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WHEN THE UNIT, UNITARY AND TOTAL GRAPHS ARE RING GRAPHS AND OUTERPLANAR

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ABSTRACT. In this paper, we investigate when the unit, unitary and total graphs are ring graphs, and also we study the case that they are outerplanar.

1. Introduction. Let G be a graph with n vertices and q edges, and let C be a cycle of G. A chord is any edge of G joining two nonadjacent vertices in C. We say C is a primitive cycle if it has no chords. Also, we say that a graph G has the primitive cycle property (PCP) if any two primitive cycles intersect in at most one edge. Let frank$\,(G)$ be the number of primitive cycles of G. The number frank$\,(G)$ is called the free rank of G and the number rank$\,(G) = q - n + r$ is called the cycle rank of G, where r is the number of connected components of G. The cycle rank of G can be expressed as the dimension of the cycle space of G. These two numbers satisfy rank$\,(G) \leq$ frank$\,(G)$, as is seen in [**6,** Proposition 2.2]. In [**6**], the authors studied and classified the family of graphs where the equality occurs. This family is precisely the family of ring graphs. The precise definition of a ring graph can be found in [**6,** Section 2]. Roughly speaking, ring graphs can be obtained starting with a cycle and subsequently attaching paths of length at least two that meet graphs already constructed in two adjacent vertices. Also, they showed that, for the graph G, the following conditions are equivalent:

(i) G is a ring graph,
(ii) rank$\,(G) =$ frank$\,(G)$,
(iii) G satisfies in PCP and G does not contain a subdivision of K_4 as a subgraph.

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Clearly ring graphs are planar. In **[6]**, the authors stated that the blocks of graph G are important for calculating the numbers $\operatorname{frank}(G)$ and $\operatorname{rank}(G)$. In fact, they proved the following lemma.

Lemma 1.1. [**6,** Lemma 2.4]. *Let G be a graph, and let $G_1, \ldots, G_r$ be its blocks. Then* $\operatorname{rank}(G) = \operatorname{frank}(G)$ *if and only if* $\operatorname{rank}(G_i) = \operatorname{frank}(G_i)$, *for all* $1 \leq i \leq r$.

In **[6]**, the authors also showed that every outerplanar graph is a ring graph. In this paper, for a graph G associated to a finite commutative ring R, we study the situations under which G is a ring graph or an outerplanar graph. In the second and third sections we study this question for unit and unitary graphs, respectively. Finally, in the last section, we provide conditions that the total graphs are ring graphs or outerplanar.

Now, we review some background of graph theory from **[5]**. A vertex v is called a cut vertex if the number of connected components in $G\backslash\{v\}$ (a subgraph of G with removing the vertex v) is larger than that of G. A maximal connected subgraph of G without cut vertices is called a block. A graph G is 2-connected if $n > 2$, where n is the number of vertices, and G has no cut vertices. Thus, a block of G is either a maximal 2-connected subgraph, a bridge or an isolated vertex. By their maximality, different blocks of G intersect in at most one vertex, which is then a cut vertex of G. Therefore, every edge of G lies in a unique block and G is the union of its blocks. An undirected graph is an outerplanar graph if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization of outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of the complete graph K_4 or the complete bipartite graph $K_{2,3}$. Clearly, every outerplanar graph is planar. For a positive integer r, a graph G is called r-regular if the degrees of all vertices are equal, where the degree of a vertex v is the number of edges adjacent to v.

Throughout the paper, R is a finite commutative ring with non-zero identity, and we denote the set of all unit elements and zero-divisor elements of R, by $U(R)$ and $Z(R)$, respectively.

2. Ring graphs and outerplanar unit graphs. The unit graph of R, denoted by $G(R)$, is the graph obtained by setting all the elements of R to be the vertices and defining distinct vertices x and y to be adjacent if and only if $x+y \in U(R)$. By [**3,** Theorem 2.4], if $2 \notin U(R)$, then the unit graph $G(R)$ is a $|U(R)|$-regular graph. Otherwise, for every $x \in U(R)$, we have $\deg(x) = |U(R)| - 1$ and, for every $x \in R \setminus U(R)$, we have that $\deg(x) = |U(R)|$.

First, we want to characterize all rings R such that $G(R)$ is a ring graph. Since a ring graph is planar, it is sufficient to focus on the planar unit graphs. By [**3,** Theorem 5.14], we have that the unit graph $G(R)$ is planar if and only if R is isomorphic to one of the following rings:

(i) $R \cong \mathbf{Z}_5$,
(ii) $R \cong \mathbf{Z}_3 \times \mathbf{Z}_3$,
(iii) $R \cong \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell} \times S$, where $\ell \geq 0$ and $S \cong \mathbf{Z}_2$, $S \cong \mathbf{Z}_3$, $S \cong \mathbf{Z}_4$, $S \cong \mathbf{F}_4$, or $S \cong \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbf{Z}_2 \right\}$.

Theorem 2.1. *The unit graph $G(R)$ is a ring graph if and only if R is isomorphic to one of the following rings:*

(i) $R \cong \mathbf{Z}_5$,
(ii) $R \cong \underbrace{\mathbf{Z}_2 \times \ldots \times \mathbf{Z}_2}_{\ell} \times S$, *where* $\ell \geq 0$ *and* $S \cong \mathbf{Z}_2$, $S \cong \mathbf{Z}_3$, $S \cong \mathbf{Z}_4$, $S \cong \mathbf{F}_4$ *or* $S \cong \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbf{Z}_2 \right\}$.

Proof. First, we assume that $G(R)$ is a ring graph. Since every ring graph is planar, we have that $G(R)$ is planar. Thus we have the following cases:

Case 1. $R \cong \mathbf{Z}_5$. The unit graph $G(R)$ is pictured in Figure 1. One can easily see that $\operatorname{rank}(G(R)) = \operatorname{frank}(G(R)) = 4$, so $G(R)$ is a ring graph.

Case 2. $R \cong \mathbf{Z}_3 \times \mathbf{Z}_3$. In this case, by considering two cycles (1,0)-(1,2)-(0,0)-(1,1)-(1,0) and (1,0)-(1,2)-(0,2)-(2,2)-(2,0)-(2,1)-(0,1)-(1,1)-(1,0), we see that $G(R)$ doesn't satisfy PCP, so it is not a ring graph.

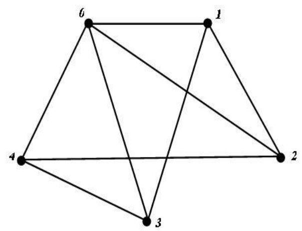

FIGURE 1

Case 3. $R \cong \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell} \times \mathbf{Z}_2$, where $\ell \geq 0$. If $\ell = 0$, then there is nothing to prove. So we may assume that $\ell > 0$. We have that $U(R) = \{(\underbrace{1, 1, \ldots, 1}_{\ell}, 1)\}$. Since $2 \notin U(R)$, $G(R)$ is a 1-regular graph, and so it is a perfect match. Thus $q = r$ and $n = 2q$. Therefore, $\operatorname{rank}(G(R)) = \operatorname{frank}(G(R)) = 0$, so in this situation, $G(R)$ is a ring graph.

Case 4. $R \cong \underbrace{\mathbf{Z}_2 \times \cdots \mathbf{Z}_2}_{\ell} \times \mathbf{Z}_3$, where $\ell \geq 0$. The case $\ell = 0$ is clear. Now, suppose that $\ell > 0$. Since $U(R) = \{(\underbrace{1, 1, \ldots, 1}_{\ell}, 1), (\underbrace{1, 1, \ldots, 1}_{\ell}, 2)\}$, $G(R)$ is a 2-regular graph. So $G(R)$ is the union of primitive cycles. In fact, every connected component of $G(R)$ is a primitive cycle. Therefore, we have that $\operatorname{frank}(G(R)) = r$. Since $q = n$, one can conclude that $\operatorname{rank}(G(R)) = r$. So $G(R)$ is again a ring graph.

Case 5. $R \cong \underbrace{\mathbf{Z}_2 \times \ldots \mathbf{Z}_2}_{\ell} \times \mathbf{Z}_4$. We know that

$$U(R) = \{(\underbrace{1, 1, \ldots, 1}_{\ell}, 1), (\underbrace{1, 1, \ldots, 1}_{\ell}, 3)\}.$$

Since $2 \notin U(R)$, $G(R)$ is a 2-regular graph. Now, similar to Case 4, it is easy to see that $G(R)$ is a ring graph.

Case 6. $R \cong \underbrace{\mathbf{Z}_2 \times \cdots \mathbf{Z}_2}_{\ell} \times \mathbf{F}_4$. Consider the field $\mathbf{F}_4 = \{0, f_1, f_2, f_3\}$.

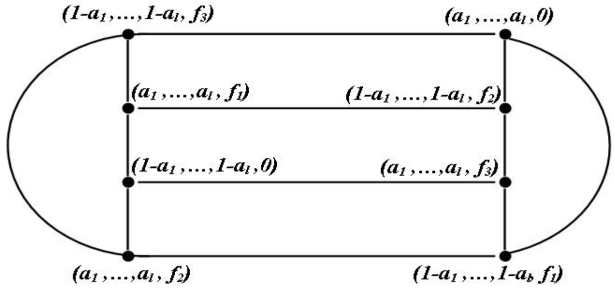

FIGURE 2

Since $U(R) = \{(\underbrace{1,1,\ldots,1}_{\ell}, f_1),\ (\underbrace{1,1,\ldots,1}_{\ell}, f_2), (\underbrace{1,1,\ldots,1}_{\ell}, f_3)\}$, $G(R)$ is a 3-regular graph. We also have $\mathrm{char}\,(\mathbf{F}_4) = 2$. Thus, the sum of each pair of distinct non-zero elements in $\mathbf{F}_4$ is non-zero. Therefore, the vertex $(a_1, a_2, \ldots, a_\ell, f_i)$ is adjacent to $(1-a_1, 1-a_2, \ldots, 1-a_\ell, f_j)$ for all $a_k \in \mathbf{Z}_2$ and $f_i, f_j \in \mathbf{F}_4$, where $1 \le k \le \ell$ and $1 \le i \ne j \le 3$. So every component of $G(R)$ has a form similar to that we show in the following figure.

Also it is easy to see that every component of this graph is a block. Now, by [**6,** Lemma 2.4], we must show that rank and free rank of every component of $G(R)$ are equal. In view of Figure 2, we have that $\mathrm{rank}\,(G_i) = \mathrm{frank}\,(G_i) = 5$ for every component G_i of $G(R)$. So in this situation, $G(R)$ is a ring graph.

Case 7. $R \cong \underbrace{\mathbf{Z}_2 \times \cdots \mathbf{Z}_2}_{\ell} \times \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbf{Z}_2 \right\}$. In this case, we have that

$$U(R) = \left\{ \left(\underbrace{1,1,\ldots,1}_{\ell}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right), \left(\underbrace{1,1,\ldots,1}_{\ell}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) \right\}.$$

Since $2 \notin U(R)$, $G(R)$ is a 2-regular graph. Similar to Case 3, $G(R)$ is a ring graph.

The converse statement follows easily. $\square$

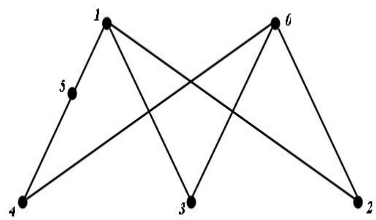

FIGURE 3

Theorem 2.2. *The unit graph $G(R)$ is outerplanar if and only if R is one of the following rings*:

$$R \cong \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell} \times S, \text{ where } \ell \geq 0 \text{ and}$$

$$S \cong \mathbf{Z}_2,\ S \cong \mathbf{Z}_3,\ S \cong \mathbf{Z}_4 \text{ or } S \cong \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbf{Z}_2 \right\}.$$

Proof. Suppose that $G(R)$ is outerplanar. Since an outerplanar graph is a ring graph, we consider the unit graphs which are ring graphs. By Theorem 2.1, we have the following cases.

Case 1. $R \cong \mathbf{Z}_5$. In view of Figure 3, $G(R)$ contains a subgraph isomorphic to $K_{2,3}$. Thus, it is not outerplanar.

Case 2. $R \cong \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell} \times \mathbf{Z}_2$, where $\ell \geq 0$. The case $\ell = 0$ is clear. If $\ell > 0$, then $G(R)$ is a perfect matching, so clearly it is outerplanar.

Case 3. $R \cong \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell} \times S$, where $\ell \geq 0$ and $S \cong \mathbf{Z}_3$, $S \cong \mathbf{Z}_4$ or $S \cong \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbf{Z}_2 \right\}$. In these cases, $G(R)$ is a union of primitive cycles, so $G(R)$ is an outerplanar graph.

Case 4. $R \cong \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell} \times \mathbf{F}_4$. In view of Figure 4, one can find a subdivision of $K_{2,3}$ in $G(R)$, and so $G(R)$ is not outerplanar.

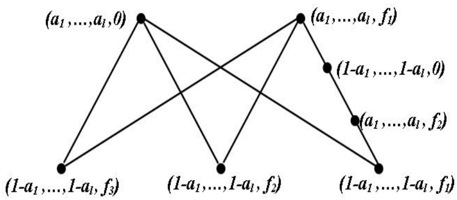

FIGURE 4

Conversely, if

$R \cong \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell} \times S$, where $\ell \geq 0$ and

$$S \cong \mathbf{Z}_2,\ S \cong \mathbf{Z}_3,\ S \cong \mathbb{Z}_4 \text{ or } S \cong \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbf{Z}_2 \right\},$$

then one can easily check that $G(R)$ is outerplanar. $\square$

3. Ring graphs and outerplanar unitary graphs. The unitary graph $G_R = \mathrm{Cay}\,(R, U(R))$ is defined to be the graph whose vertex-set is R, with an edge between x and y if $x - y \in U(R)$. It is easy to see that G_R is a $|U(R)|$-regular graph. In this section, we provide a characterization of all finite rings whose G_R are ring graphs and outerplanar. By [**1,** Theorem 8.2], G_R is planar if and only if R is one of the following rings:

$$\underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell \geq 0}, \quad \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell \geq 0} \times \mathbf{Z}_3, \quad \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell \geq 0} \times \mathbf{Z}_4$$

and

$$\underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell \geq 0} \times \mathbf{F}_4.$$

Theorem 3.1. *Let R be a finite ring. Then G_R is a ring graph if and only if G_R is a planar graph.*

Proof. Since a ring graph is planar, it is sufficient to show that every planar unitary graph is a ring graph. We have the following cases:

Case 1. $R \cong \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell \geq 0}$. The case $\ell = 0$ is clear. So we may assume that $\ell > 0$. Since $U(R) = \{(\underbrace{1, 1, \ldots, 1}_{\ell})\}$, G_R is a perfect matching. So we have that $\operatorname{frank}(G_R) = \operatorname{rank}(G_R) = 0$. Therefore, G_R is a ring graph.

Case 2. $R \cong \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell \geq 0} \times \mathbf{Z}_3$. We have that

$$U(R) = \{(\underbrace{1, 1, \ldots, 1}_{\ell}, 1), (\underbrace{1, 1, \ldots, 1}_{\ell}, 2)\}.$$

Hence, G_R is a 2-regular graph, and thus every connected component of G_R is a primitive cycle. Thus $\operatorname{frank}(G_R) = r$ and we have that $q = n$. Therefore, $\operatorname{frank}(G_R) = \operatorname{rank}(G_R) = r$, so G_R is a ring graph.

Case 3. $R \cong \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell \geq 0} \times \mathbf{Z}_4$. Similar to Case 2, one can see that G_R is a ring graph.

Case 4. $R \cong \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell \geq 0} \times \mathbf{F}_4$. Let $\mathbf{F}_4 = \{0, f_1, f_2, f_3\}$. Since $f_i - f_j \neq 0$ for all $1 \leq i \neq j \leq 3$, we have that $(a_1, a_2, \ldots, a_\ell, f_i)$ and $(1 - a_1, 1 - a_2, \ldots, 1 - a_\ell, f_j)$ are adjacent in G_R, where $a_k \in \mathbf{Z}_2$ and $1 \leq k \leq \ell$. So every connected component of the graph G_R has a similar form to that we show in the following figure.

It is easy to check that every component of this graph is a block of graph. In view of Figure 5, we have that $\operatorname{rank}(G_i) = \operatorname{frank}(G_i) = 5$. So in this case, G_R is a ring graph. $\square$

Theorem 3.2. *The unitary graph G_R is outerplanar if and only if R is isomorphic to one of the following rings:*

$$\underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell \geq 0}, \quad \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell \geq 0} \times \mathbf{Z}_3 \quad \text{and} \quad \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell \geq 0} \times \mathbf{Z}_4.$$

Proof. Since every outerplanar graph is a ring graph, we need to check the outerplanarity of G_R, whenever G_R is a ring graph. By **[1,**

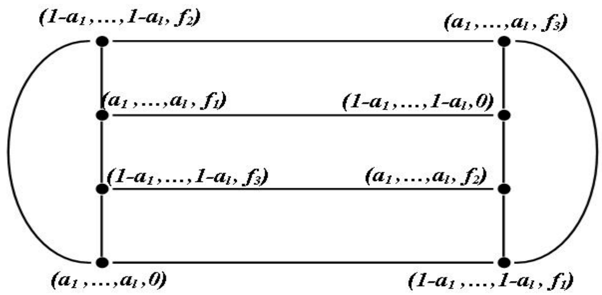

FIGURE 5

Theorem 8.2] in conjunction with Theorem 3.1, we may assume that R is isomorphic to one of the rings

$$\underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell\geq 0}, \quad \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell\geq 0} \times \mathbf{Z}_3, \quad \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell\geq 0} \times \mathbf{Z}_4 \quad \text{or}$$

$$\underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell\geq 0} \times \mathbf{F}_4.$$

If R is one of the rings

$$\underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell\geq 0}, \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell\geq 0} \times \mathbf{Z}_3 \quad \text{or} \quad \underbrace{\mathbf{Z}_2 \times \ldots \times \mathbf{Z}_2}_{\ell\geq 0} \times \mathbf{Z}_4,$$

then G_R is a 1-regular or a 2-regular graph. Thus G_R is an outerplanar graph.

If $R \cong \underbrace{\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2}_{\ell\geq 0} \times \mathbb{F}_4$, then we can find a subdivision of $K_{2,3}$ (see Figure 4). So G_R is not outerplanar.

The converse statement follows easily. □

4. Ring graphs and outerplanar total graphs. The total graph $T(\Gamma(R))$ is a graph with vertex-set R, and two distinct vertices a and b are adjacent if and only if $a + b \in Z(R)$. In this section, we investigate all finite commutative rings R such that their total graphs $T(\Gamma(R))$ are ring graphs and also outerplanar. If $T(\Gamma(R))$ is planar, then, by **[7,** Theorem 1.5], we have the following cases:

(i) If R is a local ring, then R is a field or isomorphic to one of the following rings:

$$\mathbf{Z}_4,\ \frac{\mathbf{Z}_2[X]}{(X^2)},\ \frac{\mathbf{Z}_2[X]}{(X^3)},\ \frac{\mathbf{Z}_2[X,Y]}{(X,Y)^2},\ \frac{\mathbf{Z}_4[X]}{(2X,X^2)},$$
$$\frac{\mathbf{Z}_4[X]}{(2X,X^2-2)},\ \mathbf{Z}_8,\ \frac{\mathbf{F}_4[X]}{(X^2)},\ \frac{\mathbf{Z}_4[X]}{(X^2+X+1)}.$$

(ii) If R is non-local, then R is isomorphic to the ring $\mathbf{Z}_2 \times \mathbf{Z}_2$ or $\mathbf{Z}_6$.

Theorem 4.1. *The total graph $T(\Gamma(R))$ is a ring graph if and only if one of the following statements hold*:

(i) *If R is a local ring, then R is a field or isomorphic to one of the rings $\mathbf{Z}_4$ or $\mathbf{Z}_2[X]/(X^2)$.*

(ii) *If R is non-local, then R is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$.*

Proof. First suppose that $T(\Gamma(R))$ is a ring graph. Therefore, it is planar. Now we have the following cases:

Case 1. R is local. Then, by **[7,** Theorem 1.5(a)], R is a field or it is isomorphic to one of the following rings:

$$\mathbf{Z}_4,\ \frac{\mathbf{Z}_2[X]}{(X^2)},\ \frac{\mathbf{Z}_2[X]}{(X^3)},\ \frac{\mathbf{Z}_2[X,Y]}{(X,Y)^2},\ \frac{\mathbf{Z}_4[X]}{(2X,X^2)},$$
$$\frac{\mathbf{Z}_4[X]}{(2X,X^2-2)},\ \mathbf{Z}_8,\ \frac{\mathbf{F}_4[X]}{(X^2)},\ \frac{\mathbf{Z}_4[X]}{(X^2+X+1)}.$$

Since R is finite, every non-unit element is a zero-divisor. Moreover, $Z(R)$ is the maximal ideal of R. Now, if $2 \notin Z(R)$, then since $T(\Gamma(R))$ is planar, by **[2,** Theorem 2.2 (2)], we have that $|Z(R)| \leqslant 2$. Since $|R| \leq |Z(R)|^2$, R is either a ring of order 4 or is a field, so $T(\Gamma(R))$ is a ring graph.

Suppose that $2 \in Z(R)$. If $|Z(R)| = 1$, then R is a field. Hence $T(\Gamma(R))$ is totally disconnected, so it is a ring graph. Otherwise $|Z(R)| > 1$. Since $T(\Gamma(R))$ is planar, by **[2,** Theorem 2.2 (1)], we have that $|Z(R)| \leqslant 4$. Also, since $2 \in Z(R)$, $|R| = 2^k$. Thus, $|R| = 2, 4, 8$ or 16. According to **[4]**, there are two non-isomorphic rings of order 16 with maximal ideals of order 4, namely, $\mathbf{F}_4[X]/(X^2)$ and $\mathbf{Z}_4[X]/(X^2+X+1)$. In these rings we have $T(\Gamma(R)) \cong 4K_4$ and every block of $T(\Gamma(R))$ is K_4. Since $\operatorname{rank}(K_4) \neq \operatorname{frank}(K_4)$, we have that the numbers rank and frank of these graphs are not equal, which means that they are not ring graphs.

In [**4**] it is also shown that the local rings of order 8 with $|Z(R)| = 4$ are isomorphic to one of the following rings

$$\frac{\mathbf{Z}_2[X]}{(X^3)}, \ \frac{\mathbf{Z}_2[X,Y]}{(X,Y)^2}, \ \frac{\mathbf{Z}_4[X]}{(2X,X^2)}, \frac{\mathbf{Z}_4[X]}{(2X,X^2-2)}, \ \mathbf{Z}_8.$$

In these rings we have that $T(\Gamma(R)) \cong 2K_4$, and so their ranks and free ranks are not equal. Hence, they are not ring graphs.

In the remaining two rings $\mathbf{Z}_4$ and $\mathbf{Z}_2[X]/(X^2)$, we have $T(\Gamma(R)) \cong 2K_2$, and therefore they are ring graphs.

Case 2. R is non-local. Then, since $T(\Gamma(R))$ is planar and R is finite, by [**7,** Theorem 1.5(b)], we have that R is isomorphic to the ring $\mathbf{Z}_2 \times \mathbf{Z}_2$ or $\mathbf{Z}_6$. If $R \cong \mathbf{Z}_6$, then $T(\Gamma(R))$ contains a subdivision of K_4, so it is not a ring graph. In the case that $R \cong \mathbf{Z}_2 \times \mathbf{Z}_2$, we have $T(\Gamma(R)) \cong C_4$, which is a ring graph.

Conversely, it is easy to see that if one of the parts (i) or (ii) holds, then $T(\Gamma(R))$ is a ring graph. $\square$

Theorem 4.2. $T(\Gamma(R))$ *is outerplanar if and only if* R *is a field or isomorphic to one of the rings* $\mathbf{Z}_4$, $\mathbf{Z}_2[X]/(X^2)$ *or* $\mathbf{Z}_2 \times \mathbf{Z}_2$.

Proof. Assume that $T(\Gamma(R))$ is outerplanar. Since outerplanar graphs are ring graphs, by Theorem 4.1, R is isomorphic to one of the rings $\mathbf{Z}_4$, $\mathbf{Z}_2[X]/(X^2)$, $\mathbf{Z}_2 \times \mathbf{Z}_2$ or R is a field. For two rings $\mathbf{Z}_4$ and $\mathbf{Z}_2[X]/(X^2)$, we have $T(\Gamma(R)) \cong 2K_2$, and therefore they are outerplanar. Also, $T(\Gamma(\mathbf{Z}_2 \times \mathbf{Z}_2)) \cong C_4$ which is outerplanar. If R is a field, then $T(\Gamma(R))$ is the union of a totally disconnected graph and a perfect matching, which is outerplanar.

The converse statement follows easily. $\square$

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