

QUADRAFREE FACTORISATIO NUMERORUM

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ABSTRACT. We derive expressions for the number of factorization of positive integers into squarefree factors with order not counting and for the asymptotic average of these factorizations.

1. Introduction. We study the multiplicative decomposition of natural numbers into squarefree factors, $n = n_1 n_2 \cdots n_m$, $n_1 \leq n_2 \leq \cdots \leq n_m$, where each n_i is squarefree. Let $f_2(n)$ be the number of such factorizations, where the order of the factors does not count. For example, $24 = 2 \times 2 \times 6 = 2 \times 2 \times 2 \times 3$ so $f_2(24) = 2$.

The number of all possible factorizations of n with order not counting, denoted $f(n)$, has been well studied, and a great deal is known about its behavior [3, 14]. However, other than in the case of squarefree or prime power values of n , there is no explicit formula for its evaluation, so one has normally to be content with generating the entire factorization tree and counting the nodes (the greedy algorithm or its equivalent), or asymptotic results.

This is in contrast to squarefree factorizations where a range of direct formulas and recursive expressions are presented in this paper, with sufficient structure to enable them, potentially, to be extended.

Note that Warlimount [17] develops a range of generalizations of $f(n)$ to different types of factorization, including one called “squarefree.” However, this term refers to the single nature of the occurrence of a given factor, rather than the factors themselves. For example, $12 = 4 \times 3$ would be a squarefree factorization for Warlimount but would not contribute to $f_2(n)$.

Since $f_2(n) = f(n) = B_{\phi(n)}$ for all squarefree n , it follows that $f_2(n) \neq O(n^\alpha)$ for any α with $0 < \alpha < 1$ and, since $f_2(n) \leq f(n)$ for

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all $n \in \mathbf{N}$, $f_2(n) \leq n/\log n$ for all n [5, Theorem 3.1, Proposition 4.1]. The normal order of $f_2(n)$ remains to be investigated.

The set of values of $f_2(n)$ is completely different from the $f(n)$ case [2, 13]; since $f_2((2 \cdot 3)^a) = 1 + a$ for all $a \geq 1$ (Example 4 below) and $f_2(2) = 1$, we have $|\{f_2(n) : f_2(n) \leq x\}| = \lfloor x \rfloor$.

The summary of the paper content is as follows. We begin with some explicit formulas for $f_2(n)$, when n is a prime power, is squarefree, or is a product of at most three prime powers. Then we derive a combinatorial polynomial equality, based on a given multiset using a method of Rota. This method was first used to obtain partition identities for sets. This enables a recursive formula to be derived for integers which are a power of 2 times an odd squarefree number, a square of a squarefree number or a square of a squarefree number times a coprime squarefree number. Finally, we use the method of Oppenheim, developed for $f(n)$, to derive an asymptotic expression for the average of $f_2(n)$.

Notation. By a squarefree factorization of n , we mean a tuple $s = (n_1, \dots, n_t)$ such that each n_i is squarefree, $n = n_1 \cdots n_t$ and $n_1 \leq n_2 \leq \cdots \leq n_t$. By the length of a squarefree factorization, $l(s) := t$, the number of factors. The symbols p, q and p_i represent prime numbers. The expression $\Omega(n)$ is the total number of primes dividing n , including multiplicity. Expressions of the form

$$\binom{n}{i} \quad \text{or} \quad \left\{ \begin{matrix} t \\ r \end{matrix} \right\}$$

represent the standard binomial coefficient or Stirling numbers of the first kind (being the number of ways to partition a set of t objects into r nonempty subsets) respectively. The related Bell numbers B_n are the total number of distinct subset partitions of a set of n elements,

$$(B_n)_{n \geq 0} = (1, 1, 2, 5, 15, 52, 203, 877, \dots).$$

The generating function $e^{e^x - 1}$ gives B_n as the coefficient of x^n times $n!$. We also need the Bell polynomials, $B_n(x)$. These can be expressed as the coefficients in the exponential generating function

$$e^{(e^t - 1)x} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n.$$

They satisfy $B_n = B_n(1)$. Finally, if $a \geq 0$ is an integer and u a real variable, define the so-called downward factorial function $(u)_a := u(u - 1) \cdots (u - a + 1)$ so $(u)_0 = 1$, $(u)_1 = u$.

2. Examples. We begin with a number of easy to derive properties of $f_2(n)$. Let $a, b, a_i, b_i, i \in \{1, 2\}$, be positive integers.

- (1) $f_2(p^a) = 1$,
- (2) $f_2(p_1 p_2 \cdots p_r) = B_r$,
- (3) $(a, b) = 1$ implies $f_2(ab) \geq f_2(a)f_2(b)$,
- (4) If $a \mid b$, then $f_2(a) \leq f_2(b)$,
- (5) $f_2(p_1^{a_1} p_2^{a_2}) = 1 + \min\{a_1, a_2\}$,
- (6) $f_2(p_1^{a_1} p_2^{a_2} p_3) = 2 + 3 \min\{a_1, a_2\} + \min\{a_1 - 1, a_2\}$ if $a_1 \geq a_2$,
- (7) $f_2(2^{\Omega(b)} \cdot b) = f_2(2^a \cdot b)$, $a \geq \Omega(b)$.

The first three properties are the same or similar to those of $f(n)$. For item (4), take the corresponding multiset for a given squarefree factorization of a and map this to its union with the multiset of singleton primes for b/a to get a unique multiset squarefree decomposition of b .

To see (5), let $a = \max\{a_1, a_2\}$, $b = \min\{a_1, a_2\}$ so $f_2(p_1^{a_1} p_2^{a_2}) = f_2(2^a \times 3^b)$. Then the squarefree factorizations of $2^a 3^b$ have 0 through b factors with value 6, the remaining factors being singletons. No other squarefree factorizations are possible, giving the result $b + 1$. For (6), let $f_2(p_1^{a_1} p_2^{a_2} p_3) = f_2(2^a \times 3^b \times 5)$ with $a \geq b$. If $a > b$, then there can be i 6's in any squarefree factorization of $2^a \times 3^b$, with $0 \leq i \leq b$, the remaining factors being singleton 2's or 3's. This gives, when $a > b$, $1 + b$ factors with a singleton 5, $1 + b$ with one of the 2's replaced by 10, b with one of the 3's replaced by 15, and b with one of the 6's replaced by 30. In case $a = b$, then we lose a factorization because of the situation where there are no 2's to be replaced. This gives a total of $2 + 3b + \min\{a - 1, b\}$ factors.

Finally, to see (6), simply note that in any squarefree factorization of b there must be at most $\Omega(b)$ factors, and each may be combined with at most that many 2's. Hence, each factorization must have at least $a - \Omega(b)$ single 2's.

In a similar vein, we can go beyond these examples, but the results get much more complicated. For example, if $n = p^a q^b r^c$ with p, q, r distinct primes and natural numbers $a \geq b \geq c > 0$ then, with a bit of

work, it is possible to see that

$$f_2(n) = \sum_{i=0}^b \sum_{j=0}^{\min\{i,c\}} \theta(a-i, b-i, c-j)$$

where, if $\beta(n) := 1 + 2 + \dots + n$ with $\beta(0) := 0$, we set

$$\theta(u, v, w) := \begin{cases} 1/2(w+1)(w+2) & 0 \leq w \leq v, \\ \beta(w+1) - \beta(w-v) & v < w < u, \\ (u+1)(v+1) - \beta(u+v-w) & u \leq w \leq u+v, \\ (u+1)(v+1) & u+v < w, \end{cases}$$

which is example (8).

To our knowledge, there do not exist corresponding direct formulas to (4)–(7) for the computation of $f(n)$. Going beyond example (8), would result, we consider, in even more complicated expressions.

3. Rota’s method and the combinatorial sieve. Let M be a finite multiset with one element of multiplicity $a_1 \geq 1$, one of multiplicity $a_2 \geq 1$, etc., so $|M| = a_1 + \dots + a_k$. Let U be a finite set with $|U| = u$. Then

Lemma 3.1.

$$\sum_{\pi} \frac{(u)_{N(\pi)}}{b_1! \dots b_{l_i}!} = \frac{(u)_{a_1} \dots (u)_{a_k}}{a_1! \dots a_k!},$$

where the sum is over partitions π of M such that each subset of π is a set, and for each such partition there are b_1 equal subsets of type 1, b_2 of type 2, etc., so $N(\pi) = b_1 + \dots + b_{l_i}$ where $N(\pi)$ represents the number of subsets with multiplicity in the partition π .

Proof. Let

$$M = \{p_1, \dots, p_1, p_2, \dots, p_2, \dots, \dots, p_k \dots, p_k\}$$

with $a_j \geq 1$ copies of p_j for $1 \leq j \leq k$ be the multiset. For the right hand side of the given identity, consider the set of functions $F := \{f : M \rightarrow U\}$ where each element with multiplicity has each copy mapped to a different element of U . Then

$$|F| = (u)_{a_1} \dots (u)_{a_k}.$$

For each such f , define a partition of M , called $\ker f$, by $x \sim y$ if and only if $f(x) = f(y)$.

For the left hand side, for each partition of M into disjoint sets, $M = B_1 \cup B_2 \cup \dots \cup B_l$, define a function $g : M \rightarrow U$, such that g is constant on each B_i , and such that if $B_i \neq B_j$ as sets; then $g(x) \neq g(y)$ for all $x \in B_i$ and $y \in B_j$ and $\ker g$ is the partition $B_1 \cup B_2 \cup \dots \cup B_l$ of M . Note here that since M is a multiset it is quite possible to have some $B_i = B_j$ for some $i \neq j$ in this partition, and the functions g must give have different values on such subsets. Finally, we regard as equivalent functions which have the same set of values on identical B'_i s. The number of functions in M associated in this manner with each partition $B_1 \cup B_2 \cup \dots \cup B_l$ is given by

$$\frac{u(u-1)(u-2)\dots(u-l+1)}{b_1! \dots b_m!},$$

where the b_i are the number of sets B_j which are equal so $l = b_1 + \dots + b_m$. Let G be the set of all such functions g defined in this manner.

Since each f is associated with a unique partition into sets we can define a map $\theta : F \rightarrow G$ by setting $\theta(f)(B_i) = f(x)$ for $x \in B_i$ and where $\ker f = B_1 \cup B_2 \cup \dots \cup B_l$. Then, if $f, f' \in F$, $\theta(f) = \theta(f')$ if and only if there is a group element

$$\sigma \in S_{a_1} \times \dots \times S_{a_k} =: H,$$

acting on M in such a way that each p_i is mapped to another copy of p_i , and where S_n is the symmetric group of all permutations of $n \geq 1$ symbols, such that $f(x) = f'(\sigma \cdot x)$ for all $x \in M$. Hence, θ is onto and the inverse image of each element on the left hand side has size $|H|$, and the given identity follows. □

Example 3.2. Let $n = 2^4 3^2$ with corresponding multiset $M = \{2, 2, 2, 2, 3, 3\}$, so $a_1 = 4, a_2 = 2$ and the right hand side of Lemma 1 is the polynomial in $\mathbf{Q}[u]$

$$\frac{(u)_4 (u)_2}{4! 2!} = \frac{u(u-1)(u-2)(u-3)}{4!} \cdot \frac{u(u-1)}{2!}.$$

For the left hand side, the partitions with subparts which are sets are

$$\begin{aligned} \pi_1 &= \{\{2, 3\}, \{2, 3\}, \{2\}, \{2\}\}, \\ \pi_2 &= \{\{2, 3\}, \{2\}, \{2\}, \{2\}, \{3\}\}, \\ \pi_3 &= \{\{2\}, \{2\}, \{2\}, \{2\}, \{3\}, \{3\}\}, \end{aligned}$$

giving, since $N(\pi_1) = 4$, $N(\pi_2) = 5$, $N(\pi_3) = 6$, the three corresponding terms on the left hand side

$$\frac{(u)_4}{2!2!} + \frac{(u)_5}{3!1!1!} + \frac{(u)_6}{4!2!}.$$

Using Lemma 3.1 to derive partition identities is not as straightforward as Rota’s original application to sets. However, we do have some results. First we treat the case with just one prime to a power higher than 1:

Example 3.3. Let p_1, \dots, p_k be distinct odd primes with $k \geq 1$. Then

- (1) $f_2(2^1 p_1 \cdots p_k) = \frac{1}{1!} (B_{k+1}),$
- (2) $f_2(2^2 p_1 \cdots p_k) = \frac{1}{2!} (B_{k+2} - B_{k+1} + B_k),$
- (3) $f_2(2^3 p_1 \cdots p_k) = \frac{1}{3!} (B_{k+3} - 3B_{k+2} + 5B_{k+1} + 2B_k),$
- (4) $f_2(2^4 p_1 \cdots p_k) = \frac{1}{4!} (B_{k+4} - 6B_{k+3} + 17B_{k+2} - 4B_{k+1} + 9B_k).$

To see how to obtain equations of this type we will derive (4): For $2 \leq j \leq 4$, let b_j be the number of squarefree factorizations of $n \in \mathbf{N}$ with exactly j 2’s. Let $f_2^d(n)$ be the number of these factorizations where each of the factors is different. Then $f_2(n) = f_2^d(n) + b_2 + b_3 + b_4$ (*). Note that $b_4 = f_2(p_1 \cdots p_k)$ and that, for $2 \leq j \leq 3$, $b_j = f_2(2^{4-j} p_1 \cdots p_k) - f_2(2^{3-j} p_1 \cdots p_k)$.

Define a linear mapping L from $V := \mathbf{Q}[u]$ to \mathbf{Q} in the same manner as Rota, i.e., set $L((u)_n) = 1$ for $n \geq 0$ and, recalling that the polynomials $((u)_n)_{n \geq 0}$ are a basis, extend L to the whole of V . Then apply L to the identity of Lemma 3.1 when $n = 2^4 p_1 \cdots p_k$, so

the right hand side is

$$\frac{(u)_4(u)_1 \cdots (u)_1}{4!1! \cdots 1!} = \frac{u^k(u)_4}{4!},$$

to obtain the equation

$$f_2^d(n) + \frac{b_2}{2!} + \frac{b_3}{3!} + \frac{b_4}{4!} = \frac{1}{24}L(u^k(u)_4),$$

where

$$\begin{aligned} b_2 &= f_2(2^2 p_1 \cdots p_k) - f_2(2 p_1 \cdots p_k), \\ b_3 &= f_2(2 p_1 \cdots p_k) - f_2(p_1 \cdots p_k), \\ b_4 &= f_2(p_1 \cdots p_k), \end{aligned}$$

and $u^k(u)_4 = u^{k+1}(u-1)(u-2)(u-3)$.

Now use the fact that, for all $n \geq 0$, $L(u^n) = B_n$ and the values of $f_2(\cdot)$ from (1) and (2) listed above, to obtain an equation for $f_2^d(n)$ in terms of the B_j 's, substitute this back in (*) to obtain (4).

The following gives the general form for these equations. It is derived in a similar manner to the special case set out above.

Theorem 3.4. *Let $1 \leq a \leq k$ and p_1, \dots, p_k be a set of distinct odd primes. Then*

$$f_2(2^a p_1 \cdots p_k) = \frac{1}{a!}L(u^k(u)_a) + \sum_{j=2}^a \left(\frac{j-1}{j!}\right) f_2(2^{a-j} p_1 \cdots p_k).$$

If $a > k$, then $f_2(2^a p_1 \cdots p_k) = f_2(2^k p_1 \cdots p_k)$.

We are also able to use the lemma to derive an expression for $f_2(s^2)$ where s is squarefree. We use the version of the combinatorial sieve that is easily derived using generating functions [18], i.e., given a finite set Ω of objects and a family P of properties, for each subset $S \subset P$ of properties, let N^S be the number of objects that has at least the properties in S . Then define, for each $r \geq 0$,

$$(3.1) \quad N_r := \sum_{S:|S|=r} N^S.$$

Then, for $j \geq 0$, the number of objects having exactly j of the properties is given by

$$(3.2) \quad e_j := \sum_{r \geq 0} (-1)^{r-j} \binom{r}{j} N_r.$$

Set $a_0 = 1$ and, for $k \geq 1$, if p_1, \dots, p_k is a set of distinct odd primes, set $a_k = f_2((p_1 \cdots p_k)^2)$. Then $(a_k)_{k \geq 0} = (1, 1, 3, 16, 139, 1750, \dots)$, and we have the following recurrence:

Theorem 3.5. *For all $k \geq 0$,*

$$a_k = \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} B_{k+j} - \sum_{j=1}^k a_{k-j} \binom{k}{j} B_j \left(-\frac{1}{2}\right).$$

Proof. Let $n = (p_1 \cdots p_k)^2$ and, for $j \geq 0$, let e_j represent the number of squarefree factors of n with exactly j equal factor pairs, with order not counting. Note that we need not consider factor triples or higher multiplicities. Then, if $f_2^d(n)$ is the number of squarefree factors of n with all factors different, by Lemma 3.1,

$$(3.3) \quad f_2^d(n) + \sum_{j=1}^k \frac{e_j}{(2!)^j} = L \left(\left(\frac{(u)_2}{2!} \right)^k \right).$$

Let Q be the set of squarefree factors of \sqrt{n} , excluding 1, and for every non-empty subset $S \subset Q$, let N^S be the number of squarefree factorizations of n with at least each of the factors $s \in S$ repeated. Then by equation (3.1)

$$\begin{aligned} N_r &:= \sum_{S:|S|=r} N^S = \sum_{|S|=r} f_2 \left(\frac{n}{\prod_{s \in S} s^2} \right) \\ &= \sum_{|S|=r} a_{k-\phi(\prod_{s \in S} s)} \\ &= \sum_{1 \leq t \leq k} \left(\#S : |S| = r \text{ and } \phi \left(\prod_{s \in S} s \right) = t \right) a_{k-t} \\ &= \sum_{1 \leq t \leq k} \binom{k}{t} \left\{ \begin{matrix} t \\ r \end{matrix} \right\} a_{k-t}. \end{aligned}$$

Note that $f_2(n) = f_2^d(n) + e_1 + \dots + e_k$, so by equation (3.3)

$$(3.4) \quad f_2(n) = L\left(\left(\frac{(u)_2}{2!}\right)^k\right) + \sum_{j=1}^k \left(1 - \frac{1}{2^j}\right) e_j.$$

By the binomial theorem,

$$(3.5) \quad L\left(\left((u)_2\right)^k\right) = L\left(u^k(u-1)^k\right)$$

$$(3.6) \quad = L\left(u^k \left(\sum_{j=0}^k (-1)^{k-j} \binom{k}{j} u^j\right)\right)$$

$$(3.7) \quad = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} L(u^{j+k})$$

$$(3.8) \quad = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} B_{k+j}.$$

Now using the expression derived for N_r and equation (3.2), we get

$$(3.9) \quad e_j = \sum_{j \leq r \leq k} (-1)^{r-j} \binom{r}{j} \sum_{1 \leq t \leq k} \binom{k}{t} \left\{ \begin{matrix} t \\ r \end{matrix} \right\} a_{k-t},$$

so by equation (3.9),

$$\begin{aligned} \sum_{1 \leq j \leq k} \left(1 - \frac{1}{2^j}\right) e_j &= \sum_{1 \leq j, r, t \leq k} \left(1 - \frac{1}{2^j}\right) (-1)^{r-j} \binom{r}{j} \binom{k}{t} \left\{ \begin{matrix} t \\ r \end{matrix} \right\} a_{k-t} \\ &= \sum_{1 \leq t \leq k} a_{k-t} \binom{k}{t} \\ &\quad \sum_{1 \leq r \leq k} \left\{ \begin{matrix} t \\ r \end{matrix} \right\} \sum_{1 \leq j \leq k} \left(1 - \frac{1}{2^j}\right) (-1)^{r-j} \binom{r}{j} \\ &= - \sum_{1 \leq t \leq k} a_{k-t} \binom{k}{t} \sum_{1 \leq r \leq k} \frac{(-1)^r \left\{ \begin{matrix} t \\ r \end{matrix} \right\}}{2^r} \end{aligned}$$

$$= - \sum_{1 \leq t \leq k} a_{k-t} \binom{k}{t} B_t \left(-\frac{1}{2} \right).$$

Feeding this expression into equation (3.4) and using equation (3.5), we obtain the recurrence for a_k , and the derivation is complete. \square

We can extend this recurrence to count the number of squarefree factorizations of numbers of the form $n = (p_1 \cdots p_k)^2 p_{k+1} \cdots p_{k+l}$, simply replacing a_k by $a_{k,l} := f_2(n)$, leading to

Theorem 3.6. *For all $l \geq 0$, $a_{0,l} = B_l$, and for all $k \geq 1$, $l \geq 0$, we have*

$$a_{k,l} = \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} B_{k+l+j} - \sum_{j=1}^k a_{k-j,l} \binom{k}{j} B_j \left(-\frac{1}{2} \right).$$

It should be possible to extend this method to derive a recursive expression for the evaluation of $f_2((p_1 \cdots p_k)^a)$ for $a \geq 1$, $k \geq 1$. Instead, here we derive a simple lower bound. Since there are B_k partitions of $\{p_1, \dots, p_k\}$, and we may select a of these, with order not significant, to form squarefree factorizations of $n = (p_1 \cdots p_k)^a$, we get

Proposition 3.7. *Let the primes p_i , for $1 \leq i \leq k$, be distinct and $a \geq 1$. Then*

$$f_2((p_1 \cdots p_k)^a) \geq \binom{B_k + a - 1}{a}.$$

Note that there are squarefree factorizations of $n = (p_1 \cdots p_k)^a$ which are not included in these selections, e.g., $(6, 10, 15)$ when $n = (2 \times 3 \times 5)^2$. Indeed we get equality only when $a = 1$ or $k = 2$.

4. The average of $f_2(n)$. We can express the Dirichlet series of $f_2(n)$ as an Euler product

$$\varphi(s) := \sum_{n=1}^{\infty} \frac{f_2(n)}{n^s} = \prod_{\substack{1 < n \\ n \text{ squarefree}}} (1 - n^{-s})^{-1},$$

which converges at least for $\Re s > 2$. We can rewrite this series as

$$\sum_{n=1}^{\infty} \frac{f_2(n)}{n^s} = \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\zeta(ks)}{\zeta(2ks)} - 1 \right) \right) = \exp \left(\sum_{n=1}^{\infty} \frac{c_n}{n^s} \right).$$

where $c_1 = 0$ and, for all $n \geq 1$, $0 \leq c_n \leq 1$. Indeed,

$$\log \varphi(s) = \frac{0}{1} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{2} \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{3} \frac{1}{8^s} + \dots$$

Note that, for $n > 1$, $c_n = 0$ whenever n is not a power of a squarefree number, and $1/k$ if n is the k 'th power of a squarefree number. This Dirichlet series converges for $\Re s > 1$, has poles at $s = 1, 1/2, 1/3, \dots$ and at the points $s = 1/(2m) + i\gamma/(2m)$, $m \in \mathbf{N}$ for each zero of $\zeta(s)$, $1/2 + i\gamma$. It has an essential singularity at $s = 0$.

To obtain the average of $f_2(n)$, we will use the method of Oppenheim [14] developed for $f(n)$. Indeed, the method applies almost in its entirety, and we only give sufficient detail to show how the constants in the leading asymptotic expressions are different in this case.

We need the following standard result in the case $k = 1$.

Lemma 4.1. [1, Lemma 3, page 281]. *Let $c > 0$ and $u > 0$ be real numbers. Then*

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{u^{-s}}{s(s+1)\cdots(s+k)} ds = \begin{cases} \frac{1}{k!}(1-u)^k & 0 < u \leq 1, \\ 0 & u > 1. \end{cases}$$

We also need as a property the Bessel function $I_\alpha(z)$ [4, Chapter 10]. For $\alpha > 0$ and $\zeta \in \mathbf{C} \setminus (-\infty, 0]$, let

$$I_\alpha(z) := \sum_{m=1}^{\infty} \frac{(z/2)^{2m+\alpha}}{m!\Gamma(m+\alpha+1)}$$

be the usual definition of the modified Bessel function of the first kind. It is one of the two linearly independent power series solutions to the differential equation $x^2y'' + xy' - (x^2 + \alpha^2)y = 0$, is holomorphic in the given open subset of \mathbf{C} , goes to infinity with real positive z and, for $\alpha > 0$, $\lim_{x \rightarrow 0+} I_\alpha(x) = 0$, whereas $\lim_{x \rightarrow 0+} I_0(x) = 1$. If Γ is any contour enclosing the origin in a positive direction (anti-clockwise),

then we can represent this function as

$$I_\nu(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{e^{z(t+\frac{1}{t})/2}}{t^{\nu+1}} dt.$$

Asymptotic expressions for the function and its derivative are

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} (-1)^n \frac{a_n(\nu)}{z^n}, \quad \arg z < \pi/2,$$

$$I'_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} (-1)^n \frac{b_n(\nu)}{z^n}, \quad \arg z < \pi/2, \text{ where}$$

$$a_0(\nu) = 1,$$

$$a_n(\nu) = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2) \cdots (4\nu^2 - (2n - 1)^2)}{n!8^n}, \quad n > 0,$$

$$b_0(\nu) = 1, \quad b_1(\nu) = \frac{(4\nu^2 + 3)}{8},$$

$$b_n(\nu) = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2) \cdots (4\nu^2 - (2n - 3)^2)(4\nu^2 + 4n^2 - 1)}{n!8^n}, \quad n > 1.$$

The properties of $I_\alpha(z)$ which we need are summarized in the following lemma.

Lemma 4.2. *If x, c and α are real and positive, then*

$$(a) \quad I_\alpha(x) = \frac{x^\alpha}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{z+(x^2)/(4z)}}{z^{\alpha+1}} dz,$$

and, as $x \rightarrow \infty$,

$$(b) \quad I_\alpha(x) = \frac{e^x}{\sqrt{2\pi x}} \left(1 - \frac{4\alpha^2 - 1}{8x} + O\left(\frac{1}{x^2}\right) \right).$$

Theorem 4.3. *Let $a_n = f_2(n)$ and $A_2(n) = \sum_{1 \leq n \leq x} a_n$. Then, as $x \rightarrow \infty$, we get*

$$A_2(x) \sim c_1 \frac{6^{1/4} x e^{((2\sqrt{\log x})/(\sqrt{\zeta(2)}))}}{4\pi (\log x)^{3/4}}.$$

Proof.

(i) If $s > 1$, we have

$$(4.1) \quad \log \varphi(s) = \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\zeta(ms)}{\zeta(2ms)} - 1 \right).$$

This series is uniformly convergent on compact subsets of $s > 0$ which avoid the points $s = 1/m$ and $s = 1/(2m) + i\gamma/m$ where $1/2 + i\gamma$ are the complex zeros of $\zeta(s)$. Thus, $\varphi(s)$ is holomorphic on the right half plane away from these points.

(ii) For fixed $m \in \mathbf{N}$, on an open neighborhood of $s = 1/m$, there exists a holomorphic function $\varphi_m(s)$ such that we can write

$$\begin{aligned} \varphi(s) &= \exp \left(\frac{1}{\zeta(2)m(ms-1)} + \varphi_m(s) \right) \\ &= \exp \left(\frac{1}{\zeta(2)(s-1)} + \varphi_1(s) \right). \end{aligned}$$

The function $\mathcal{P}_1(s)$ is holomorphic on $|s-1| < 1/2$, so we have a convergent power series representation on this disk given by

$$\mathcal{P}_1(s) = \sum_{n=0}^{\infty} \frac{\mathcal{P}_1^{(n)}(1)}{n!} (s-1)^n.$$

Therefore, for some real coefficients $(\alpha_n : n \geq 0)$ we have

$$\mathcal{P}(s) = c_1 e^{1/(\zeta(2)(s-1))} (1 + \alpha_1(s-1) + \alpha_2(s-1)^2 + \dots),$$

where $c_1 = e^{\mathcal{P}_1(1)}$ and $\mathcal{P}_1(1) = 0.395895\dots$. To derive this last constant, let as $s \rightarrow 1+$,

$$\begin{aligned} \mathcal{P}_1(s) &= \Delta(s) + \Phi(s), \\ \Phi(s) &:= \sum_{m=2}^{\infty} \frac{1}{m} \left(\frac{\zeta(ms)}{\zeta(2ms)} - 1 \right), \text{ so} \\ \Phi(1) &= 0.35203769\dots, \\ \Delta(s) &:= \frac{\zeta(s)}{\zeta(2s)} - \frac{1}{\zeta(2)(s-1)} - 1, \end{aligned}$$

$$\begin{aligned} \zeta(s) &= \frac{1}{s-1} + \gamma + O(|s-1|), \\ \zeta(2s) &= \zeta(2) + 2\zeta'(\xi)(s-1), \quad 2 < \xi < 2s, \text{ so} \\ \Delta(s) &= \frac{\gamma}{\zeta(2)} - \frac{2\zeta'(\xi)}{\zeta(2)^2} - 1 + O(|s-1|), \text{ giving} \\ \mathcal{P}_1(1) &= \frac{\gamma}{\zeta(2)} - \frac{2\zeta'(2)}{\zeta(2)^2} - 1 + \Phi(1) = 0.395895\dots \end{aligned}$$

- (iii) Now we *claim* that the order of $\mathcal{P}(s)$ on $s = 1 + it$, for some ε with $0 < \varepsilon < 1$, satisfies $\mathcal{P}(1 + it) = O(|t|^\varepsilon)$ uniformly for $t \geq e$. Write

$$\begin{aligned} &\exp(\mathcal{P}(1 + it)) \\ &= \frac{\zeta(1 + it)}{\zeta(2 + 2it)} + \sum_{m=2}^{\infty} \frac{1}{m} \left(\frac{\zeta(m(1 + it))}{\zeta(2m(1 + it))} - 1 \right) - 1 \\ &\leq \frac{6}{\pi^2} \log t + C \end{aligned}$$

for $t \geq e$ and some absolute constant $C > 0$, where we have used Lindelöf’s estimate $\zeta(1 + it) \ll \log t$, which requires a simple application of the Euler-Maclaurin summation [12].

- (iv) By Lemma 4.1 with $u = 1/x$, we get

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)} ds = \begin{cases} x-1 & x \geq 1, \\ 0 & 0 < x \leq 1. \end{cases}$$

Therefore,

$$B(x) := \sum_{1 \leq n \leq x} a_n(x-n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)} \varphi(s) ds.$$

Now let $a > 0$ be a small real number. By Cauchy’s theorem, we can deform the contour of integration $(2 - i\infty, 2 + i\infty)$ to a new contour Γ which consists of the five components $C_1 = (1 - i\infty, 1 - it_0]$, $C_2 = [1 - it_0, 1 - ia]$, $C_3 = \{1 + ae^{i\theta} := \pi/2 \leq \theta \leq \pi/2\}$, $C_4 = [1 + ia, 1 + it_0]$ and $C_5 = [1 + it_0, 1 + i\infty)$ with corresponding integrals having values V_j with $1 \leq j \leq 5$.

Using the estimate of item (iii), we get $V_1 = O(x^2)$ and $V_5 = O(x^2)$.

Also,

$$|V_2| \leq \frac{c_1 x^2}{2\pi} \int_0^e \frac{1}{|1+it||2+it|} |e^{\varphi_1(1+it)}| dt \leq c_2 x^2$$

with the same bound holding for $|I_4|$.

For the integral V_3 , make the change of variables $w = 1/(s - 1)$ so the contour of integration is now $w = e^{i\theta}/a$ with θ going from $-\pi/2$ to $+\pi/2$. It follows that, for $b > 0$,

$$V_3 = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{x^{2+(1/w)}}{(w+1)(2w+1)} \varphi\left(1 + \frac{1}{w}\right) dw + O(x^2).$$

Therefore, using the change of variables $w = \zeta(2)u$ and Lemma 4.2,

$$\begin{aligned} I_3 &= c_1 \frac{x^2}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\exp((\log x/w) + (w/\zeta(2)))}{(w+1)(2w+1)} \\ &\quad \times \exp\left(\mathcal{P}_1\left(1 + \frac{1}{w}\right)\right) dw + O(x^2) \\ &= \frac{c_1 x^2}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\exp((\log x/w) + (w/\zeta(2)))}{(w+1)(2w+1)} \\ &\quad \times \left(1 + \frac{\alpha_1}{w} + \frac{\alpha_2}{w^2} + \dots\right) dw + O(x^2) \\ &= \frac{c_1 x^2}{4\pi i \zeta(2)} \int_{b-i\infty}^{b+i\infty} \exp\left(u + \frac{\log x/\zeta(2)}{u}\right) \frac{1}{u^2} \left(1 + \frac{\beta_1}{u} + \dots\right) du + O(x^2) \\ &= \frac{c_1 x^2}{2\zeta(2)} \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\exp(u + (\log x/\zeta(2))/u)}{u^{n+2}} du + O(x^2) \\ &= \frac{c_1 x^2}{2\zeta(2)} \sum_{n=0}^{\infty} \frac{\zeta(2)^{(n+1)/2} I_{n+1}(2\sqrt{\log x}/\sqrt{\zeta(2)})}{2^{n+1}(\log x)^{(n+1)/2}} + O(x^2). \end{aligned}$$

Using a similar analysis to Oppenheim [14],

$$V_3 \sim c_1 \frac{6^{1/4}}{8\pi} \frac{x^2 e^{(2\sqrt{\log x}/\sqrt{\zeta(2)})}}{(\log x)^{3/4}},$$

we get the asymptotic expansion for the derivative

$$A_2(x) \sim c_1 \frac{6^{1/4}}{4\pi} \frac{x e^{(2\sqrt{\log x}/\sqrt{\zeta(2)})}}{(\log x)^{3/4}}. \quad \square$$

5. Concluding remarks. (1) Corresponding to $f(n)$, there is a function, often called $m(n)$, the number of multiplicative compositions, or the number of factorizations of n with order counting. This function has been studied by many authors, starting with Kalmár. See [6, 7, 8, 9, 10]. So here we could consider $m_2(n)$, the number of squarefree factorizations of n with order counting. Here the Dirichlet series is simpler, and the average easier to derive than that for $f_2(n)$:

$$\sum_{n=1}^{\infty} \frac{m_2(n)}{n^s} = 1 + \sum_{j=1}^{\infty} \left(\frac{\zeta(s)}{\zeta(2s)} - 1 \right)^j = \frac{\zeta(2s)}{2\zeta(2s) - \zeta(s)}.$$

for $\Re s > \eta$, so

$$\sum_{n \leq x} m_2(n) \sim x^\eta \frac{\zeta(2\eta)}{\eta(4\zeta'(2\eta) - \zeta'(\eta))}$$

where η is the real root of $2\zeta(2s) = \zeta(s)$ with $\eta > 1$, $\eta = 1.57802\dots$

(2) There are a number of issues regarding $f_2(n)$ which must await further study. For example, the property of numbers which is analogous to the highly composite numbers of Ramanujan [15] or the “highly factorable” numbers of Cranfield, Pomerance and Erdős [3]. The highly squarefree factorable numbers are champions for the function $f_2(n)$. I conjecture, based on a small amount of numerical evidence, that there are an infinite number of these champions which have the form $n = p_1 \cdots p_m$, a squarefree number consisting of the product of an initial sequence of primes.

(3) Finally, corresponding to $f_2(n)$, there is the function $f_k(n)$, the number of factorizations of n into k -free integers with order not counting, so there exists a $k \leq \Omega(n)$ with $f(n) = f_k(n)$. It is expected that the formula for the average of $f_k(n)$ would have $\sqrt{\zeta(k)}$ in the denominator of the exponential part.

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