Σ/f is an orbifold. Its underlying space is a compact surface. Let $\pi : \Sigma \to \Sigma/f$ be the quotient map. For a multiple point $p \in \Sigma$ of f, $\pi(p)$ is a branch point of Σ/f. Thus, we can speak of the valency of a branch point of Σ/f. Also, we can speak of the valency of a boundary curve of Σ/f.

In order to prove Theorem 3.4, we need the following theorems.

Theorem 2.2. [**8**] *Let f be a periodic map on Σ_g of period n, and let $(m_i, \lambda_i, \sigma_i)$ be the valency of branch points p_i $(i = 1, \ldots, k)$ of Σ_g/f with respect to f. We denote by g' the genus of Σ_g/f.*

There is a periodic map f whose data is $[n, g'; (\sigma_1, \lambda_1), \ldots, (\sigma_k, \lambda_k)]$ if and only if the following conditions are satisfied:

(i) $\{2(g-1)\}/n = 2(g'-1) + \sum_{i=1}^{k}(1 - 1/\lambda_i)$,
(ii) $\sum_{j=1}^{k}(\sigma_j/\lambda_i)n \equiv 0 \pmod{n}$.

We consider two data sets to be the same if they differ by reordering the pairs $(\sigma_1, \lambda_1), \ldots, (\sigma_k, \lambda_k)$. Nielsen also proved that this data set determines a periodic map up to conjugacy. Equation (i) is the *Riemann-Hurwitz formula*.

Matsumoto and Montesinos proved the following theorem.

Theorem 2.3 ([**9,** Section 15], [**3,** Theorem 13.3], [**5,** Theorem 2.1, Corollary 3.3.1, Corollary 3.7.1]). *Any pseudo-periodic map of Σ_{g+1} is isotopic to a pseudo-periodic map f such that*:

(i) *There exists a system of disjoint annular neighborhoods $\{A_i\}_{i=1}^r$ of the precise system of cut curves $\mathcal{C} = \bigcup_{i=1}^r C_i$ subordinate to f such that $f(\mathcal{A}) = \mathcal{A}$, where $\mathcal{A} = \bigcup_{i=1}^r A_i$;*
(ii) *the map $f|_{\Sigma_g - \mathcal{A}} : \Sigma_g - \mathcal{A} \to \Sigma_g - \mathcal{A}$ is periodic;*
(iii) *let $(m_i^0, \lambda_i^0, \delta_i^0)$ and $(m_i^1, \lambda_i^1, \delta_i^1)$ be the second valencies of the boundary curves $\partial_0 A_i$ and $\partial_1 A_i$ of A_i with respect to f, respectively. $\partial_0 A_i$ and $\partial_1 A_i$ are regarded as boundary curves of $\overline{\Sigma_g - \mathcal{A}}$.*
Then, If C_i is non-amphidrome,
(a) $m_i^0 = m_i^1$,
(b) *$s(C_i) + \delta_i^0/\lambda_i^0 + \delta_i^1/\lambda_i^1$ is an integer.*
If C_i is amphidrome,
(a) *$m_i^0 = m_i^1 =$ an even number,*
(b) *$\delta_i^0 = \delta_i^1$ and $\lambda_i^0 = \lambda_i^1$,*
(c) *$s(C_i)/2 + \delta_i/\lambda_i$ is an integer,*
(λ_i denotes $\lambda_i^0 = \lambda_i^1$ and δ_i denotes $\delta_i^0 = \delta_i^1$).

3. The conjugacy classes of roots of the Dehn twist about a nonseparating curve. In this section we will prove Theorem 3.4.

Let C be a nonseparating curve on Σ_{g+1}, and let t_C be a representative of the Dehn twist about C. By isotopy, we may assume that $t_C(C) = C$. $t_C|_{\Sigma_{g+1}-C}$ is isotopic to the identity in the complement of C. Suppose that $[h]$ is a root of $[t_C]$ of degree $n > 1$. Since

$$[t_C] = [h]^n = [h][h]^n[h]^{-1} = [h][t_C][h]^{-1} = [t_{h(C)}],$$

we see that $h(C)$ is isotopic to C. Changing h by isotopy, we may assume that $h(C) = C$. Since $t_C = h^n$ and $h(C) = C$, $h|_{\Sigma_{g+1}-\{C\}}$

must be isotopic to a periodic map of order n. Therefore, h is a pseudo-periodic map, and an admissible system of cut curves $\mathcal{C}$ is C.

From Theorem 2.3, changing h by isotopy, we may assume that there exists an annular neighborhood A of C such that $h(A) = A$ and that $h|_{\Sigma_{g+1}-A}$ is a periodic map of order n. Let $(m^0, \lambda^0, \delta^0)$ and $(m^1, \lambda^1, \delta^1)$ be the second valencies of $\partial_0 A$ and $\partial_1 A$ with respect to h, respectively.

Claim 3.1. *C is non-amphidrome with respect to h.*

Proof. For contradiction, we assume that C is amphidrome with respect to h.

We will find the screw number $s(C)$ of h. Let b be $\Sigma_{g+1} - C$, and let α, β and n_b be the smallest positive integers such that $h^\alpha(\overrightarrow{C}) = \overrightarrow{C}$, $h^\beta(b) = b$, and $(h^\beta|_b)^{n_b}$ is isotopic to $id|_b$, respectively.Since $h(C) = C$ and C is amphidrome, we have $\alpha = m^0 = m^1 = 2$. Moreover, we have $\beta = 1$ and $n_b = L$. Thus, since $h^n(\overrightarrow{C}) = t_C(\overrightarrow{C}) = \overrightarrow{C}$ and $\alpha = 2$, we may write n as $2k$. By definition of $L > 0$, L is a divisor of n ($z := n/L \in \mathbf{Z}_{\geq 1}$). Since $t_C = h^n = (h^L)^z = (t_C^e)^z = t_C^{ez}$, we see that $e = z = 1$ and $L = 2k = n$. From the above arguments, we have

$$s(C) = e\alpha/L = 1/k.$$

By Theorem 2.3, we have

$$\delta/\lambda = (2k-1)/2k, \tag{1}$$

(Here λ denotes $\lambda^0 = \lambda^1$, and δ denotes $\delta^0 = \delta^1$.) However, since $n = 2k$ and the action of h^2 is the rotation of angle $2\pi\delta/\lambda$ in the circle, δ/λ must be equal to δ/k. This contradicts (1). Therefore, we see that C is non-amphidrome with respect to h. □

Lemma 3.2. $\delta^0 + \delta^1 = n - 1 \quad (1 \leq \delta^\nu \leq n-2,\ \nu = 0, 1).$

Proof. In order to prove Lemma 3.2 we will use Theorem 2.3. We will determine the screw number $s(C)$ of h. Since $h(C) = C$ and C is non-amphidrome (by Claim 3.1), we have $\alpha = m^0 = m^1 = 1$. Thus, we have $s(C) = 1/n$ by the argument of Claim 3.1. Furthermore, we find that $\lambda^\nu =$ order of $h|_{\overrightarrow{\partial_\nu A}}$ (by $m^\nu\lambda^\nu =$ order of $h|_{\overrightarrow{\partial_\nu A}}$ for $\nu = 0, 1$). By Theorem 2.3, we have $\delta^0/\lambda^0 + \delta^1/\lambda^1 = (n-1)/n$.

Let ∂_ν be the boundary components of $\overline{\Sigma_{g+1} - A}$ which correspond to $\partial_\nu A$ ($\nu = 0, 1$). Then, since $h|_{\overline{\Sigma_{g+1}-A}}$ is a periodic map of order $n(= L)$, the period λ^ν of $h|_{\partial_\nu A}$ is equal to n. We note that δ^ν is not equal to 0. If $\delta^\nu = 0$, then $n = \lambda^\nu$ is equal to 1 by the definition of λ^ν. This contradicts $n > 1$. □

Proposition 3.3. *For a root $[h]$ of $[t_C]$ of degree n in $\mathcal{M}_{g+1}$, the conjugacy class of $[h]$ in $\mathcal{M}_{g+1}$ is completely determined by n and the conjugacy class of $h|_{\overline{\Sigma_{g+1}-A}}$.*

Proof. We prove Proposition 3.3 by using Theorem 2.1.

Let G_C be the oriented graph G_C whose vertices and edges correspond to connected components of $\Sigma_{g+1} - C$ and C, respectively. Since C is non-amphidrome with respect to h, we find that the action of h on the oriented graph G_C is identity. Therefore, for h, we have $\mathcal{C} = C$, $\alpha = 1$, $\beta = 1$, $n_b = n$, that C is non-amphidrome with respect to h, and that the action of h on G_C is identity. These are the same data as h^{-1}.

Since $s(C)$ of h is equal to $1/n$ from the proof of Lemma 3.2, we see that $s(C)$ of h^{-1} is equal to $-1/n$. Therefore, h^{-1} is negative twist. If we restrict our attention to roots of t_C^{-1}, by using Theorem 2.1, the conjugacy class of the root $[h^{-1}]$ of $[t_C^{-1}]$ is completely determined by n and the conjugacy class of the periodic map $h^{-1}|_{\Sigma_{g+1}-C}$. Therefore, the conjugacy class of $[h]$ is completely determined by n and the conjugacy class of $h|_{\overline{\Sigma_{g+1}-A}}$. This completes the proof of Proposition 3.3. □

Theorem 3.4. *Let $[h]$ be a root of $[t_C]$ of degree n in $\mathcal{M}_{g+1}$.*

There is a representative h whose data is $[n, g', (\sigma^0, \sigma^1); (\sigma_1, \lambda_1), \ldots, (\sigma_k, \lambda_k)]$ if and only if the following conditions are satisfied:

(i) $2g/n = 2g' + \sum_{i=1}^k (1 - 1/\lambda_i)$,
(ii) $\sum_{i=1}^k \sigma_i n/\lambda_i + \sigma^0 + \sigma^1 \equiv 0 \pmod{n}$,
(iii) $\sigma^0 + \sigma^1 + \sigma^0\sigma^1 \equiv 0 \pmod{n}$,

where n, g', σ^ν, σ_i and λ_i are nonnegative integers such that

(1) $1 < n$, $0 \le g' \le g-1$, *each* $1 \le \sigma_i \le \lambda_i - 1$, *each* $1 \le \sigma^\nu \le n-2$, *and each λ_i divides n,*

(2) $\gcd(\sigma^0, n) = \gcd(\sigma^1, n) = 1$ *and each* $\gcd(\sigma_i, \lambda_i) = 1$.

Moreover, this data set determines a root of $[t_C]$ *up to conjugacy in* $\mathcal{M}_g$.

We consider two data sets to be the same if they differ by interchanging σ^0 and σ^1 or reordering the pairs $(\sigma_1, \lambda_1), \ldots, (\sigma_k, \lambda_k)$. McCullough and Rajeevsarathy also got a similar data set in **[7]**. We follow the notation $[n, g', (\sigma^0, \sigma^1); (\sigma_1, \lambda_1), \ldots, (\sigma_k, \lambda_k)]$ of **[7]**.

Proof. We first show that there are data satisfying the condition for a representative h. From the above arguments, we may assume that there exists an annulus A of C such that $h(A) = A$ and $h|_{\overline{\Sigma_{g+1} - A}}$ is a periodic map of order n. Therefore, by pasting a disk D_ν to $\partial_\nu(\overline{\Sigma_{g+1} - A})$ $(\nu = 0, 1)$, we can extend a periodic map f of order n on $S_g \cong \Sigma_g \cong \overline{\Sigma_{g+1} - A} \cup D_0 \cup D_1$ preserving D_ν. Since C is non-amphidrome with respect to h, a center point q^ν of D_ν is a fixed point of f. We denote by $\widehat{q}^\nu$ the branch point $\pi(q^\nu)$ on S_g/f, where $\pi : S_g \to S_g/f$ is the quotient map. By Lemma 3.2, the second valency of $\widehat{q}^\nu$ with respect to f is $(1, n, \delta^\nu)$ $(\nu = 0, 1)$ such that $\delta^0 + \delta^1 = n - 1$ $(1 \leq \delta^\nu \leq n - 2)$.

Let $\widehat{p}_i$ and $\widehat{q}^\nu$ $(i = 1, \ldots, k,\ \nu = 0, 1)$ be branch points on S_g/f, respectively. Let $(m_i, \lambda_i, \sigma_i)$ and $(1, n, \sigma^\nu)$ be the valencies of $\widehat{p}_i$ and $\widehat{q}^i$, respectively. By the definition of the valency, we see that $\sigma^\nu \delta^\nu \equiv 1 \pmod{n}$. From $1 \leq \delta^\nu \leq n - 2$, we have $1 \leq \sigma^\nu \leq n - 2$. Since $n - 1 = \delta^0 + \delta^1 \equiv 1/\sigma^0 + 1/\sigma^1 \pmod{n}$, we have

$$\sigma^0 + \sigma^1 + \sigma^0 \sigma^1 \equiv 0 \pmod{n}.$$

From Theorem 2.2 we have parts (i) and (ii) of Theorem 3.4.

We next show that there is a representative h for data satisfying the condition. If there are such integers, then by Theorem 2.2, there is a periodic map $f : \Sigma_g \to \Sigma_g$ such that the valencies of branch points $\widehat{p}_i$ and $\widehat{q}^\nu$ $(i = 1, \ldots, k,\ \nu = 0, 1)$ with respect to f are $(m_i, \lambda_i, \sigma_i)$ and $(1, n, \sigma^\nu)$, respectively. We note that a lift q^ν of $\widehat{q}^\nu$ to Σ_g is a fixed point of f, so there is a disk neighborhood D_ν of q^ν such that $f(D_\nu) = D_\nu$ and $f|_{\overline{\Sigma_g - D_0 \cup D_1}}$ is a periodic map of $\overline{\Sigma_g - D_0 \cup D_1}$. From the valency of $\widehat{q}^\nu$ with respect to f and part (iii) of Theorem 3.4, we find that the second valency of ∂D_ν with respect to f is $(1, n, \delta^\nu)$ $(\nu = 0, 1)$ such that $\delta^0 + \delta^1 = n-1$ $(1 \leq \delta^\nu \leq n-2)$. When we attach an annulus A, we obtain $S_{g+1} \cong \Sigma_{g+1}$. Let C be a simple closed curve on A which is parallel to $\partial_0 A$ and $\partial_1 A$. By the conditions of the second valencies of ∂D_0 and ∂D_1 we can extend $f|_{\overline{\Sigma_g - D_0 \cup D_1}}$ to a homeomorphism $\overline{h}$ of S_{g+1} such that $(\overline{h})^n$ is isotopic to t_C^{1-n}. Then, since by the construction of $\overline{h}$, $\overline{h}$ and t_C commute with each other, $\overline{h}t_C$ is a representative of a root of $[t_C]$ with the data set.

Finally, we prove the last part of Theorem 3.4. From Proposition 3.3, the conjugacy class of $[h]$ in $\mathcal{M}_{g+1}$ is completely determined by the conjugacy class of $h|_{\overline{\Sigma_{g+1} - A}}$ and n. Moreover, the conjugacy class of $h|_{\overline{\Sigma_{g+1} - A}}$ and n correspond to the conjugacy class of f by a homeomorphism preserving $\{D_0, D_1\}$ and period n. If we restrict our attention to the conjugacy class of f by a homeomorphism preserving $\{D_0, D_1\}$ and period, for Theorem 2.2 we see that the data set determines a root of $[t_C]$ up to conjugacy in $\mathcal{M}_{g+1}$.

This completes the proof of Theorem 3.4. □

Corollary 3.5. *Suppose that there is a root of $[t_C]$ of degree n. Then, $3 \leq n \leq 2g+1$, and n is odd.*

Proof. By (iii) and (2) of Theorem 3.4, we see that n is odd.

For $n = 3$, if $g' = 0$, $k = g$ and $\sigma^0 = \sigma^1 = 1$, we can select σ_i $(i = 1, \ldots, g)$ which satisfy the condition (2). It means that there always exists the root of degree 3 for $g \geq 1$.

Suppose $n > 2g+1$. By condition (i), we have $1 > 2g/n = 2g' + \sum_{i=1}^{k}(1 - 1/\lambda_i)$ so $g' = 0$ and $k = 1$. From conditions (2) and (3) we have $n\sigma_1/\lambda_1 \equiv \sigma^0\sigma^1 \pmod{n}$. Therefore, we see that $0 \equiv n\sigma_1 \equiv \sigma^0\sigma^1\lambda_1 \pmod{n}$. This means that $\sigma^0\sigma^1\lambda_1/n = \sigma^0\sigma^1/m_1$ is an integer (we note that $m_1\lambda_1 = n$). Since $\gcd(\sigma^0, n) = \gcd(\sigma^1, n) = 1$ and $m_1\lambda_1 = n$, we see that m_1 must be equal to 1. This, in turn, means that $\lambda_1 = n$. Thus, we have $2g/n = 1 - 1/n$ so $n = 2g+1$. This contradicts $n > 2g+1$. Since Margalit and Schleimer constructed the root of degree $2g+1$, we have $n \leq 2g+1$. □

4. Dehn twist expression of the root of degree 3. We give an explicit enumerate of the root of $[t_C]$ of degree 3 by using the *star* *relation* given by Gervais [**2**]. In this section, we denote by t_C the isotopy class of the Dehn twist about C on Σ_{g+1}.

We consider the torus with three boundary components d_1, d_2, d_3, and let a_1, a_2, a_3 and b be simple closed curves in Figure 1.

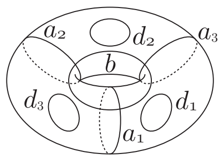

FIGURE 1. The curves of star relation.

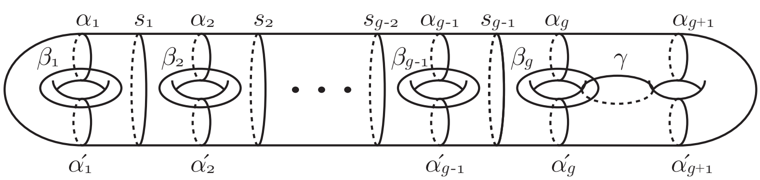

FIGURE 2. The curves α_i, α'_i, β_i $(i = 1, \ldots, g+1)$, γ and s_j $(j = 1, \ldots, g-1)$.

The *star relation* is as follows:

$$(t_{a_1} t_{a_2} t_{a_3} t_b)^3 = t_{d_1} t_{d_2} t_{d_3}.$$

If $a_1 = a_2$, then t_{d_3} is trivial, and the relation becomes

$$(t_{a_1}^2 t_{a_3} t_b)^3 = t_{d_1} t_{d_2}$$

Let α_i, α'_i, β_i $(i = 1, \ldots, g+1)$ and γ be nonseparating simple closed curves, and let s_j $(j = 1, \ldots, g-1)$ be the separating simple closed curve in Figure 2.

We define

$$\begin{aligned} \rho_1 &= (t_{\alpha_1} t_{\beta_1})^2 \\ \rho_i &= t_{\alpha_i}^2 t_{\alpha'_i} t_{\beta_i} \quad (i = 2, \ldots, g-1) \\ \rho_g &= t_{\alpha_g} t_\gamma t_{\alpha'_g} t_{\beta_g} \end{aligned}$$

and

$$\widehat{h} = \begin{cases} \rho_g\rho_{g-1}^{-1}\rho_{g-2}\cdots\rho_3^{-1}\rho_2\rho_1^{-1} & \text{(if } g+1 \text{ is odd)},\\ \rho_g\rho_{g-1}^{-1}\rho_{g-2}\cdots\rho_3\rho_2^{-1}\rho_1 & \text{(if } g+1 \text{ is even)}. \end{cases}$$

We note that $\rho_1^3 = t_{s_1}$, that $\rho_1\ldots,\rho_g$ commute with each other and that $\widehat{h}$ and $t_{\alpha_{g+1}}$ commute with each other. Then, by the star relation, we have $\widehat{h}^3 = t_{\alpha_{g+1}}^2$. When we define

$$h = t_{\alpha_{g+1}}\widehat{h}^{-1},$$

h is the root of $t_{\alpha_{g+1}}$ of degree 3.

5. A root of elementary matrices. In this section we give an alternative proof of the latter part of Corollary 3.5.

The action of $\mathcal{M}_{g+1}$ on $\mathrm{H}_1(\Sigma_{g+1};\mathbf{Z})$ preserves the algebraic intersection forms, so it induces a representation $\phi : \mathcal{M}_{g+1} \to \mathrm{Sp}(2(g+1),\mathbf{Z})$, which is well known to be surjective. Suppose $g+1=2$. An element 4×4 matrix $A \in \mathrm{Sp}(4,\mathbf{Z})$ satisfies that $AJ{}^tA = J$, where ${}^tA$ is transpose of A and J is

$$J = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Let α_1 be a nonseparating simple closed curve in Figure 2, and let S be

$$S = \rho(t_{\alpha_1}) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \left(S^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}\right).$$

We assume that $S = A^2$, where $A(\in \mathrm{Sp}(4,\mathbf{Z}))$ is

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

Since $A^{-1} = S^{-1}A$, we have ${}^tAJ = JA^{-1} = JS^{-1}A$. By

$$
{}^tAJ = \begin{pmatrix} -a_{31} & -a_{41} & a_{11} & a_{21} \\ -a_{32} & -a_{42} & a_{12} & a_{22} \\ -a_{33} & -a_{43} & a_{13} & a_{23} \\ -a_{34} & -a_{44} & a_{14} & a_{24} \end{pmatrix}
$$

and

$$
JS^{-1}A = \begin{pmatrix} -a_{11}+a_{31} & -a_{12}+a_{32} & -a_{13}+a_{33} & -a_{14}+a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ -a_{11} & -a_{12} & -a_{13} & -a_{14} \\ -a_{21} & -a_{22} & -a_{23} & -a_{24} \end{pmatrix}
$$

we have

$$
a_{11}/2 = a_{31}. \tag{2}
$$

Since $SA = AS$, we have

$$
\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11}+a_{31} & a_{12}+a_{32} & a_{13}+a_{33} & a_{14}+a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}
$$

$$
= \begin{pmatrix} a_{11}+a_{13} & a_{12} & a_{13} & a_{14} \\ a_{21}+a_{23} & a_{22} & a_{23} & a_{24} \\ a_{31}+a_{33} & a_{32} & a_{33} & a_{34} \\ a_{41}+a_{43} & a_{42} & a_{43} & a_{44} \end{pmatrix}
$$

We have $a_{12} = a_{13} = a_{14} = 0$. By $S = A^2$, we have

$$
a_{11} = \pm 1. \tag{3}
$$

By equations (2) and (3) we have $a_{31} = a_{11}/2 = \pm 1/2$. This contradicts $A \in \mathrm{Sp}(4, \mathbf{Z})$. Similar arguments apply to the case $g > 2$.

6. Roots in the mapping class group of a surface. In this section we consider a root of the Dehn twist about a nonseparating curve on a surface with boundary components and punctures.

Let $D_1, \dots, D_b$ be disjoint b open disks in Σ_{g+1}, and let $x_1, \dots, x_p$ be p marked points in Σ_{g+1}. We denote by $\mathcal{M}_{g+1,p}^b$ the group of isotopy classes of orientation-preserving homeomorphisms of Σ_{g+1} permuting p marked points and fixing $D_1, \dots, D_b$ pointwise, modulo isotopies which do not move the marked points and fix $D_1, \ldots, D_b$ pointwise. Therefore, we regard $\mathcal{M}^b_{g+1,p}$ as a subgroup of $\mathcal{M}_{g+1}$. It is well known that $\mathcal{M}^b_{g+1,p}$ is isomorphic to the mapping class group of a surface of genus $g+1$ with b boundary components and p punctures. If $b=0$, we omit b from the notation. Let C be a nonseparating curve C on Σ_{g+1} $(g+1 \geq 2)$ disjoint from $D_1, \ldots, D_b$ and $x_1, \ldots, x_p$.

Theorem 6.1. *If $b>0$, then $t_C \in \mathcal{M}^b_{g+1,p}$ has no roots.*

Proof. Suppose that there is a root $[h]$ of $[t_C]$ of degree n in $\mathcal{M}^b_{g+1,p}$. Since $[h](C) = C$ from the property of roots, $[h]^n|_{\Sigma_{g+1}-C} = id$.

From the arguments of Section 3, there is a representation h such that there is an annular neighborhood A of C, and $h|_{\overline{\Sigma_{g+1}-A}}$ is a periodic map of $\overline{\Sigma_{g+1}-A}$ satisfying $h|_{D_i} = id|_{D_i}$. By pasting two disks to two boundary curves of $\overline{\Sigma_{g+1}-A}$, we get a periodic map f on $S_g \cong \Sigma_g$ of order n such that $f|_{D_i} = id|_{D_i}$. However, since $\mathcal{M}^b_{g,p}$ $(b>0)$ is torsion free, nth power of the isotopy class of f is not an identity in $\mathcal{M}^b_{g,p}$. This means $[h]^n|_{\overline{\Sigma_{g+1}-A}} \neq id$. This contradicts $[h^b_p]^n|_{\Sigma^b_{g+1,p}-C} = id$. □

Theorem 6.2. *If $p \not\equiv 0, 1 \pmod{2g+1}$, $t_C \in \mathcal{M}_{g+1,p}$ has no roots of degree $2g+1$. In particular, if $p \equiv 2 \pmod 3$, then $t_C \in \mathcal{M}_{2,p}$ has no roots.*

Proof. Let h be a representative of a root of $[t_C]$ of degree $2g+1$ in $\mathcal{M}_{g+1}$, and let A be an annular neighborhood of C such that $h|_{\overline{\Sigma_{g+1}-A}}$ is a periodic map on $\overline{\Sigma_{g+1}-A}$ of order $2g+1$. By the proofs of Theorem 3.4 and Corollary 3.5, $h|_{\overline{\Sigma_{g+1}-A}}$ has only one fixed point in $\overline{\Sigma_{g+1}-A}$. Therefore, if $p \equiv r \pmod{2g+1}$ and $1 < r < 2g+1$, then there is no $\mathbf{Z}_{2g+1}$-action on $\overline{\Sigma_{g+1}-A}$. This means that $t_C \in \mathcal{M}_{g+1,p}$ has no roots of degree $2g+1$. □

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