ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 44, Number 2, 2014

SERIES REPRESENTATIONS FOR THE STIELTJES CONSTANTS

MARK W. COFFEY

ABSTRACT. The Stieltjes constants $\gamma_k(a)$ appear as the coefficients in the regular part of the Laurent expansion of the Hurwitz zeta function $\zeta(s,a)$ about $s=1$. We present series representations of these constants of interest to theoretical and computational analytic number theory. A particular result gives an addition formula for the Stieltjes constants. As a byproduct, expressions for derivatives of all orders of the Stieltjes coefficients are given. Many other results are obtained, including instances of an exponentially fast converging series representation for $\gamma_k = \gamma_k(1)$. Some extensions are briefly described, as well as the relevance to expansions of Dirichlet L functions.

1. Introduction and statement of results. The Stieltjes (or generalized Euler) constants $\gamma_k(a)$ appear as expansion coefficients in the Laurent series for the Hurwitz zeta function $\zeta(s,a)$ about its simple pole at $s=1$ [**5, 13, 20, 25, 30**]. These constants are important in analytic number theory and elsewhere, where they appear in various estimations and as a result of asymptotic analyses. They are also of much use in developing a binomial sum $S_\gamma(n)$ introduced by the author in the study of a critical subsum in application of the Li criterion for the Riemann hypothesis [**9, 27**]. The constants $\gamma_k(1)$ are important in relation to the derivatives of the Riemann ξ function $\xi(s)=\pi^{-s/2}\Gamma(s/2+1)(s-1)\zeta(s)$ at $s=1$, where Γ is the Γ function and $\zeta(s)$ is the Riemann ζ function [**16, 21, 22, 31**], hence, some of the relevance to the Li criterion.

Despite the fact that many relations are known for the Stieltjes constants, there are many open questions, including those concerned with

2010 AMS *Mathematics subject classification.* Primary 11M35, 11M06, 11Y60, Secondary 05A10.

Keywords and phrases. Stieltjes constants, Riemann zeta function, Hurwitz zeta function, Laurent expansion, Stirling numbers of the first kind, Dirichlet L functions, Lerch zeta function.

Received by the editors on March 2, 2010, and in revised form on December 28, 2011.

DOI:10.1216/RMJ-2014-44-2-443

their arithmetic nature. Even good estimations of their magnitudes is still lacking. The evaluation of $\gamma_1(a)$ and $\gamma_2(a)$ for rational arguments has been given very recently **[6]**. In addition, formulas for obtaining the Stieltjes constants to arbitrary precision remain of interest.

On the subject of the magnitudes $|\gamma_k(a)|$, known estimates **[4, 39]** are highly conservative, and there is much room for improvement. We present one approach for such estimation. Our result is probably less important by itself than for the possibilities for extension and improvement that it suggests.

In this paper, we present various series representations of the Stieltjes coefficients. Most of these are fast converging, so have application to high precision computation. In particular, we make use of the Stirling numbers of the first kind $s(k,j)$ and their properties. Therefore, our treatment reflects a fusion of analytic number theory and enumerative combinatorics. Moreover, Proposition 9 gives instances of an exponentially fast converging series representation for γ_k. Such very fast converging series are relatively rare, and this may be the first result of its kind for the Stieltjes constants.

The Hurwitz zeta function, initially defined by $\zeta(s,a) = \sum_{n=0}^\infty (n+a)^{-s}$ for $\operatorname{Re} s > 1$, has an analytic continuation to the whole complex plane **[4, 22, 35]**. In the case of $a=1$, $\zeta(s,a)$ reduces to the Riemann zeta function $\zeta(s)$. In this instance, by convention, the Stieltjes constants $\gamma_k(1)$ are simply denoted γ_k **[5, 20, 25, 28, 30, 39]**. We recall that $\gamma_0(a) = -\psi(a)$, where $\psi = \Gamma'/\Gamma$ is the digamma function. We also recall that $\gamma_k(a+1) = \gamma_k(a) - (\ln^k a)/a$, and more generally that for $n \geq 1$ an integer

$$\gamma_k(a+n) = \gamma_k(a) - \sum_{j=0}^{n-1} \frac{\ln^k(a+j)}{a+j}, \tag{1.1}$$

as follows from the functional equation $\zeta(s,a+n) = \zeta(s,a) - \sum_{j=0}^{n-1}(a+j)^{-s}$.

Unless specified otherwise below, letters j, k, ℓ, m and n denote nonnegative integers. We point out the common alternative notations $S_n^{(m)}$ and $\left[{n \atop m}\right]$ for Stirling numbers of the first kind $s(n,m)$ **[1, 15, 19]**, with the relation $\left[{n \atop m}\right] = (-1)^{n+m} s(n,m)$ holding. We also make use of $P_1(t) \equiv B_1(t-[t]) = t - [t] - 1/2$, the first periodized Bernoulli polynomial (e.g., **[21, 35]**). Obviously, we have $|P_1(t)| \leq 1/2$. A glossary of notation is included at the end of the paper.

Proposition 1. (Addition formula for the Stieltjes constants). *Let* Re $a > 0$ *and* $|a| > |b|$, *and as usual let* $'$ *denote differentiation with respect to the argument of a function, with* $^{(j)}$ *denoting* j*-fold differentiation. Then*:

(i)

$$\gamma_\ell(a+b) = \gamma_\ell(a) + (-1)^\ell \sum_{j=2}^{\infty} \frac{b^{j-1}}{(j-1)!} \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} \times s(j,k+1)k!\zeta^{(\ell-k)}(j,a). \tag{1.2}$$

(ii) *We have*

$$\gamma_\ell'(a) = (-1)^{\ell+1}[\zeta^{(\ell)}(2,a) + \ell\zeta^{(\ell-1)}(2,a)], \tag{1.3}$$

$$\gamma_\ell''(a) = (-1)^\ell[2\zeta^{(\ell)}(3,a) + 3\ell\zeta^{(\ell-1)}(3,a) + \ell(\ell-1)\zeta^{(\ell-2)}(3,a)], \tag{1.4}$$

and

$$\gamma_\ell'''(a) = (-1)^{\ell+1}[6\zeta^{(\ell)}(4,a) + 11\ell\zeta^{(\ell-1)}(4,a) + 6\ell(\ell-1)\zeta^{(\ell-2)}(4,a) + \ell(\ell-1)(\ell-2)\zeta^{(\ell-3)}(4,a)]. \tag{1.5}$$

Corollary 1.

$$\gamma_1'(1) = \zeta(2)[\gamma + \ln(2\pi) + 12\zeta'(-1)], \tag{1.6}$$

where $\zeta(2) = \pi^2/6$ *and* $\gamma = -\psi(1)$ *is Euler's constant, and*

$$\gamma_1'\left(\frac{1}{2}\right) = 3\zeta(2)[\gamma + \ln(2\pi) + 12\zeta'(-1)] + 4\zeta(2)\ln 2. \tag{1.7}$$

Corollary 2. *For $j \geq 1$, we have*

$$\gamma_\ell^{(j)}(a) = (-1)^\ell \sum_{k=0}^{\ell} (-1)^k k! \binom{\ell}{k} s(j+1,k+1)\zeta^{(\ell-k)}(j+1,a). \tag{1.8}$$

Proposition 2.

(i) *For* $\operatorname{Re} a > 1$,

$$\begin{aligned}\gamma_n(a) = &-\frac{\ln^{n+1}(a-1)}{n+1} - n! \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \sum_{m=0}^{n} \frac{(-1)^m}{m!} \\ &\times s(k+1,n-m+1)\zeta^{(m)}(k+1,a),\end{aligned} \tag{1.9}$$

(ii) *For* $\operatorname{Re} a > 1/2$,

$$\begin{aligned}\gamma_n(a) = &-\frac{\ln^{n+1}(a-1/2)}{n+1} - n! \sum_{k=1}^{\infty} \frac{1}{4^k(2k+1)!} \sum_{m=0}^{n} \frac{(-1)^m}{m!} \\ &\times s(2k+1,n-m+1)\zeta^{(m)}(2k+1,a),\end{aligned} \tag{1.10}$$

(iii) *For* $\operatorname{Re} a > 1/2$,

$$\begin{aligned}\gamma_n(a) = &-\frac{\ln^{n+1}(a-1/2)}{n+1} \\ &+ n! \sum_{k=1}^{\infty} \frac{(-1)^k}{4^k(2k+1)!} \sum_{m=0}^{n} \frac{(-1)^m}{m!} s(2k+1,n-m+1)\zeta^{(m)} \\ &\times (2k+1,a) - 2n! \sum_{k=1}^{\infty} \frac{1}{16^k(2k+1)!} \\ &+ \sum_{m=0}^{n} \frac{(-1)^m}{m!} s(4k+1,n-m+1)\zeta^{(m)}(4k+1,a).\end{aligned} \tag{1.11}$$

Proposition 3. (Asymptotic relation). *Let B_j denote the Bernoulli numbers. For* $\operatorname{Re} a > 0$ *and* $a \to \infty$ *we have*

$$\gamma_\ell(a) \sim -\frac{\ln^{\ell+1} a}{\ell+1} + \frac{1}{2a}\ln^\ell a$$

$$- \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} a^{-2m} \sum_{k=0}^{\ell} \binom{\ell}{k} k! s(2m, k+1) \ln^{\ell-k} a. \tag{1.12}$$

Proposition 4. *For any integer $N \geq 0$ and* $\operatorname{Re} a > 0$, *we have*

$$\gamma_\ell(a) = \sum_{n=0}^{N} \frac{\ln^\ell(n+a)}{n+a} - \frac{\ln^{\ell+1}(N+a)}{\ell+1}$$

$$+ \sum_{r=2}^{\infty} \frac{(-1)^r}{r!} \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} k! s(r, k+1) \Big[(-1)^\ell \zeta^{(\ell-k)}(r, a) \tag{1.13}$$

$$- (-1)^k \sum_{n=0}^{N} \frac{\ln^{\ell-k}(n+a)}{(n+a)^r} \Big].$$

Proposition 5. *Let $a > 0$. Then there exists $a_1^* > 1$ such that $\gamma_1(a)$ is monotonically increasing for $a < a_1^*$ and $\gamma_1(a)$ is monotonically decreasing for $a > a_1^*$. The approximate numerical value of a_1^* is $a_1^* \simeq 1.39112$.*

Proposition 6. *Let* $\operatorname{Re} z > 0$. *Then we have*

$$\frac{1}{j!} \sum_{k=j}^{\infty} \frac{z^k}{(k-j)!} \int_0^1 \gamma_k(a)\, da = \frac{(-1)^j}{z}. \tag{1.14}$$

Proposition 7. *Put as in* **[28, 39]**, $C_n(a) \equiv \gamma_n(a) - (1/a)\ln^n a$ *for* $0 < a \leq 1$. *Then we have*

$$|C_n(a)| \leq \frac{en!}{\sqrt{n}2^n}, \quad n \geq 1. \tag{1.15}$$

Our methods extend to many other analytic functions. Additionally, although we present explicit results for expansions at $s = 1$, this is not a restriction. For those functions possessing functional equations, we also gain expansions typically at $s = 0$. More generally, expansions about arbitrary points in the complex plane are usually possible.

Another instance for which our methods apply is for the Lerch zeta function $\Phi(z, s, a)$. As an example, we consider the Lipschitz-Lerch transcendent

$$L(x, s, a) = \sum_{n=0}^{\infty} \frac{e^{2\pi i n x}}{(n + a)^s} = \Phi(e^{2\pi i x}, s, a), \tag{1.16}$$

for complex a different from a negative integer. We take in (1.16) x real and nonintegral, so that convergence obtains for $\operatorname{Re} s > 0$. Else, for x an integer in (1.16), we reduce to the Hurwitz zeta function. The functions Φ and L possess integral representations and functional equations.

A case of particular interest for (1.16) is when $x = 1/2$. Then we obtain the alternating Hurwitz zeta function,

$$L\left(\frac{1}{2}, s, a\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + a)^s} = 2^{-s}\left[\zeta\left(s, \frac{a}{2}\right) - \zeta\left(s, \frac{a+1}{2}\right)\right]. \tag{1.17}$$

Therefore, expansion at $s = 1$ yields expressions for *differences* of Stieltjes constants $\gamma_k(a/2) - \gamma_k[(a + 1)/2]$. More generally, for $x \notin \mathbf{Z}$, L is nonsingular at $s = 1$, and we may write

$$L(x, s, a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \ell_n(x, a)(s - 1)^n. \tag{1.18}$$

We have

Proposition 8. *Let $a > 0$ and $|\xi| < a$. Then we have the addition formula*

$$\begin{aligned}\ell_n(x, a + \xi) = \ell_n(x, a) &\\ + \sum_{k=2}^{\infty} \frac{\xi^{k-1}}{(k-1)!} \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} &(n-j)!s(k, n-j+1)\left(\frac{d}{ds}\right)^j \\ L(x, s + k - 1, a)\big|_{s=1} . &\end{aligned} \tag{1.19}$$

As a result, we obtain the following corollary for derivatives with respect to a of ℓ_n.

Corollary 3.

$$\ell_n^{(k)}(x,a) = \sum_{j=0}^{\infty}(-1)^j \binom{n}{j}(n-j)!s(k+1,n-j+1)\left(\frac{d}{ds}\right)^j \times L(x,s+k,a)\big|_{s=1}. \tag{1.20}$$

Let $\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}dt$ be the incomplete Gamma function, ${}_pF_q$ the generalized hypergeometric function, $\mathrm{Ei}\,(z) = -\int_{-z}^\infty e^{-t}(dt/t)$ the exponential integral, and erf the error function **[1, 18]**. One may seek a summation form for γ_k with very fast convergence. As a foretaste of a family of other results, we offer the following.

Proposition 9. *We have*: (i)

$$\frac{\gamma}{2} = \sum_{n=1}^{\infty}\frac{1}{n}[1-\mathrm{erf}\,(\sqrt{\pi}n)] - \sum_{n=1}^{\infty}\mathrm{Ei}\,(-\pi n^2) - 1 + \frac{1}{2}\ln(4\pi), \tag{1.21}$$

and

(ii)

$$\begin{aligned}\gamma_1 &= \frac{\pi^2}{16} + \frac{\gamma}{2}\psi\left(\frac{1}{2}\right) + \frac{1}{8}\psi^2\left(\frac{1}{2}\right) - 1 + \frac{1}{2}\ln\pi - \frac{1}{8}\ln^2\pi \\ &\quad - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left[4n\,{}_2F_2\left(\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2};-n^2\pi\right)\right. \\ &\qquad\qquad \left. + \psi\left(\frac{1}{2}\right) - 2\ln n - \mathrm{erf}\,(n\sqrt{\pi})\ln\pi\right] \\ &\quad + \sum_{n=1}^{\infty}\left\{\frac{\gamma^2}{4} + \frac{\pi^2}{24} - \frac{1}{2}n^2\pi\,{}_3F_3(1,1,1;2,2,2;-n^2\pi)\right. \\ &\qquad\qquad \left. + \frac{1}{4}\ln(n^2\pi)[2\gamma+\ln(\pi n^2)] + \frac{1}{2}\ln\pi\ \mathrm{Ei}\,(-n^2\pi)\right\},\end{aligned} \tag{1.22}$$

where $\psi(1/2) = -\gamma - 2\ln 2$.

We introduce the polylogarithm function, initially defined by $\mathrm{Li}_s(z) = \sum_{k=1}^\infty z^k/k^s$ for $|z| \leq 1$ and $\mathrm{Re}\, s > 1$, and analytically continued thereafter. We illustrate a method that more generally leads to expressions for the sums $\gamma_k(a) + \gamma_k(1-a)$. We have:

Proposition 10. *Let* $0 < a < 1$. *We have*: (i)

$$-\ln\pi + \psi\left(\frac{1}{2}\right) - \pi\cot\pi a - 2\psi(a) = \gamma + \ln\pi + 2\left.\frac{\partial}{\partial s}\right|_{s=0}[\mathrm{Li}_s(e^{2\pi i a}) + \mathrm{Li}_s(e^{-2\pi i a})], \tag{1.23}$$

and

(ii)

$$\begin{aligned}\gamma_1(a) + \gamma_1(1-a) &= \frac{\pi^2}{12} + \frac{\ln^2\pi}{4} - \frac{\ln\pi}{2}\psi\left(\frac{1}{2}\right) + \frac{1}{4}\psi^2\left(\frac{1}{2}\right) \\ &\quad + \frac{1}{2}\left[\ln\pi - \psi\left(\frac{1}{2}\right)\right][\psi(a) + \psi(1-a)] - \frac{1}{4}(\gamma + \ln\pi)^2 \\ &\quad - (\gamma + \ln\pi)\left.\frac{\partial}{\partial s}\right|_{s=0}[\mathrm{Li}_s(e^{2\pi i a}) + \mathrm{Li}_s(e^{-2\pi i a})] \\ &\quad + \left.\frac{\partial^2}{\partial s^2}\right|_{s=0}[\mathrm{Li}_s(e^{2\pi i a}) + \mathrm{Li}_s(e^{-2\pi i a})].\end{aligned} \tag{1.24}$$

Lastly, we present expressions with Stirling numbers for rapidly converging approximations to $\gamma = \gamma_0$. Historically this subject [**32**] has been important, especially due to Appell's use of them in his attempted proof of the irrationality of Euler's constant [**3**]. In addition, we present an exact representation for the difference between the Stieltjes constants and a finite sum. Near the end of the paper, we further discuss these subjects. We have

Proposition 11. *Define polynomials for* $n \geq 1$

$$P_{n+1}(y) \equiv \frac{1}{n!}\int_0^y x(1-x)(2-x)\cdots(n-1-x)\,dx, \tag{1.25}$$

and their values $p_{n+1} \equiv P_{n+1}(1)$. *Put* $r_n^{(k)} = \gamma_k - D_n^{(k)}$, *where*

$$D_n^{(k)} \equiv \sum_{m=1}^{n} \frac{\ln^k m}{m} - \frac{1}{k+1} \ln^{k+1}(n+1). \tag{1.26}$$

Then we have:

(i)

$$P_{n+1}(y) = \frac{(-1)^n}{n!} \sum_{k=1}^{n} (-1)^k \frac{s(n-1,k-1)}{k(k+1)} \left[((k-1)y + y + 1)(1-y)^{k-1}(y-1) + 1\right], \tag{1.27}$$

(ii)

$$\begin{aligned} P_{n+1}(y) &= \frac{(-1)^{n+1}}{n!} \sum_{k=0}^{n} \frac{s(n,k)}{k+1} y^{k+1} \\ &= \frac{(-1)^{n+1}}{n!} \left[\sum_{k=1}^{n-1} \frac{s(n,k)}{k+1} y^{k+1} + \delta_{n0} + \frac{1}{n+1} \right], \end{aligned} \tag{1.28}$$

where δ_{jk} *is the Kronecker symbol,*

(iii) *the special case*

$$p_{n+1} = \frac{(-1)^n}{n!} \sum_{k=1}^{n} (-1)^k \frac{s(n-1,k-1)}{k(k+1)} = \frac{(-1)^{n+1}}{n!} \sum_{k=1}^{n} \frac{s(n,k)}{k+1}, \tag{1.29}$$

and

(iv)

$$r_n^{(k)} = \sum_{m=n+1}^{\infty} I_{km}, \tag{1.30}$$

where

$$I_{km} = \sum_{n=1}^{\infty} \frac{1}{(n+1)m^{n+1}} \left[(-1)^n \ln^k m + \frac{1}{n!} \sum_{j=0}^{k-1} \frac{k!}{(k-j-1)!} s(n+1, j+2) \ln^{k-j-1} m \right]. \tag{1.31}$$

2. Proof of propositions.

Proof of Proposition 1. We let $(z)_k = \Gamma(z+k)/\Gamma(z)$ be the Pochhammer symbol. We make use of the following.

Lemma 1. *We have*

$$\left(\frac{d}{ds}\right)^{\ell}(s)_j\bigg|_{s=1} = (-1)^{j+\ell}\ell! s(j+1,\ell+1). \tag{2.1}$$

We have the standard expansion

$$(s)_j = s(s+1)\cdots(s+j-1) = \sum_{k=0}^{j}(-1)^{j+k}s(j,k)s^k, \tag{2.2}$$

giving

$$\left(\frac{d}{ds}\right)^{\ell}(s)_j = \sum_{k=\ell}^{j}(-1)^{j+k}s(j,k)k(k-1)\cdots(k-\ell+1)s^k, \tag{2.3}$$

so that

$$\begin{aligned}\left(\frac{d}{ds}\right)^{\ell}(s)_j\bigg|_{s=1} &= (-1)^j\sum_{k=\ell}^{j}(-1)^k s(j,k)(-1)^{\ell+1}k(1-k)_{\ell-1}\\ &= (-1)^j(\ell-1)!\sum_{k=\ell}^{j}(-1)^k s(j,k)k\binom{k-1}{\ell-1}\\ &= (-1)^j\ell!\sum_{k=\ell}^{j}(-1)^k s(j,k)\binom{k}{\ell}.\end{aligned} \tag{2.4}$$

It is known that [**19,** page 265]

$$\sum_{k=\ell}^{j}(-1)^k s(j,k)\binom{k}{\ell} = (-1)^{\ell}s(j+1,\ell+1). \tag{2.5}$$

Therefore, Lemma 1 follows.

In order to obtain Proposition 1, we apply a formula of Wilton **[38]** for the Hurwitz zeta function,

$$\zeta(s,a+b)=\zeta(s,a)+\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}\frac{\Gamma(s+j)}{\Gamma(s)}\zeta(s+j,a)b^j,$$

$$s\neq 1,\quad |b|<|a|,\ \mathrm{Re}\,a>0, \tag{2.6}$$

along with the product rule

$$\left(\frac{d}{ds}\right)^{\ell}(s)_j\zeta(s+j,a)\Bigg|_{s=1}=\sum_{k=0}^{\ell}\binom{\ell}{k}\left[\left(\frac{d}{ds}\right)^{k}(s)_j\right]\zeta^{(\ell-k)}(s+j,a)\Bigg|_{s=1}. \tag{2.7}$$

The defining Laurent expansion for the Stieltjes constants is

$$\zeta(s,a)=\frac{1}{s-1}+\sum_{k=0}^{\infty}\frac{(-1)^k\gamma_k(a)}{k!}(s-1)^k,\quad s\neq 1. \tag{2.8}$$

Therefore, the Wilton formula (2.6) gives

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\gamma_k(a+b)(s-1)^k=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\gamma_k(a)(s-1)^k$$
$$+\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}(s)_j\zeta(s+j,a)b^j,$$

$$|b|<|a|,\quad \mathrm{Re}\,a>0. \tag{2.9}$$

We take ℓ derivatives of this equation and use (2.7) and Lemma 1, yielding

$$(-1)^{\ell}\gamma_\ell(a+b)=(-1)^{\ell}\gamma_\ell(a)+\sum_{j=1}^{\infty}\frac{b^j}{j!}\sum_{k=0}^{\ell}\binom{\ell}{k}$$

$$(-1)^k s(j+1,k+1)k!\zeta^{(\ell-k)}(j+1,a). \tag{2.10}$$

Therefore, Proposition 1 (i) follows.

For part (ii), we form the limit difference quotient

$$\gamma_\ell'(a)=\lim_{b\to 0}\frac{1}{b}[\gamma_\ell(a+b)-\gamma_\ell(a)], \tag{2.11}$$

giving

$$\gamma_\ell'(a) = (-1)^\ell \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} s(2,k+1) k! \zeta^{(\ell-k)}(2,a). \tag{2.12}$$

The Stirling number truncates the summation, with $s(2,k+1) = (-1)^{k+1}$, $k = 0, 1$, and otherwise is 0 for $k \geq 2$.

We next have

$$\begin{aligned}\gamma_\ell''(a) &= \lim_{b\to 0} \frac{1}{b^2}[\gamma_\ell(a+2b) - 2\gamma_\ell(a+b) + \gamma_\ell(a)] \\ &= \lim_{b\to 0} \frac{1}{b^2}\{[\gamma_\ell(a+2b) - \gamma_\ell(a)] - 2[\gamma_\ell(a+b) - \gamma_\ell(a)]\} \\ &= (-1)^\ell [2\zeta^{(\ell)}(3,a) + 3\ell\zeta^{(\ell-1)}(3,a) + \ell(\ell-1)\zeta^{(\ell-2)}(3,a)]. \qquad (2.13)\end{aligned}$$

Similarly higher derivatives of $\gamma_\ell(a)$ may be determined and part (ii) has been shown.

From part (ii), we have $\gamma_1'(1) = \zeta'(2) + \zeta(2)$, wherein by the functional equation of the Riemann zeta function, or otherwise, we have the relations

$$\begin{aligned}\zeta'(2) &= \zeta(2)(\gamma + \ln 2 - 12 \ln A + \ln \pi) \\ &= \zeta(2)[\gamma + \ln(2\pi) - 1 + 12\zeta'(-1)], \qquad (2.14)\end{aligned}$$

where $\ln A = 1/12 - \zeta'(-1)$ and A is Glaisher's constant. Corollary 1 follows.

For Corollary 2 we simply shift the summation index $j \to j+1$ in the Taylor series (1.2) and read off the derivatives. Otherwise, we could make use of the forward difference operator $b^{-n}\Delta_b^n[f](x) = b^{-n}\sum_{k=0}^n \binom{n}{k}(-1)^{n-k} f(x+kb)$.

Remarks. The auxiliary relation (2.5) can be proved in a number of ways. One is with induction by using a recursion relation satisfied by $s(j,k)$ and by the binomial coefficient. Other methods include the use of integral representations either for the Stirling numbers of the first kind or for the binomial coefficient. For a proof using boson operators, see **[23]** (Identity 1).

It is readily checked that at $\ell = 0$ Proposition 1 (i) yields the identity $\gamma_0(a+b) = -\psi(a+b)$. For we have

$$\begin{aligned}
\gamma_0(a+b) &= \gamma_0 + \sum_{j=1}^{\infty}(-b)^j \zeta(j+1,a) \\
&= \gamma_0(a) + \sum_{j=1}^{\infty} \frac{(-b)^j}{j!} \int_0^{\infty} \frac{t^j e^{-(a-1)t}}{e^t - 1}\, dt \\
&= \gamma_0(a) + \int_0^{\infty} \frac{e^{-at}(e^{-bt}-1)}{1-e^{-t}}\, dt \\
&= \gamma_0(a) + \psi(a) - \psi(a+b) = -\psi(a+b). \qquad (2.15)
\end{aligned}$$

Herein we used a standard integral representation for $\zeta(s,a)$ and for the digamma function [**1,** page 259] or [**18,** page 943].

Similarly, we may write

$$\gamma_1(a+b) = \gamma_1(a) - \sum_{j=1}^{\infty}(-b)^j[\zeta'(j+1,a) + H_j\zeta(j+1,a)], \qquad (2.16)$$

where $H_j \equiv \sum_{k=1}^{j} 1/k$ is the usual harmonic number. Again, integral forms of this relation may be given.

Our integral representation [**8,** Proposition 3 (a)],

$$\gamma_k(a) = \frac{1}{2a}\ln^k a - \frac{\ln^{k+1} a}{k+1} + \frac{2}{a}\,\mathrm{Re}\int_0^{\infty} \frac{(y/a - i)\ln^k(a-iy)}{(1+y^2/a^2)(e^{2\pi y}-1)}\, dy, \qquad \mathrm{Re}\, a > 0, \qquad (2.17)$$

could also be used to prove Proposition 1 part (ii).

The approximate numerical value $\gamma_1'(1) \simeq 0.707385812532$ suggests that $\gamma_1'(1) = \zeta'(2) + \zeta(2)$ can be written as $1/\sqrt{2}$ together with a series of systematic correction terms.

Of course, the Wilton formula has built in the relation $(\partial/\partial a)\zeta(s,a) = -s\zeta(s+1,a)$, and more generally that

$$\left(\frac{\partial}{\partial a}\right)^j \zeta(s,a) = (-1)^j (s)_j \zeta(s+j,a). \qquad (2.18)$$

Proof of Proposition 2. These results are based upon **[10]** (Proposition 1, parts (i), (ii), and (iv)). We have, for each indicated domain of a,

$$\zeta(s,a)=\frac{(a-1)^{1-s}}{s-1}-\frac{1}{\Gamma(s)}\sum_{k=1}^{\infty}\frac{\Gamma(s+k)}{(k+1)!}\zeta(s+k,a),$$
$$\operatorname{Re} a>1, \tag{2.19}$$

$$\zeta(s,a)=\frac{(a-1/2)^{1-s}}{s-1}-\frac{1}{\Gamma(s)}\sum_{k=1}^{\infty}\frac{\Gamma(s+2k)}{4^k(2k+1)!}\zeta(s+2k,a),$$
$$\operatorname{Re} a>1/2, \tag{2.20}$$

and

$$\begin{aligned}\zeta(s,a)&=2^{s-2}\frac{(2a-1)^{1-s}}{s-1}\\&\quad+\frac{1}{\Gamma(s)}\left[\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(s+2k)}{4^k(2k+1)!}\zeta(s+2k,a)\right.\\&\quad\left.-\sum_{k=1}^{\infty}\frac{\Gamma(s+4k)}{16^k(4k+1)!}\zeta(s+4k,a)\right],\quad \operatorname{Re} a>1/2.\end{aligned} \tag{2.21}$$

Lemma 1 immediately carries over to

$$\left(\frac{d}{ds}\right)^{\ell}(s)_{pj}\bigg|_{s=1}=(-1)^{pj+\ell}\ell!s(pj+1,\ell+1),\quad p\geq 1. \tag{2.22}$$

We then expand equations (2.19)–(2.21) in powers of $s-1$ using the product rule and the Proposition follows. In the case of part (iii), we have first moved the $k=0$ term on the right side of equation (2.21) to the left side and multiplied the resulting equation by 2. $\square$

Remarks. Part (ii) of Proposition 2 gives the general n case beyond the low order instances given explicitly in terms of generalized harmonic numbers $H_n^{(r)}$ in **[11,** Proposition 7**]**. After all, it is well known that $s(n+1,1)=(-1)^n n!$, $s(n+1,2)=(-1)^{n+1}n!H_n$, and $s(n+1,3)=(-1)^n n![H_n^2-H_n^{(2)}]/2$, where $H_n\equiv H_n^{(1)}$. This part of the Proposition has also been obtained by Smith **[33]**.

Part (i) of Proposition 2 exhibits slow convergence. In contrast, parts (ii) and (iii) are very attractive for computation.

The Maślanka representation for the Riemann zeta function is written in terms of certain Pochhammer polynomials $P_k(s)$ **[29]**. Therefore, it is also possible to write expressions for γ_j from this representation in terms of sums including Stirling numbers of the first kind.

Likewise, the Stark-Keiper formula for ζ may be used to develop expressions for the Stieltjes constants. For $N > 0$ an integer, this representation reads

$$\zeta(s,N) = -\frac{1}{s-1}\sum_{k=1}^{\infty}\left(N+\frac{s-1}{k+1}\right)\frac{(-1)^k}{k!}(s)_k\zeta(s+k,N). \tag{2.23}$$

Serviceable series for the Stieltjes constants using the Stirling numbers of the first kind can also be written using the Taylor-series based expressions

$$\zeta(s,a) = a^{-s} + \sum_{n=0}^{\infty}\frac{(-a)^n}{n!}(s)_n\zeta(s+n), \quad |a|<1, \tag{2.24}$$

and

$$\zeta\left(s,a+\frac{1}{2}\right) = \sum_{n=0}^{\infty}\frac{(-a)^n}{n!}(s)_n(2^{s+n}-1)\zeta(s+n), \\ |a|<1/2. \tag{2.25}$$

In general, Dirichlet L functions may be written as a combination of Hurwitz zeta functions. For instance, for χ a character modulo m and $\operatorname{Re} s \geq 1$ we have

$$L(s,\chi) = \sum_{k=1}^{\infty}\frac{\chi(k)}{k^s} = \frac{1}{m^s}\sum_{k=1}^{m}\chi(k)\zeta\left(s,\frac{k}{m}\right). \tag{2.26}$$

For χ a nonprincipal character, convergence obtains herein for $\operatorname{Re} s \geq 0$. Therefore, our results are very pertinent to derivatives and expansions of Dirichlet L series about $s=1$, especially for real-Dirichlet-character combinations of low order Stieltjes constants, **[11]** may be consulted.

Proof of Proposition 3. We have the representation valid for $\operatorname{Re} s > -(2n-1)$,

$$\zeta(s,a) = a^{-s} + \frac{a^{1-s}}{s-1} + \sum_{k=1}^{n} (s)_{k-1} \frac{B_k}{k!} a^{-k-s+1}$$

$$+ \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^t - 1} - \sum_{k=0}^{n} \frac{B_k}{k!} t^{k-1} \right) e^{-at} t^{s-1} dt. \tag{2.27}$$

If we take $n \to \infty$ in this equation we in fact obtain an analytic continuation of the Hurwitz zeta function to the whole complex plane. We then develop the result

$$\zeta(s,a) = \frac{1}{2} a^{-s} + \frac{a^{1-s}}{s-1} + \sum_{j=2}^{\infty} (s)_{j-1} \frac{B_j}{j!} a^{-j-s+1}, \tag{2.28}$$

as a series in powers of $s-1$, where simply the factor

$$\left(\frac{d}{ds} \right)^{\ell-k} a^{-j-(s-1)} \Bigg|_{s=1} = a^{-j} (-1)^{\ell-k} \ln^{\ell-k} a. \tag{2.29}$$

We use the product rule for derivatives of the summation term together with Lemma 1 and find

$$\gamma_\ell(a) \sim \frac{1}{2a} \ln^\ell a - \frac{\ln^{\ell+1} a}{\ell+1}$$

$$+ \sum_{j=2}^{\infty} (-1)^{j-1} \frac{B_j}{j!} a^{-j} \sum_{k=0}^{\ell} \binom{\ell}{k} s(j,k+1) k! \ln^{\ell-k} a. \tag{2.30}$$

Since $B_{2n+1} = 0$ for $n \geq 1$, the stated form of Proposition 3 follows. $\square$

Proof of Proposition 4. We employ the representation [**2,** page 270] based upon Euler-Maclaurin summation and integration by parts, for $N \geq 0$, and $\operatorname{Re} s > -m$, with $m = 1, 2, \ldots$,

$$\zeta(s,a) = \sum_{n=0}^{N} \frac{1}{(n+a)^s} + \frac{(N+a)^{1-s}}{s-1}$$

$$- \sum_{r=1}^{m} \frac{(s)_r}{(r+1)!} \left[\zeta(s+r,a) - \sum_{n=0}^{N} \frac{1}{(n+a)^{s+r}} \right]$$

$$-\frac{(s)_{m+1}}{(m+1)!}\sum_{n=N}^{\infty}\int_0^1 \frac{u^{m+1}}{(n+a+u)^{s+m+1}}\,du. \tag{2.31}$$

Taking $m \to \infty$ and developing the result in powers of $s-1$ using Lemma 1 gives Proposition 4. $\square$

Remarks. In practice in using Proposition 4, there will be a tradeoff in selecting N and the cutoff or otherwise estimating the remainder neglected in the sum over r. We expect the convergence to be poor as $\operatorname{Re} a \to 0$ in this Proposition. We anticipate that a suitable procedure for computations is to take values with $\operatorname{Re} a > 1$ and then to use relation (1.1) as needed.

Similarly, we could employ the representation [**18,** page 1073] for integers $N \geq 0$ and $\operatorname{Re} a > 0$

$$\begin{aligned}\zeta(s,a) &= \sum_{n=0}^{N}\frac{1}{(n+a)^s} + \frac{(N+a)^{1-s}}{s-1}\\ &\quad - s\sum_{n=N}^{\infty}\int_n^{n+1}\frac{(t-n)}{(t+a)^{s+1}}\,dt, \quad \operatorname{Re} s > 1,\end{aligned} \tag{2.32}$$

where the integral may be easily expressed in closed form. Or we could use the representation for integers $N \geq 0$ and $\operatorname{Re} a > 0$ [**22,** page 16]

$$\begin{aligned}\zeta(s,a) &= \sum_{n=0}^{N}\frac{1}{(n+a)^s} + \frac{1}{s-1}\left(N+\frac{1}{2}+a\right)^{1-s}\\ &\quad + s\int_{N+1/2}^{\infty}\frac{P_1(t)}{(t+a)^{s+1}}\,dt, \quad \operatorname{Re} s > 0.\end{aligned}$$

As we easily have that the integral in this equation is bounded by

$$|s|\left|\int_{N+1/2}^{\infty}\frac{dt}{(t+a)^{s+1}}\right| = \frac{1}{|N+1/2+a|^s},$$

the integral converges uniformly for s in any compact subset of the half plane $\operatorname{Re} s > 0$ (and for arbitrary a).

Proof of Proposition 5. The function $\gamma_1(a) \to -\infty$ as $a \to 0^+$ and as $a \to \infty$. Indeed, the asymptotic form as $a \to \infty$ is $\gamma_1(a) \sim -(1/2)\ln^2 a$ as can be seen from (2.17). From Proposition 1 (ii) we have the derivative

$$(2.33)\quad \gamma_1'(a) = \zeta'(2,a) + \zeta(2,a) = \zeta'(2,a) + \psi'(a) = \sum_{n=0}^{\infty} \frac{1 - \ln(n+a)}{(n+a)^2},$$

where the term $\zeta(2,a) = \psi'(a)$ is the trigamma function. The function $\gamma_1'(a) \to \infty$ as $a \to 0^+$ and $\to 0$ through negative values as $a \to \infty$. The sole zero of $\gamma_1'(a)$ occurs at a_1^*, where $\gamma_1(a_1^*) > 0$. Therefore, γ_1 has the single global maximum as claimed. □

Remark. The approximate value $\gamma_1(a_1^*) \simeq 0.0379557$.

Proof of Proposition 6. As described below, we have for $\operatorname{Re} s < 1$

$$(2.34)\qquad \int_0^1 \zeta^{(j)}(s,a)\, da = 0.$$

We differentiate (2.8) j times with respect to s and apply this equation, putting $z = 1 - s$, giving Proposition 6. □

That (2.34) holds as can be seen by first noting that by the functional equation of the Hurwitz zeta function we have for $\operatorname{Re} s < 1$ and $0 < a < 1$

$$(2.35)\qquad \zeta(s,a) = 2^s \pi^{s-1} \Gamma(1-s) \sum_{n=1}^{\infty} \sin\left(2\pi n a + \frac{\pi s}{2}\right) n^{s-1}.$$

By using the product rule, various forms of $\zeta^{(j)}(s,a)$ follow, and these give (2.34).

Remarks. Equation (2.34) could also be found on the basis of various integral representations for ζ holding for $\operatorname{Re} s < 1$.

Proposition 6 gives a generalization of Proposition 4 of **[13]** when $j > 0$.

As a byproduct of our proof of Proposition 6, we have:

Corollary 4.

$$(2.36)\qquad \sum_{k=0}^{\infty} \frac{\gamma_{k+n}(a)}{k!} = (-1)^n [\zeta^{(n)}(0,a) + n!].$$

This is an extension of the “beautiful sum” at $a = 1$ attributed to Marichev **[36]**.

As a special case, we have

Corollary 5. *For* $\mathrm{Re}\, a > 0$*,*

$$\sum_{k=0}^{\infty} \frac{\gamma_{k+1}(a)}{k!} = -\zeta'(0,a) - 1 = \frac{1}{2}\ln(2\pi) - \ln\Gamma(a) - 1. \tag{2.37}$$

Indeed, Corollary 4 is consistent with the relation (cf. **[13,** page 610]) $R_j(a) = -(-1)^j \zeta^{(j)}(0,a)$, where

$$\frac{R'_{j+1}(a)}{j+1} = -\gamma_j - \frac{1}{a}\ln^j a - \sum_{n=1}^{\infty}\left[\frac{\ln^j(n+a)}{n+a} - \frac{\ln^j a}{n}\right], \quad j \geq 0. \tag{2.38}$$

By integrating this equation, we have

(2.39)

$$\begin{aligned} R_j(a) - R_j(1) &= j(1-a)\gamma_{j-1} - \ln^j a \\ &\quad - \sum_{n=1}^{\infty}\left[\ln^j(n+a) - \ln^j(n+1) - \frac{j}{n}\int_1^a \ln^{j-1} a\, da\right]. \end{aligned}$$

We record an expression for $(-1)^{j+1}R_j(a)$ in the following.

Lemma 2. *We have, for integers* $j \geq 1$*,*

$$\begin{aligned} \zeta^{(j)}(s,a) &= (-1)^j a^{1-s}\sum_{k=0}^{j}\binom{j}{k}(j-k)!\frac{\ln^k a}{(s-1)^{j-k+1}} \\ &\quad + \frac{(-1)^j}{2}a^{-s}\ln^j a + (-1)^j\int_0^{\infty}\frac{P_1(x)}{(x+a)^{s+1}}\ln^{j-1}(x+a)\,dx \\ &\quad - (-1)^j s\int_0^{\infty}\frac{P_1(x)}{(x+a)^{s+1}}\ln^j(x+a)\,dx, \end{aligned} \tag{2.40}$$

giving

$$\zeta^{(j)}(0,a) = a\sum_{k=0}^{j}\binom{j}{k}(j-k)!(-1)^{k+1}\ln^k a + \frac{(-1)^j}{2}\ln^j a$$
$$+ (-1)^j j\int_a^\infty \frac{\ln^{j-1} x}{x} P_1(x-a)\,dx. \tag{2.41}$$

Lemma 2 follows from a direct calculation using the integral representation [**39**] (2.3) valid for $\operatorname{Re} s > -1$,

$$\zeta(s,a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} - s\int_0^\infty \frac{P_1(x)}{(x+a)^{s+1}}\,dx. \tag{2.42}$$

Equation (2.40) may also be proved by induction. The $a=1$ reduction of (2.42) is well known [**35,** page 14].

Proof of Proposition 7. We have from [**39,** pages 153–154]

$$C_n(a) = \int_1^\infty P_1(x-a)\frac{\ln^{n-1} x}{x^2}(n-\ln x)\,dx. \tag{2.43}$$

We put $e_n \equiv \exp(n/2)$. We note that the generic function $\ln^n x/x^2$ is nonnegative and monotonically increasing for $1 < x < e_n$ and nonnegative and monotonically decreasing for $e_n < x < \infty$. We have that $P_1(x-a)$ is bounded and integrable and we split the integrals in (2.43) at e_{n-1} and e_n. By the second mean value theorem for integrals (e.g., [**18,** pages 1097–1098]) we have

$$\int_1^{e_n} \frac{\ln^n x}{x^2} P_1(x-a)\,dx = \frac{\ln^n e_n}{e_n^2}\int_\eta^{e_n} P_1(x-a)\,dx, \tag{2.44}$$

for some η with $1 \le \eta \le e_n$. Therefore, we obtain

$$\int_1^{e_n} \frac{\ln^n x}{x^2} P_1(x-a)\,dx = \frac{n^n}{2^n e^n}\int_\eta^{e_n} P_1(x-a)\,dx \le \frac{1}{6}\frac{n^n}{2^n e^n} \tag{2.45}$$

Here we have used the standard Fourier series for $P_1(x)$ [**39,** page 151] or [**1,** page 805],

$$P_1(x) = -\sum_{n=1}^\infty \frac{\sin(2n\pi x)}{n\pi}, \tag{2.46}$$

giving

$$\int_b^c P_1(x-a)\,dx = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\cos[2n\pi(x-a)]}{n^2\pi^2}\bigg|_b^c, \tag{2.47}$$

so that

$$\left|\int_b^c P_1(x-a)\,dx\right| \le \frac{2}{2\pi^2}\zeta(2) = \frac{1}{6}. \tag{2.48}$$

Similarly, we have

$$\int_{e_n}^{\infty} \frac{\ln^n x}{x^2} P_1(x-a)\,dx = \frac{\ln^n e_n}{e_n^2}\int_{e_n}^{\xi} P_1(x-a)\,dx, \tag{2.49}$$

for some ξ with $e_n \le \xi < \infty$. This gives

$$\int_{e_n}^{\infty} \frac{\ln^n x}{x^2} P_1(x-a)\,dx = \frac{n^n}{2^n e^n}\int_{e_n}^{\xi} P_1(x-a)\,dx \le \frac{1}{6}\frac{n^n}{2^n e^n}. \tag{2.50}$$

Combining the four integral contributions of (2.43) yields Proposition 7, as

$$|C_n(a)| \le \frac{1}{3}\left[\frac{(n-1)^{n-1}n}{2^{n-1}e^{n-1}} + \frac{n^n}{2^n e^n}\right] \le \frac{en^n}{2^n e^n}. \tag{2.51}$$

□

Remarks. Zhang and Williams previously found the better bound

$$|C_n(a)| \le \frac{[3+(-1)^n](2n)!}{n^{n+1}(2\pi)^n}. \tag{2.52}$$

However, our presentation shows possibility for improvement. If, for instance, a better bound is found for the integrals $\int_b^c P_1(x-a)\,dx$, then we may expect a tighter estimation.

Let us note also that the use of (2.40) for $\zeta^{(j)}(pk+1,a)$ together with the expressions of Proposition 2 provides many other opportunities for the estimation of $|\gamma_k(a)|$ and $|C_k(a)|$.

Proof of Proposition 8. We apply the formula of Klusch **[24]** (2.5),

$$L(x,s,a+\xi) = \sum_{k=0}^{\infty}(-1)^k\frac{(s)_k}{k!}L(x,s+k,a)\xi^k. \tag{2.53}$$

This formula may be obtained by Taylor expansion, expansion of an integral representation, or by binomial expansion in (1.16). We take n derivatives of (2.53) using the product rule. We evaluate at $s=1$ using Lemma 1 and the defining relation (1.17) of ℓ_n. Separating the $k=0$ term yields Proposition 8 and then Corollary 3. $\square$

Remark. As very special cases, we have

$$\ell_0\left(\frac{1}{2},a\right)=\frac{1}{2}\left[\psi\left(\frac{1+a}{2}\right)-\psi\left(\frac{a}{2}\right)\right],$$
$$\ell_1\left(\frac{1}{2},a\right)=\frac{1}{2}\left\{\ln 2\left[\psi\left(\frac{a}{2}\right)-\psi\left(\frac{1+a}{2}\right)\right]+\gamma_1\left(\frac{1+a}{2}\right)-\gamma_1\left(\frac{a}{2}\right)\right]\right\},$$

and

$$\begin{aligned}\ell_2\left(\frac{1}{2},a\right)&=\frac{1}{4}\left\{\ln^2 2\left[\psi\left(\frac{1+a}{2}\right)-\psi\left(\frac{a}{2}\right)\right]\right.\\ &\left.+2\ln 2\left[\gamma_1\left(\frac{a}{2}\right)-\gamma_1\left(\frac{a+1}{2}\right)\right]+\gamma_2\left(\frac{a}{2}\right)-\gamma_2\left(\frac{a+1}{2}\right)\right\}.\end{aligned}\tag{2.54}$$

As another example, from

$$\begin{aligned}L\left(\frac{1}{4},s,a\right)&=\sum_{n=0}^{\infty}\frac{i^n}{(n+a)^s}\\ &=4^{-s}\left\{\zeta\left(s,\frac{a}{4}\right)-\zeta\left(s,\frac{a+2}{4}\right)\right.\\ &\left.+i\left[\zeta\left(s,\frac{a+1}{4}\right)-\zeta\left(s,\frac{a+3}{4}\right)\right]\right\},\end{aligned}\tag{2.55}$$

by taking the real and imaginary parts of derivatives evaluated at $s=1$, we obtain expressions for $\gamma_k(a/4)-\gamma_k[(a+2)/4]$ and $\gamma_k[(a+1)/4]-\gamma_k[(a+3)/4]$. Evidently for a a rational number such relations always exist.

Proof of Proposition 9. From the classical theta function-based representation of the Riemann zeta function **[14]** (2) we have

$$\Gamma(s/2)\zeta(s)=\frac{\pi^{s/2}}{s(s-1)}+\sum_{n=1}^{\infty}n^{-s}\Gamma\left(\frac{s}{2},\pi n^2\right)$$

$$+\pi^{s-1/2}\sum_{n=1}^{\infty} n^{s-1}\Gamma\left(\frac{1-s}{2},\pi n^2\right). \tag{2.56}$$

We expand both sides of this equation in powers of $s-1$, and equate the coefficients of $(s-1)^0$ and $(s-1)^1$ on both sides to obtain parts (i) and (ii), respectively. The coefficient of $(s-1)^{n-1}$ for $n \geq 0$ of the term $\pi^{s/2}/[s(s-1)]$ is given by $\pi^{1/2}(-1)^n \sum_{j=0}^{n}(-1)^j \ln^j \pi/2^j$, and the simple polar term from the left side of (2.56) is canceled by the $n=0$ term. We use the special function relations $\Gamma(0,z) = -\mathrm{Ei}\,(-z)$ and $\Gamma(1/2,z) = \sqrt{\pi}[1-\mathrm{erf}\,(\sqrt{z})]$. The incomplete Gamma function **[18]** is given for $\mathrm{Re}\,x > 0$ by

$$\Gamma(\alpha,x) = \int_x^{\infty} e^{-t}t^{\alpha-1}\,dt = \frac{2x^{\alpha}e^{-x}}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{t^{1-2\alpha}e^{-t^2}}{t^2+x}\,dt, \tag{2.57}$$

where the latter form holds for $\mathrm{Re}\,\alpha < 1$. It is convenient in finding derivatives of this function to use the relation

$$\Gamma(\alpha,x) = \Gamma(\alpha) - \frac{x^{\alpha}}{\alpha}\,{}_1F_1(\alpha;\alpha+1;-x), \quad -\alpha \notin \mathbf{N}, \tag{2.58}$$

where ${}_1F_1$ is the confluent hypergeometric function **[1, 18]**. It follows that

$$\begin{aligned}\frac{d}{d\alpha}\,{}_1F_1(\alpha;\alpha+1;-x) = \frac{1}{\alpha}&\left[{}_1F_1(\alpha;\alpha+1;-x)\right.\\ &\left.-{}_2F_2(\alpha,\alpha;\alpha+1,\alpha+1;-x)\right],\end{aligned} \tag{2.59}$$

and

$$\begin{aligned}\frac{d}{d\alpha}\Gamma(\alpha,x) = \Gamma(\alpha)\psi(\alpha) + \frac{x^{\alpha}}{\alpha^2}&\left[-\alpha\ln x\;{}_1F_1(\alpha;\alpha+1;-x)\right.\\ &\left.+\,{}_2F_2(\alpha,\alpha;\alpha+1,\alpha+1;-x)\right].\end{aligned} \tag{2.60}$$

When $\alpha \to 0$ in (2.60), the singular terms $1/\alpha^2$ cancel. In fact, we have the expansions

$$\Gamma'(\alpha) = \Gamma(\alpha)\psi(\alpha) = -\frac{1}{\alpha^2} + \frac{\gamma^2}{2} + \frac{\pi^2}{12} + O(\alpha), \tag{2.61}$$

and

$$\frac{x^{\alpha}}{\alpha^2}\frac{(\alpha)_j}{(\alpha+1)_j}\left[-\alpha\ln x + \frac{(\alpha)_j}{(\alpha+1)_j}\right]\frac{(-x)^j}{j!} \tag{2.62}$$

$$= \frac{(1 - j\ln x)}{j^3(j-1)!}(-x)^j + O(\alpha), \quad \alpha \to 0.$$

Summing the latter relation on j as $\alpha \to 0$ then gives

$$\sum_{j=1}^{\infty} \frac{(1 - j\ln x)}{j^3(j-1)!}(-x)^j = -x {}_3F_3(1,1,1;2,2,2;-x) + [\gamma + \Gamma(0,x) + \ln x]\ln x. \tag{2.63}$$

We apply these relations at $\alpha = 0$ and $1/2$ and Proposition 9 follows. □

Remarks. By making a change of variables in the theta function-based representation **[14]** (2) we may include a parameter in the series representation (2.56), and so in Proposition 9 too. Beyond this, we may include a parameter $b > 0$ and a set of polynomials $p_j(s)$ with zeros lying only on the critical line in representing the Riemann zeta and xi functions **[12]** (Proposition 3). Thereby, we obtain a generalization of Proposition 9.

In equation (1.21), the sum terms provide a small $\simeq 0.0230957$ correction to produce the value $\gamma/2 \simeq 0.288607$. Although there are infinite sum corrections in Proposition 9, the expressions such as (1.21) and (1.22) may have some attraction for computation. This is due to the very fast decrease of the summands with n. For (1.21), using known asymptotic forms, the summand terms have exponential decrease $\sim e^{-n^2\pi}[2/(\pi n^2) + O(1/n^4)]$. Similarly for (1.22), the summand has exponential decrease in n. Therefore, high order approximations for the constants may be obtained with relatively few terms.

We note an integral representation for the term $-\sum_{n=1}^{\infty} \mathrm{Ei}\,(-\pi n^2)$ in Proposition 9 in the following.

Lemma 3. *Put the function $\theta_3(y) = 1 + 2\sum_{n=1}^{\infty} y^{n^2}$. Then we have*

$$-\sum_{n=1}^{\infty} \mathrm{Ei}\,(-\pi n^2) = -\frac{1}{2}\int_0^1 \left[\theta_3\left(e^{\pi/(x-1)}\right) - 1\right]\frac{dx}{x-1} = -\frac{1}{2}\int_1^{\infty} [1 - \theta_3(e^{-\pi u})]\frac{du}{u}. \tag{2.64}$$

Proof. Let L_n^α be the Laguerre polynomial of degree n and parameter α. We use the relation (e.g., [**18,** page 1038])

$$\Gamma(\alpha, x) = x^\alpha e^{-x} \sum_{k=0}^{\infty} \frac{L_k^\alpha(x)}{k+1}, \quad \alpha > -1, \tag{2.65}$$

at $\alpha = 0$. We then have

$$\begin{aligned} -\sum_{n=1}^{\infty} \mathrm{Ei}\,(-\pi n^2) &= \sum_{n=1}^{\infty} e^{-\pi n^2} \sum_{k=0}^{\infty} \frac{L_k(\pi n^2)}{k+1} \\ &= \sum_{n=1}^{\infty} e^{-\pi n^2} \sum_{k=0}^{\infty} \int_0^1 x^k\,dx \sum_{k=0}^{\infty} L_k(\pi n^2) \\ &= -\sum_{n=1}^{\infty} e^{-\pi n^2} \int_0^1 \frac{e^{x\pi n^2/(x-1)}}{x-1}\,dx, \qquad (2.66) \end{aligned}$$

where we used the generating function of the Laguerre polynomials (e.g., [**18,** page 1038]). The interchange of summation and integration is justified by the absolute convergence of the integral. Using the definition of θ_3 completes Lemma 3. $\square$

3. Discussion: Generalization of Proposition 9. We may generalize Proposition 9 to the context of generalized Stieltjes constants coming from the evaluation of Dirichlet L series derivatives at $s = 1$. For this we require a number of definitions. We use [**22,** Chapter 1, subsection 4.2].

Let χ be a primitive character modulo k. We need two theta functions,

$$\theta(\tau, \chi) = \sum_{n=-\infty}^{\infty} \chi(n) e^{-\pi\tau n^2/k},$$

for χ an even character,

$$\theta_1(\tau, \chi) = \sum_{n=-\infty}^{\infty} n\chi(n) e^{-\pi\tau n^2/k},$$

for χ an odd character, and the Gauss sum

$$g(\chi) = \sum_{j=1}^{k} \chi(j) e^{2\pi i j/k}.$$

We let $\delta = 0$ or 1 depending upon whether $\chi(-1) = 1$ or -1, respectively. We put

$$\xi(s,\chi) = (\pi k^{-1})^{-(s+1)/2}\Gamma\left(\frac{s+\delta}{2}\right)L(s,\chi).$$

Then by **[22,** page 15] we find for χ an even character,

$$\begin{aligned}
2\xi(s,\chi) &= \int_1^\infty \tau^{s/2-1}\theta(s,\chi)\,d\tau + \frac{\sqrt{k}}{g(\overline{\chi})}\int_1^\infty \tau^{-(s+1)/2}\theta(\tau,\bar{\chi})\,d\tau \\
&= \pi^{-s/2-1}k^{s/2+1}\sum_{n=-\infty}^{\infty}\chi(n)n^{-s-2}\Gamma\left(\frac{s}{2}+1,\pi\frac{n^2}{k}\right) \\
&\quad + \frac{\sqrt{k}}{g(\overline{\chi})}\pi^{(s-1)/2}k^{(1-s)/2} \\
&\quad + \sum_{n=-\infty}^{\infty}\overline{\chi}(n)n^{s-1}\Gamma\left(\frac{1-s}{2},\pi\frac{n^2}{k}\right),
\end{aligned}$$

and for χ an odd character,

$$\begin{aligned}
2\xi(s,\chi) &= \int_1^\infty \tau^{(s-1)/2}\theta_1(s,\chi)\,d\tau + \frac{i\sqrt{k}}{g(\overline{\chi})}\int_1^\infty \tau^{-s/2}\theta_1(\tau,\overline{\chi})\,d\tau \\
&= \pi^{-(s+1)/2}k^{(s+1)/2}\sum_{n=-\infty}^{\infty}n\chi(n)n^{-(s+1)} \\
&\quad \times\Gamma\left(\frac{s+1}{2}+1,\pi\frac{n^2}{k}\right) + \frac{i\sqrt{k}}{g(\overline{\chi})}\pi^{s/2-1}k^{1-s/2} \\
&\quad \times\sum_{n=-\infty}^{\infty}n\overline{\chi}(n)n^{s-2}\Gamma\left(1-\frac{s}{2},\pi\frac{n^2}{k}\right).
\end{aligned}$$

Then one may multiply the expressions for $\xi(s,\chi)$ by $(\pi k^{-1})^{(s+1)/2}$ and proceed as in the proof of Proposition 9 in developing both sides in powers of $s-1$.

In the case of the Hurwitz zeta function we employ the theta function

$$\theta(\tau,a) = \sum_{n\neq 0} e^{-\pi\tau(n+a)^2},$$

with its functional equation

$$\theta(1/\tau, a) = \sqrt{\tau} e^{-\pi a^2/\tau} \theta(\tau, -ia/\tau).$$

Proof of Proposition 10. We make use of expressions in the paper of Fine. In particular, see his expressions for the function $H(a,s)$ **[17,** pages 362–363]. We have

(3.1)
$$\begin{aligned}\pi^{(s-1)/2}\Gamma\left(\frac{1-s}{2}\right)[\zeta(1-s,a)+\zeta(1-s,1-a)] &= 2\sum_{n=1}^{\infty}\cos(2\pi na) \\ \times \int_0^\infty e^{-\pi n^2 t} t^{s/2-1}\,dt &= 2\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\sum_{n=1}^{\infty}\frac{\cos(2\pi na)}{n^s} \\ &= \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)[\mathrm{Li}_s(e^{2\pi ia})+\mathrm{Li}_s(e^{-2\pi ia})].\end{aligned}$$

Initially, the left side of this equation is valid for $\mathrm{Re}\, s < 0$. With analytic continuation, the function is $H(a,s) + 2/s$ is entire in s. Therefore, we may expand both sides of the representation (3.1) about $s = 0$. As it must, the singular term $-2/s$ effectively cancels from both sides of (3.1). Part (i) of Proposition 10 results from the constant term s^0 on both sides of (3.1), and using the reflection formula for the digamma function $\psi(1-a) = \psi(a) + \pi \cot \pi a$. Similarly, part (ii) results from the term s^1 on both sides of (3.1). □

Remarks. Proposition 10 and (3.1) properly reduce as they should for $a = 1/2$. In this case, we have the alternating zeta function $\sum_{n=1}^\infty (-1)^n/n^s = (2^{1-s} - 1)\zeta(s)$. Then, both sides of part (i) of Proposition 10 yield $\gamma - \ln \pi + 2 \ln 2$.

Proof of Proposition 11. By using the definition (1.25) we have

$$\begin{aligned}P_{n+1}(y) &= \frac{1}{n!}\int_0^y x(1-x)_{n-1}\,dx \\ &= \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^{n+k-1}s(n-1,k)\int_0^y x(1-x)^k\,dx\end{aligned}$$

$$= \frac{(-1)^n}{n!} \sum_{k=0}^{n-1} (-1)^{k-1} \frac{s(n-1,k)}{(k+2)(k+1)}$$
$$\times [(ky+y+1)(1-y)^k(y-1)+1], \tag{3.2}$$

whence part (i) follows. Of the many ways to perform the elementary integral here, it may be evaluated as a special case of the incomplete Beta function (e.g., [**18,** page 950])

$$B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1} dt$$
$$= \frac{x^p}{p} {}_2F_1(p, 1-q; p+1; x), \quad \mathrm{Re}\, p > 0, \ \mathrm{Re}\, q > 0. \tag{3.3}$$

For part (ii) we again use a generating function relation for $s(n,k)$, writing

$$P_{n+1}(y) = -\frac{1}{n!} \int_0^y (-x)_n \, dx$$
$$= \frac{(-1)^{n+1}}{n!} \sum_{k=0}^{n} s(n,k) \int_0^y x^k dx. \tag{3.4}$$

The second line of (1.27) simply follows from the values $s(n,0) = \delta_{n0}$ and $s(n,n) = 1$.

The identity of part (iii) follows simply from putting $y = 1$ in parts (i) and (ii).

For part (iv), we have the well-known expression

$$\gamma_k = \lim_{N \to \infty} \left(\sum_{m=1}^{N} \frac{\ln^k m}{m} - \frac{1}{k+1} \ln^{k+1} N \right), \tag{3.5}$$

so that $\lim_{n\to\infty} r_n^{(k)} = 0$. We form

$$D_n^{(k)} = \sum_{m=1}^{n} \left(\frac{\ln^k m}{m} - \int_m^{m+1} \frac{\ln^k x}{x} dx \right)$$
$$= \sum_{m=1}^{n} \int_0^1 \left[\frac{\ln^k m}{m} - \frac{\ln^k (x+m)}{x+m} \right] dx. \tag{3.6}$$

Therefore, we have

$$r_n^{(k)} = \sum_{m=n+1}^{\infty} \int_0^1 \left[\frac{\ln^k m}{m} - \frac{\ln^k(x+m)}{x+m}\right] dx. \tag{3.7}$$

We now use [**11**] (5.7)

$$\frac{\ln^n y}{y} - \frac{\ln^n(x+y)}{x+y} = \sum_{k=1}^{\infty} \frac{x^k}{y^{k+1}} \left[(-1)^k \ln^n y \right.$$
$$\left. + \frac{1}{k!} \sum_{j=0}^{n-1} \frac{n!}{(n-j-1)!} s(k+1, j+2) \ln^{n-j-1} y\right]. \tag{3.8}$$

Performing the integration of (3.7) gives part (iv). □

4. Discussion related to Proposition 11. By defining the function $f_k(x, m) = (x+m)\ln^k m - m\ln^k(x+m)$, it is possible to further decompose the remainders $r_n^{(k)}$ of (1.30) by writing

$$r_n^{(k)} = \sum_{m=n+1}^{\infty} \int_0^1 f_k(x,m) \left[\frac{1}{m(x+m)} - \frac{1}{m(m+1)}\right] dx$$
$$+ \int_0^1 \sum_{m=n+1}^{\infty} \frac{f_k(x,m)}{m(m+1)}\, dx. \tag{4.1}$$

The representation (3.8) can then be used three times in this equation. This process of adding and subtracting terms can be continued, building an integral term with denominator $m(m+1)\cdots(m+j)(m+x)$. For $k=0$, there is drastic reduction to the original construction of Ser [**32**].

Proposition 11 (iv) can be easily extended to expressions for $r_n^{(k)}(a) = \gamma_k(a) - D_n^{(k)}(a)$.

As pointed out by Ser, it is possible to write many summations and generating function relations with the polynomials P_n. As simple examples, we have

$$\sum_{n=1}^{\infty} P'_{n+1}(y) z^n = -\sum_{n=1}^{\infty} (-1)^n \binom{y}{n} z^n = 1-(1-z)^y, \tag{4.2}$$

and

$$\sum_{n=1}^{\infty} \frac{P'_{n+1}(y)}{n} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \binom{y}{n} = \psi(y+1) + \gamma = H_y. \tag{4.3}$$

With the form (1.29) we easily verify a generating function:

$$\begin{aligned}
\sum_{n=1}^{\infty} p_{n+1} z^{n-1} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \sum_{k=1}^{n} \frac{s(n,k)}{k+1} z^{n-1} \\
&= -\frac{1}{z} \sum_{k=1}^{\infty} \frac{1}{k+1} \sum_{n=k}^{\infty} \frac{s(n,k)}{n!} (-z)^n \\
&= -\frac{1}{z} \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \ln^k(1-z) \\
&= \frac{1}{z} + \frac{1}{\ln(1-z)}. \qquad (4.4)
\end{aligned}$$

One may also write a great many hypergeometric series of the form $\sum_{n=1}^{\infty} (P'_{n+1}(y))/(n^j) z^n$, that we omit.

In contrast to an expression of Ser **[32]**, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} p_{n+1}(1-e^z)^{n-1} &= \frac{1}{z} + \frac{1}{1-e^z} - \frac{1}{2} \\
&= \frac{1}{z} - \frac{1}{2}\coth\left(\frac{z}{2}\right) \qquad (4.5) \\
&= \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} z^{2k-1}, \quad |z| < 2\pi,
\end{aligned}$$

where the latter series may be found from **[18,** page 35]. This relation serves to connect the numbers p_{n+1} with the values $\zeta(2k)$. Equivalently, using an integral representation for $\zeta(2k)$, we have

$$\sum_{n=2}^{\infty} p_{n+1}(1-e^z)^{n-1} = 2\int_0^{\infty} \sinh(zv) \frac{dv}{e^{2\pi v}-1}. \tag{4.6}$$

As well, we have many extensions including

$$\sum_{n=2}^{\infty} p_{n+1}(1-e^z)^{n-1} e^{-tz(n-1)}! = \frac{1}{\ln[1+e^{z(1-t)}-e^{-tz}]} + \frac{e^{tz}}{1-e^z} - \frac{1}{2}. \tag{4.7}$$

We may inquire as to the asymptotic form of p_{n+1} as $n \to \infty$. Using only the leading term of the result of [**37**], we have

$$p_{n+1} \sim -\frac{1}{n} \sum_{k=1}^{n} \frac{(-1)^k}{(k+1)} \frac{\ln^{k-1} n}{(k-1)!} \sim \frac{1}{n \ln^2 n}, \quad n \to \infty. \tag{4.8}$$

There are corrections to this result which overall decrease it. Therefore, one may wonder if it can be proved that the expression on the right serves as an upper bound for all $n \geq 2$.

Motivated by [**1,** page 824], we conjectured that

$$p_{n+1} \sim \frac{1}{n(\ln n + \gamma)^2}, \quad n \to \infty, \tag{4.9}$$

is an improved asymptotic form and upper bound. This asymptotic form has been verified by Knessl [**26**]. In fact, he has obtained a full asymptotic series for p_{n+1} in the form

$$p_{n+1} \sim \frac{1}{n \ln^2 n} \left(1 + \sum_{j=1}^{\infty} \frac{A_j}{\ln^j n} \right), \tag{4.10}$$

where A_j are constants that may be explicitly determined from certain logarithmic-exponential integrals. We have $A_1 = -2\gamma$ and $A_2 = 3\gamma^2 - \pi^2/2$. Knessl develops (4.10) from the exact representation

$$p_{n+1} = \int_0^\infty \frac{1}{(1+u)^n} \frac{du}{(\ln^2 u + \pi^2)}, \quad n \geq 1. \tag{4.11}$$

From (4.11), we have the following

Corollary 6. *We have the integral representation*

$$\gamma = \int_{-\infty}^{\infty} e^z \frac{\ln(1+e^{-z})}{z^2 + \pi^2} \, dz. \tag{4.12}$$

Proof. We have [**18,** page 943]

$$\begin{aligned} \gamma = -\psi(1) &= \int_0^1 \left(\frac{1}{\ln u} + \frac{1}{1-u} \right) du \\ &= \int_0^1 \left(\frac{1}{\ln(1-v)} + \frac{1}{v} \right) dv \end{aligned}$$

$$= \int_0^1 \sum_{n=1}^\infty p_{n+1} v^{n-1}\, dv$$

$$= \sum_{n=1}^\infty \frac{p_{n+1}}{n}, \tag{4.13}$$

where we used (4.4). Now we use Knessl's representation (4.11), giving

$$\gamma = \sum_{n=1}^\infty \frac{1}{n} \int_0^\infty \frac{1}{(1+u)^n} \frac{du}{(\ln^2 u + \pi^2)}$$

$$= -\int_0^\infty \ln\left(\frac{u}{1+u}\right) \frac{du}{(\ln^2 u + \pi^2)}$$

$$= -\int_{-\infty}^\infty \frac{e^z}{(z^2+\pi^2)} [z - \ln(1+e^z)]\, dz$$

$$= \int_{-\infty}^\infty \frac{e^z}{(z^2+\pi^2)} \ln(1+e^{-z})\, dz. \tag{4.14}$$

□

Other properties and applications of the p_{n+1} constants are given in the very recent work [**7**].

5. Summary and very brief discussion. Our methods apply to a wide range of functions of interest to special function theory and analytic number theory including, but not limited to, the Lerch zeta function and Dirichlet L functions. Two-dimensional extensions would be to Epstein and double zeta functions. Our results include an addition formula for the Stieltjes coefficients, as well as expressions for their derivatives. The series representations for $\gamma_k(a)$ have very rapidly convergent forms, making them applicable for multiprecision computation. Byproducts of our results include some summation relations for the Stieltjes coefficients. We have given a means for estimating the magnitude of these coefficients, although this remains an outstanding problem.

Acknowledgments. I thank R. Smith for useful correspondence. I thank C. Knessl for reading the manuscript and for access to his results on the constants p_{n+1}.

Glossary

Symbol	Quantity
A	Glaisher's constant
B_j	Bernoulli number
$B(x,y)$	Beta function
$B_z(x,y)$	incomplete Beta function
$\binom{\ell}{k}$	binomial coefficient
χ	Dirichlet character
$C_n(a)=\gamma_n(a)-\frac{1}{a}\ln^n a$	normalized Stieltjes coefficient of the Hurwitz zeta function
Δ_b^n	forward difference operator
Ei	exponential integral
${}_2F_1$	Gauss hypergeometric function
${}_pF_q$	generalized hypergeometric function
$\Gamma(s)$	Gamma function
$\Gamma(a,z)$	incomplete Gamma function
$\gamma_k=\gamma_k(1)$	Stieltjes coefficient of the Riemann zeta function
$\gamma_k(a)$	Stieltjes coefficient of the Hurwitz zeta function
$g(\chi)$	Gauss sum
erf	error function
Li_s	polylogarithm function
$L(s,\chi)$	Dirichlet L function
$\psi(a)=-\gamma_0(a)$	digamma function
$\psi'(a)=\frac{d}{da}\psi(a)$	trigamma function
$\gamma=\gamma_0=-\psi(1)$	Euler constant
$L_n^\alpha(x)$	Laguerre polynomial
$L(x,s,a)$	Lipshitz-Lerch transcendent
$L(1/2,s,a)$	alternating Hurwitz zeta function
$\ell_n(x,a)$	Taylor coefficient about $s=1$ of the Lipshitz-Lerch transcendent
$P_1(t)=B_1(t-[t])$	first periodic Bernoulli polynomial
$P_{n+1}(y)$	normalized integrated Pochhammer symbol
$p_{n+1}=P_{n+1}(1)$	constants given by $p_{n+1}=-\frac{1}{n!}\int_0^1(-x)_n dx$.
$(z)_n=\Gamma(z+n)/\Gamma(z)$	Pochhammer symbol (rising factorial)
$s(n,k)=S_n^{(k)}$ $=(-1)^{n+k}\left[{n \atop k}\right]$	Stirling number of the first kind
$\zeta(s)$	Riemann zeta function
$\zeta(s,a)$	Hurwitz zeta function
$\theta(\tau,\chi)$	theta function with an even character
$\theta_1(\tau,\chi)$	theta function with an odd character
$\theta_3(y)$	Jacobi theta function
$\xi(s)$	Riemann xi function (completed Riemann zeta function)
$\xi(s,\chi)$	completed Dirichlet L function

REFERENCES

1. M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions*, National Bureau of Standards, Washington, 1964.

2. T.M. Apostol, *Introduction to analytic number theory*, Springer Verlag, New York, 1976; corrected fourth printing, 1995.

3. P. Appell, *Sur la nature arithmétique de la constante d'Euler*, Comp. Rend. Acad. Sci. **182** (1926), 897–899, 949.

4. B.C. Berndt, *On the Hurwitz zeta function*, Rocky Mountain J. Math. **2** (1972), 151–157.

5. W.E. Briggs, *Some constants associated with the Riemann zeta-function*, Michigan Math. J. **3** (1955), 117–121.

6. M.W. Coffey, *Functional equations for the Stieltjes constants*, arXiv:1402.3746, 2014.

7. ______, *Certain logarithmic integrals, including solution of monthly problem* 11629*, zeta values, and expressions for the Stieltjes constants*, arXiv:1201.3393, 2012.

8. ______, *The Stieltjes constants, their relation to the* η_j *coefficients, and representation of the Hurwitz zeta function*, Analysis **30** (2010), 383–409.

9. ______, *Series of zeta values, the Stieltjes constants, and a sum* $S_\gamma(n)$, arXiv/math-ph/:0706.0345v2, 2009.

10. ______, *On some series representations of the Hurwitz zeta function*, J. Comp. Appl. Math. **216** (2008), 297–305.

11. ______, *On representations and differences of Stieltjes coefficients, and other relations*, Rocky Mountain J. Math. **41** (2011), 1815–1846, arXiv:0809.3277.

12. ______, *Theta and Riemann xi function representations from harmonic oscillator eigenfunctions*, Phys. Lett. **362** (2007), 352–356.

13. ______, *New results on the Stieltjes constants*: *Asymptotic and exact evaluation*, J. Math. Anal. Appl. **317** (2006), 603–612, arXiv:math-ph/0506061.

14. ______, *Relations and positivity results for derivatives of the Riemann* ξ *function*, J. Comp. Appl. Math. **166** (2004), 525–534.

15. L. Comtet, *Advanced combinatorics*, D. Reidel, Dordrecht, 1974.

16. H.M. Edwards, *Riemann's zeta function*, Academic Press, New York, 1974.

17. N.J. Fine, *Note on the Hurwitz zeta-function*, Proc. Amer. Math. Soc. **2** (1951), 361–364.

18. I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series, and products*, Academic Press, New York, 1980.

19. R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete mathematics*, 2nd ed., Addison Wesley, New York, 1994.

20. G.H. Hardy, *Note on Dr. Vacca's series for* γ, Quart. J. Pure Appl. Math. **43** (1912), 215–216.

21. A. Ivić, *The Riemann zeta-function*, Wiley, New York, 1985.

22. A.A. Karatsuba and S.M. Voronin, *The Riemann zeta-function*, Walter de Gruyter, New York, 1992.

23. J. Katriel, *Stirling number identities*: *Interconsistency of q-analogues*, J. Phys. **31** (1998), 3559–3572.

24. D. Klusch, *On the Taylor expansion of the Lerch zeta-function*, J. Math. Anal. Appl. **170** (1992), 513–523.

25. J.C. Kluyver, *On certain series of Mr. Hardy*, Quart. J. Pure Appl. Math. **50** (1927), 185–192.

26. C. Knessl, private communication, 2009.

27. C. Knessl and M.W. Coffey, *An asymptotic form for the Stieltjes constants $\gamma_k(a)$ and for a sum $S_\gamma(n)$ appearing under the Li criterion*, Math. Comp. **80** (2011), 2197–2217.

28. R. Kreminski, *Newton-Cotes integration for approximating Stieltjes* (*generalized Euler*) *constants*, Math. Comp. **72** (2003), 1379–1397.

29. K. Maślanka, *A hypergeometric-like representation of zeta-function of Riemann*, arXiv/math-ph/:0105007v1, 2001.

30. D. Mitrović, *The signs of some constants associated with the Riemann zeta function*, Michigan Math. J. **9** (1962), 395–397.

31. B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monat. Preuss. Akad. Wiss. **671** (1859–1860).

32. J. Ser, Inter. Math. (1925), 126–127.

33. R. Smith, private communication, 2008.

34. T.J. Stieltjes, *Correspondance d'Hermite et de Stieltjes*, Volumes 1 and 2, Gauthier-Villars, Paris, 1905.

35. E.C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed., Oxford University Press, Oxford, 1986.

36. E.W. Weisstein, *Stieltjes constants*, from MathWorld–A Wolfram Web Resource, `http://mathworld.wolfram.com/StieltjesConstants.html`.

37. H.S. Wilf, *The asymptotic behavior of the Stirling numbers of the first kind*, J. Comb. Theor. **64** (1993), 344–349.

38. J.R. Wilton, *A note on the coefficients in the expansion of $\zeta(s,x)$ in powers of* $s-1$, Quart. J. Pure Appl. Math. **50** (1927), 329–332.

39. N.-Y. Zhang and K.S. Williams, *Some results on the generalized Stieltjes constants*, Analysis **14** (1994), 147–162.

Department of Physics, Colorado School of Mines, Golden, CO 80401
Email address: **mcoffey@mines.edu**