ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 43, Number 6, 2013

MINIMAL NUMBER OF POINTS WITH BAD REDUCTION FOR ELLIPTIC CURVES OVER $\mathbf{P}^1$

JOHANNES SPRANG

ABSTRACT. In this work we use elementary methods to discuss the question of the minimal number of points with bad reduction over $\mathbf{P}_k^1$ for elliptic curves $E/k(T)$ which are non-constant, respectively have non-constant j-invariant.

1. Introduction. It is a well-known fact going back to J. Tate (see [**3**, Chapter 5, Section 8]) that there are no elliptic curves over $\mathbf{Q}$ with good reduction everywhere. This was generalized by Fontaine [**2**] to abelian varieties over $\mathbf{Q}$. In [**6**] Schoof answers the question about the existence of non-zero abelian varieties over $\mathbf{Q}$ with bad reduction at just one prime.

In the case of function fields there are trivial examples of elliptic curves with good reduction everywhere coming from elliptic curves over the field of constants. Thus, it is natural to impose further conditions to exclude those trivial examples.

Beauville proves in [**1**] that every semi-stable elliptic surface $S \to \mathbf{P}^1_{\mathbf{C}}$ has at least four singular fibers. He also shows in this paper that every non-isotrivial elliptic surface $S \to \mathbf{P}_k^1$ for k an algebraic closed field of characteristic $p > 3$ has at least three singular fibres by applying the Riemann-Hurwitz theorem to the induced j-map. In a remark he claims that similar methods may be used to prove that there are no elliptic curves over $\mathbf{A}_k^1$ with non-constant j-invariant and good reduction everywhere in characteristics 2 and 3.

In this paper we take a more elementary approach to determine the minimal number of points with bad reduction over $\mathbf{P}_k^1$ for elliptic curves E over $k(T)$, if we require E to be non-constant, respectively $j(E) \notin k$. In characteristics different from 2 and 3 a non-constant elliptic curve has at least 2 points with bad reduction. If we furthermore demand $j(E) \notin k$, then there are at least 3 points with bad reduction.

2010 AMS *Mathematics subject classification.* Primary 11R58, 14H52.
Received by the editors on December 23, 2010, and in revised form on June 29, 2011.

DOI:10.1216/RMJ-2013-43-6-2017

In characteristics 2 and 3 we show that every non-constant elliptic curve has at least 1 point with bad reduction. If we further require E to have non-constant j-invariant, i.e., $j(E) \notin k$, then we obtain at least 2 points with bad reduction. By giving examples we will show that the obtained bounds are sharp.

2. Notation and conventions. In this section k denotes an algebraically closed field. Let E be an elliptic curve over a function field $K/k(T)$. We call E *constant*, if there exists a Weierstrass equation for E with coefficients in k. We say E has *constant j-invariant*, if $j(E) \in k$. Clearly every constant elliptic curve has constant j-invariant. Conversely, every elliptic curve with constant j-invariant gets constant over some finite extension of K. For the following we refer to [**4**, Chapter 10]. Let $S = \operatorname{Spec} A$ be an affine Dedekind scheme (of dimension one) with function field $K(S)$ and $E/K(S)$ an elliptic curve. We say E has good reduction at $s \in S$, if E admits a smooth model over $\operatorname{Spec} \mathcal{O}_{S,s}$.

In the following, we will consider affine open subsets $S = \operatorname{Spec} A \subseteq \mathbf{P}^1_k$ and elliptic curves E over $K(S) = k(T)$. Since $\operatorname{Pic}(A) = 0$, we can choose a globally minimal Weierstrass equation W for E over S. Then E has good reduction at $s \in S$, if and only if $\nu_s(\Delta_W) = 0$. Here ν_s denotes the valuation associated to $\mathcal{O}_{S,s}$.

3. Statement of the results.

Theorem 1. *Let k be an algebraically closed field with characteristic different from* 2 *and* 3*. Further let $E/k(T)$ be an elliptic curve. Then*:

(a) *there are at least three points with bad reduction over $\mathbf{P}^1_k$ if E has non-constant j-invariant*,

(b) *and at least two points with bad reduction if E is non-constant.*

The bounds given in (a) *and* (b) *are sharp.*

Theorem 2. *Let k be an algebraically closed field of characteristic* 2 *or* 3 *and let $E/k(T)$ be an elliptic curve. Then*:

(a) *there are at least two points with bad reduction over $\mathbf{P}^1_k$ if E has non-constant j-invariant*,

(b) *and at least one point with bad reduction if E is assumed to be non-constant.*

The bounds given in (a) *and* (b) *are sharp.*

4. Proofs.

4.1. Proofs for characteristic different from 2 and 3. In this section k denotes an algebraically closed field. We prove the results in characteristic different from 2 and 3 by applying Mason's *abc*-inequality to the discriminant of a globally minimal Weierstrass equation:

Definition 3. For $x \in k(T)$, we call

$$H(x) := \sum_{\nu} -\min(0, \nu(x)) \tag{1}$$

where the sum is over all canonical valuations on $k(T)$, the *height* of x. For more detailed background we refer to [**5**, Chapters I, VI].

Lemma 4 (Mason's ABC-inequality). *Let $\mathfrak{V}$ be a finite set of canonical valuations ν on $k(T)$. Let $\gamma_1, \gamma_2, \gamma_3 \in k(T)^\times$ with $\gamma_1 + \gamma_2 + \gamma_3 = 0$ and such that $\nu(\gamma_1) = \nu(\gamma_2) = \nu(\gamma_3)$ for each valuation ν not in $\mathfrak{V}$.*

(a) *If* $\operatorname{char}(k) = 0$*, we have either $\gamma_1/\gamma_2 \in k$ or*

$$H\left(\frac{\gamma_1}{\gamma_2}\right) \leq |\mathfrak{V}| - 2.$$

(b) *If* $\operatorname{char}(k) = p > 0$*, we have either $\gamma_1/\gamma_2 \in k(T)^p$ or*

$$H\left(\frac{\gamma_1}{\gamma_2}\right) \leq |\mathfrak{V}| - 2.$$

Here H denotes the height function on $k(T)$. Compare [**5**, Chapters I, VI].

Proof. See [**5**, page 14, Lemma 2; page 97 Lemma 10]. ◻

Lemma 5. *Let k be an algebraically closed field of characteristic different from 2 and 3. For all $A, B \in k[T, T^{-1}]$ with $A^3 - B^2 \in k[T, T^{-1}]^\times$ we have $A, B \in k[T, T^{-1}]^\times \cup \{0\}$.*

Proof. In the following we may assume $AB \neq 0$ since otherwise we are done. Let $A, B \in k[T, T^{-1}]$ such that $A^3 - B^2 = aT^l$ for $a \in k^\times$, $l \in \mathbf{Z}$. For the polynomials

$$q_A := \frac{A}{T^{\nu_T(A)}}, \qquad q_B := \frac{B}{T^{\nu_T(B)}}$$

let us define:

$$\mathfrak{V}_A := \{\nu_g : g \in \text{support}\,(q_A)\}$$
$$\mathfrak{V}_B := \{\nu_g : g \in \text{support}\,(q_B)\}.$$

Since A and B are coprime in $k[T, T^{-1}]$, we see

$$\mathfrak{V}_A \cap \mathfrak{V}_B = \varnothing \tag{2}$$

and, for $\mathfrak{V} := \mathfrak{V}_A \cup \mathfrak{V}_B \cup \{\nu_\infty, \nu_T\}$ we have:

$$|\mathfrak{V}| \leq \deg q_A + \deg q_B + 2. \tag{3}$$

The definition of the height function gives the following estimates:

$$H\left(\frac{A^3}{B^2}\right) \geq \sum_{\nu \in \mathfrak{V}_B} - \min\left(0, \nu\left(A^3\right) - \nu\left(B^2\right)\right) \tag{4}$$
$$= 2\deg q_B$$

$$H\left(\frac{B^2}{A^3}\right) \geq \sum_{\nu \in \mathfrak{V}_A} - \min\left(0, \nu\left(B^2\right) - \nu\left(A^3\right)\right) \tag{5}$$
$$= 3\deg q_A.$$

(1) Now we first suppose [char$\,(k) = 0$ *and* $(A^3/B^2), (B^2/A^3) \notin k$] or [char$\,(k) = p > 0$ and $(A^3/B^2), (B^2/A^3) \notin k(T)^p$], so that we may apply Mason's inequality:

$$H\left(\frac{B^2}{A^3}\right) \leq |\mathfrak{V}| - 2, \qquad H\left(\frac{A^3}{B^2}\right) \leq |\mathfrak{V}| - 2. \tag{6}$$

We get

$$3\deg q_A + 2\deg q_B \overset{(4),(5),(6)}{\leq} 2(|\mathfrak{V}|-2) \overset{(3)}{\leq} 2\deg q_A + 2\deg q_B$$

as well as:

$$2\deg q_B \overset{(4)}{\leq} H\left(\frac{A^3}{B^2}\right) \overset{(6)}{\leq} (|\mathfrak{V}|-2) \overset{(3)}{\leq} \deg q_A + \deg q_B.$$

Together this yields $\deg q_B \leq \deg q_A \leq 0$ and from the definition of q_A and q_B we get $A, B \in k[T, T^{-1}]^\times$ as desired.

(2) If we have [char $(k) = 0$ *and* $(A^3/B^2), (B^2/A^3) \in k$], we get $q_A^3/q_B^2 \in k$. And since q_A and q_B are coprime as polynomials, we have $q_A, q_B \in k$.

(3) The remaining case [char $(k) = p > 0$ *and* $(A^3/B^2), (B^2/A^3) \in k(T)^p$] may be reduced to the first case. Let $s := \sup(s : (A^3/B^2) \in k(T)^{p^s})$. We may assume $s < \infty$ since otherwise we would have $(A^3/B^2) \in k$ and therefore $A^3, B^2 \in k[T, T^{-1}]^\times$. Since we have $(A^3/B^2) \in k(T)^{p^s}$, and since p is different from 2 and 3, there are $\widetilde{q_A}, \widetilde{q_B}$ and some $n \geq 0$ such that:

$$\widetilde{q_A}^{p^s} = q_A, \quad \widetilde{q_B}^{p^s} = q_B \text{ and } 3\nu_T(A) - 2\nu_T(B) = np^s.$$

After dividing $A^3 - B^2 = aT^l$ by $T^{2\nu_T(B)}$ we have:

$$(\widetilde{q_A}^3 T^n - \widetilde{q_B}^2)^{p^s} = aT^{l-2\nu_T(B)}.$$

Multiplying $\widetilde{q_A}^3 T^n - \widetilde{q_B}^2 \in k[T, T^{-1}]^\times$ by an appropriate T^m such that $3 \mid (m+n)$ and $2 \mid m$ yields:

$$(\widetilde{q_A} T^{(n+m)/3})^3 - (\widetilde{q_B} T^{m/2})^2 \in k[T, T^{-1}]^\times.$$

By the choice of s we have $(\widetilde{q_A}^3 T^{n+m})/(\widetilde{q_B}^2 T^m) = (\widetilde{q_A}^3 T^n \widetilde{q_B}^2) \notin k(T)^p$, and thus the first case applies. □

Corollary 6. *For $l \in \mathbf{Z}$ and $u \in k^\times$ consider $A, B \in k[T, T^{-1}]$ with $A^3 - B^2 = uT^l$.*

(a) *If $AB \neq 0$, then $(A, B) \in \{(aT^{2n}, bT^{3n}) : a, b \in k^\times, n \in \mathbf{Z}\}$.*

(b) *If* $A = 0$*, then* $(A, B) \in \{(0, bT^n) : b \in k^\times, n \in \mathbf{Z}\}$.

(c) *If* $B = 0$*, then* $(A, B) \in \{(aT^n, 0) : a \in k^\times, n \in \mathbf{Z}\}$.

Proof. By Lemma 5, we have $A, B \in k[T, T^{-1}]^\times \cup \{0\}$, say $A = aT^{l_A}, B = bT^{l_B}$ with $a, b \in k$, $l_a, l_B \in \mathbf{Z}$. Thus we have an equation of the form:

$$a^3T^{3l_A} - b^2T^{2l_B} = uT^l$$

Now the corollary follows from the fact that $\{T^i\}_{i\in\mathbf{Z}}$ is a k-basis for $k[T, T^{-1}]$. □

Corollary 7. *Let* $A, B \in k[T]$ *satisfying* $A^3 - B^2 \in k^\times$*. Then we have* $A, B \in k$*.*

Proof. Setting $l = 0$ in Corollary 6 we see that the result even holds for $A, B \in k[T, T^{-1}]$. □

Using the two corollaries above we can prove the main results in all characteristics different from 2 and 3.

Proof of Theorem 1 (a). We show that every elliptic curve $E/k(T)$ with bad reduction restricted to two points has $j(E) \in k$. Without loss of generality, we may assume these points to be $\{0, \infty\}$, so let $E/k(T)$ be an elliptic curve with good reduction everywhere over $\operatorname{Spec} k[T, T^{-1}]$. Since we are in characteristic different from 2 and 3, we can choose for E a globally minimal Weierstrass equation over $\operatorname{Spec} k[T, T^{-1}]$ of the form

$$W : y^2 = x^3 - 3Ax + 2B$$

with $A, B \in k[T, T^{-1}]$. Since E has good reduction everywhere over $\operatorname{Spec} k[T, T^{-1}]$, we have $\Delta_W = 1728(A^3 - B^2) \in k[T, T^{-1}]^\times$. But this is just possible for (A, B) as in Corollary 6. It is easy to see that E has constant j-invariant by inserting the results from Corollary 6 into the formula $j(E) = 12^3(A^3)/(A^3 - B^2)$.

If we remove three points we can just write down an elliptic curve with non-constant j-invariant and good reduction everywhere. □

Example 8. The elliptic curve $E/k(T)$ defined by the Weierstrass equation in Legendre form:

$$W: \quad y^2 = x(x-1)(x-T)$$

has j-invariant $j(E) = 2^8(T^2-T+1)^3/(T^2(T-1)^2)$, so E has non-constant j-invariant. From the discriminant $\Delta_W = 16T^2(T-1)^2$ we see that W is a globally minimal Weierstrass equation for E over $\mathbf{P}^1_k \setminus \{0,1,\infty\}$ and that it has good reduction outside $\{0,1,\infty\}$.

Proof of Theorem 1 (b). Let E be an elliptic curve over $k(T)$. Since we are in characteristic different from 2 and 3 we can choose for E a globally minimal Weierstrass equation over $\mathbf{A}^1_k = \operatorname{Spec} k[T]$ of the form:

$$W: y^2 = x^3 - 3Ax + 2B$$

with $A, B \in k[T]$. If E has good reduction everywhere over $\mathbf{A}^1_k$, we have $\Delta_W = 1728(A^3 - B^2) \in k^\times$, but due to Corollary 7 this is just possible for A, B constant. Thus E is constant. $\square$

The following example shows that there are non-constant elliptic curves, if we allow bad reduction in one more point.

Example 9. The elliptic curve E over $k(T)$ defined by

$$W: \quad y^2 = x^3 - 3T^2x$$

has j-invariant $j = 1728$ and discriminant $\Delta_W = 1728T^6$. Thus W is a globally minimal Weierstrass equation for E over $\operatorname{Spec} k[T, T^{-1}]$ and has good reduction everywhere over $\mathbf{P}^1_k \setminus \{0,\infty\}$ since $\Delta_W \in k[T, T^{-1}]^\times$. It is easy to see that E is non-constant: Suppose we had a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with constant coefficients. Then a suitable substitution would give us a constant two-parameter Weierstrass equation

$$W': \quad y'^2 = x'^3 + ax' + b$$

with constant coefficients $a, b \in k$. Comparing the Weierstrass equations W and W' yields $-3T^2 = au^4$ for some $u \in k(T)$, which is impossible.

4.2. Proofs for characteristics 2 and 3. In characteristics 2 and 3 the non-existence of elliptic curves E with good reduction everywhere over $\mathbf{A}^1_k$ and non-constant j-invariant can be obtained by applying the following lemma to a globally minimal Weierstrass equation of E.

Lemma 10. *Let k be an algebraically closed field of characteristic $p > 0$. Let*

$$f(Y) = Y^{p^r} - Y^n Q_1^m - Q_1^{m+1} Q - c \in (k[T])\,[Y] \tag{7}$$

with $Q_1, Q \in k[T]$, Q_1 non-constant, $c \in k^\times$, $r, m, n \geq 1$ integers with $m < p^r$ such that m and n are coprime to p. Then $f(Y)$ has no roots in $k[T]$.

Proof. Define

$$\begin{aligned} w: \quad k[T] &\longrightarrow \mathbf{Z} \cup \{\infty\} \\ g &\longmapsto \sup\left\{s : g \in k[T]^{p^s}\right\} \end{aligned}$$

(for non constant $g \in k[T]$ we have $w(g) < \infty$). We now prove Lemma 10 by induction over $w(Q_1)$.

As a first case we consider $\underline{s := w(Q_1) < r}$. Let f be an arbitrary polynomial of the form (7) as given in Lemma 10 with $w(Q_1) < r$. Let us further assume we had a solution $A \in k[T]$ to $f(Y) = 0$:

$$A^{p^r} - c = Q_1^m\,(A^n + Q_1 Q) \quad \text{in } k[T].$$

From this equation we can see that Q_1 is coprime to A and thus Q_1 is also coprime to $A^n + Q_1 Q$. Since $A^{p^r} - c \in k[T]^{p^r}$, we have $Q_1^m(A^n + Q_1 Q) \in k[T]^{p^r}$. Since k is algebraically closed, we obtain from $(Q_1, A^n + Q_1 Q) = 1$ that $Q_1^m \in k[T]^{p^r}$. And since m is coprime to p we get $Q_1 \in k[T]^{p^r}$ contradicting $w(Q_1) < r$.

$\underline{s - 1 \to s}$. We may assume $s \geq r$ since $s < r$ has already been treated. Suppose we have proved Lemma 10 for all polynomials Q_1

with $w(Q_1) < s$. Now consider a polynomial f of the form (7) as in the lemma with $w(Q_1) = s$. Further, assume we had an $A \in k[T]$ with $f(A) = 0$:

$$A^{p^r} - c = Q_1^m \left(A^n + Q_1 Q\right).$$

Again, since $(Q_1, A^n + Q_1 Q) = 1$ and $(m, p) = 1$ we have $Q_1 \in k[T]^{p^r}$. Let $\widetilde{Q}_1 \in k[T]$ with $\widetilde{Q}_1^{p^r} = Q_1$; further let $\widetilde{c} \in k$ be such that $\widetilde{c}^{p^r} = c$. Thus, we have:

$$\left(\frac{A - \widetilde{c}}{\widetilde{Q}_1^m}\right)^{p^r} = A^n + Q_1 Q. \tag{8}$$

Setting $g := (A - \widetilde{c})/(\widetilde{Q}_1^m)$ and inserting $A = \widetilde{c} + g\widetilde{Q}_1^m$ in (8) yields:

$$\begin{aligned} g^{p^r} &= (\widetilde{c} + g\widetilde{Q}_1^m)^n + Q_1 Q \\ &= \widetilde{c}^n + n\widetilde{c}^{n-1} g \widetilde{Q}_1^m + \underbrace{\sum_{i=2}^{n} \binom{n}{i} \widetilde{c}^{n-i} g^i \widetilde{Q}_1^{im} + \widetilde{Q}_1^{p^r} Q}_{=:\widetilde{Q}_1^{m+1}\widetilde{Q} \text{ using } p^r > m}. \end{aligned}$$

Thus we have a solution $g \in k[T]$ of the equation

$$0 = Y^{p^r} - \underbrace{n\widetilde{c}^{n-1}}_{=:c'} Y \widetilde{Q}_1^m - \widetilde{Q}_1^{m+1}\widetilde{Q} - \widetilde{c}^n,$$

which is of the form (7) as considered in Lemma 10 (the constant c' doesn't matter since we can pull it into $\widetilde{Q}_1$). And because of $w(\widetilde{Q}_1) = w(Q_1) - r < w(Q_1)$ this solution contradicts the induction hypothesis. □

Lemma 10 may be used to prove part (a) of Theorem 2, i.e., every elliptic curve $E/k(T)$ with non-constant j-invariant has at least two points with bad reduction.

Proof of Theorem 2 (a). It suffices to show that there is no elliptic curve with non-constant j-invariant and with good reduction everywhere over $\mathbf{A}_k^1$. Suppose we had such an elliptic curve E. We first prove this for characteristic 2. Let

$$W : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

$a_i \in k[T]$ be a globally minimal Weierstrass equation for E over $\mathbf{A}^1_k$. We have:

$$\begin{aligned}\Delta_W &= a_3^4 + a_1^3 a_3^3 + a_1^4\left(a_1^2 a_6 + a_1 a_3 a_4 + a_2 a_3^2 + a_4^2\right),\\ j(E) &= \frac{a_1^{12}}{\Delta_W}\end{aligned} \tag{9}$$

By our assumptions on E we must have $\Delta_W \in k^\times$ and $j(E) \notin k$. Then $j(E) \notin k$ implies $a_1 \notin k$. But by (9) $a_3 \in k[T]$ would be a solution to

$$Y^4 + a_1^3 Y^3 + a_1^4(a_1^2 a_6 + a_1 a_3 a_4 + a_2 a_3^2 + a_4^2) - \Delta_W = 0,$$

contradicting Lemma 10 since $a_1 \notin k$.

In characteristic 3 we can choose a globally minimal Weierstrass equation for E over $\mathbf{A}^1_k$ of the form

$$W : y^2 = x^3 + a_2 x^2 + a_4 x + a_6$$

$a_i \in k[T]$. Then W has discriminant

$$\Delta_W = -a_4^3 + a_2^2 a_4^2 - a_2^3 a_6 \tag{10}$$

and j-invariant $j(E) = (a_2^6)/\Delta_W$. By our assumptions on E we must have $\Delta_W \in k^\times$ as well as $j(E) \notin k$. And $j(E) = (a_2^6)/\Delta_W$ implies $a_2 \notin k$. But, by (10), $a_4 \in k[T]$ would be a solution of

$$Y^3 - Y^2 a_2^2 + a_2^3 a_6 + \Delta_W = 0,$$

again contradicting Lemma 10 since $a_2 \notin k$.

It remains to give an example of an elliptic curve with non-constant j-invariant and bad reduction at just two points. ◻

Example 11. (a) Let k be an algebraically closed field of characteristic 2. Then the elliptic curve $E/k(T)$ given by

$$W : y^2 + xy = x^3 + x^2 + T$$

has discriminant $\Delta_W = T$, and hence W is a globally minimal Weierstrass equation for E with bad reduction restricted to $\{0, \infty\}$. The j-invariant of E is $1/T$ so E has non-constant j-invariant.

(b) Let k be an algebraically closed field of characteristic 3. Then the elliptic curve $E/k(T)$ given by

$$W : y^2 = x^3 + Tx^2 - T$$

has discriminant $\Delta_W = T^4$, and hence W is a globally minimal Weierstrass equation with bad reduction restricted to $\{0, \infty\}$. The j-invariant of E is T^2 so E has non-constant j-invariant.

Although we do not have in general a two-parameter Weierstrass equation $y^2 = x^3+Ax+B$ in characteristics 2 and 3, we can nevertheless choose simple equations:

Proposition 12. *Let K be any field of characteristic* 2 *or* 3*.* (a) *Let E/K be a elliptic curve given by a Weierstrass equation*

$$\widetilde{W} : \widetilde{y}^2 + \widetilde{a}_1\widetilde{x}\widetilde{y} + \widetilde{a}_3\widetilde{y} = \widetilde{x}^3 + \widetilde{a}_2\widetilde{x}^2 + \widetilde{a}_4\widetilde{x} + \widetilde{a}_6.$$

Then we can find substitutions to a simpler Weierstrass equation W:

(i) *if* $\operatorname{char} K = 3$ *and* $j(E) \neq 0$:

$$W : y^2 = x^3 + a_2x^2 + a_6$$

with $\Delta_W = -a_2^3a_6$ and $j(E) = -(a_2^3)/(a_6)$. The only substitutions preserving this form are: $x = u^2x'$, $y = u^3y'$ with $u \in K^\times$.

(ii) *If* $\operatorname{char} K = 3$ *and* $j(E) = 0$:

$$W : y^2 = x^3 + a_4x + a_6$$

with $\Delta_W = -a_4^3$ and $j(E) = 0$. The only substitutions preserving this form are: $x = u^2x' + r$, $y = u^3y'$ with $u \in K^\times$, $r \in K$.

(iii) *If* $\operatorname{char} K = 2$ *and* $j(E) \neq 0$:

$$W : y^2 + xy = x^3 + a_2x^2 + a_6$$

with $\Delta_W = a_6$ and $j(E) = 1/a_6$. The only substitutions preserving this form are: $x = x'$, $y = y' + sx'$ with $s \in K$.

(iv) *If* $\operatorname{char} K = 2$ *and* $j(E) = 0$:

$$W : y^2 + a_3 y = x^3 + a_4 x + a_6$$

with $\Delta_W = a_3^4$ *and* $j(E) = 0$. *The only substitutions preserving this form are*: $x = u^2 x' + s^2$, $y = u^3 y' + u^2 s x' + t$ *with* $u \in K^\times$ *and* $s, t \in K$.

(b) *If we have* $K = k(T)$ *in* (a) *with* $j(E) \in k$, *then we can even choose a globally minimal Weierstrass equation for* E *over* $\mathbf{A}_k^1$ *of the indicated simple form* W *as given above.*

Proof. Part (a) is well known and may be proven by just writing down explicit substitutions. See for example [**7**, Appendix A, Proposition 1.1 and Proof of Proposition 1.2].

Part (b) is an easy consequence of the proof of (a). Therefore, start with a globally minimal Weierstrass equation for E over $\mathbf{A}_k^1$:

$$\widetilde{W} : \widetilde{y^2} + \widetilde{a_1}\widetilde{x}\widetilde{y} + \widetilde{a_3}\widetilde{y} = \widetilde{x^3} + \widetilde{a_2}\widetilde{x^2} + \widetilde{a_4}\widetilde{x} + \widetilde{a_6}$$

with $\widetilde{a}_i \in k[T]$ and check that the explicit substitutions given in [loc. cit.] change the discriminant just by an element in $k^\times$ and that we still obtain coefficients in $k[T]$. □

Corollary 13. *Let* k *be an algebraically closed field of characteristic* 3. *Let* $E/k(T)$ *be an elliptic curve with* $j(E) \in k$ *and good reduction everywhere over* $\mathbf{A}_k^1$. *Then* E *is already constant, if* $j(E) \neq 0$.

Proof. This is an immediate consequence of Proposition 12 (a) (i). □

Part (b) of Theorem 2, which says that every elliptic curve $E/k(T)$ with good reduction everywhere is constant, may be proven by direct calculations involving Weierstrass equations as in Proposition 12:

Proof of Theorem 2 (b). Let E be an elliptic curve with good reduction everywhere over $\mathbf{P}_k^1$. By the proof of part (a) we have $j(E) \in k$. Let T, respectively T^{-1}, be uniformizers at $0 \in \mathbf{P}_k^1$, respectively $\infty \in \mathbf{P}_k^1$. We prove Proposition 12 by choosing globally minimal Weierstrass equations for E over $\mathbf{P}_k^1 \setminus \{\infty\} = \operatorname{Spec} k[T]$ as well as for $\mathbf{P}^1_k \setminus \{0\} = \operatorname{Spec} k[T^{-1}]$. Then, comparing these equations over $\mathbf{P}^1_k \setminus \{\infty, 0\}$, yields substitutions giving constant elliptic curves.

We start with characteristic 3. By Corollary 13 we may assume $j(E) = 0$ and by Proposition 12 (b) we may choose a globally minimal Weierstrass equation W for E over $\operatorname{Spec} k[T] = \mathbf{P}^1_k \setminus \{\infty\}$:

$$W : y^2 = x^3 + a_4 x + a_6$$

with $a_6 \in k[T]$ and $a_4 \in k^\times$, since $-a_4^3 = \Delta_W \in k^\times$. We also obtain a globally minimal Weierstrass equation W' for E over $\operatorname{Spec} k[T^{-1}] = \mathbf{P}^1_k \setminus \{0\}$:

$$W' : y'^2 = x'^3 + a'_4 x' + a'_6$$

with $a'_4 \in k^\times$, $a'_6 \in k[T^{-1}]$, say $a'_6 = \sum_{i=0}^{k} b_i T^{-i}$. Since W' and W define the same elliptic curve over $k(T)$ there exists a change of variables between W and W'. By Proposition 12 (a) (ii) such a substitution must be of the form $y = u^3 y'$, $x = u^2 x' + r$ for some $u, r \in k(T)$. Inserting gives further restrictions:

$$u^4 = \frac{a_4}{a'_4} \in k^\times, \qquad r^3 + a_4 r - (u^6 a'_6 - a_6) = 0.$$

In particular, we see $u \in k^\times$. Applying Gauss's lemma to $X^3 + a_4 X - (u^6 a'_6 - a_6) = 0$ shows that such a $r \in k(T)$ must already be in $k[T, T^{-1}]$, say $r = \sum_{i=-m}^{n} r_i T^i \in k[T, T^{-1}]$. By setting $r_+ := \sum_{i=0}^{n} r_i T^i$ and $r_- := \sum_{i=-m}^{-1} r_i T^i$ we get:

$$\underbrace{(r_+^3 + a_4 r_+ - u^6 b_0 + a_6)}_{\in k[T]} + \underbrace{(r_-^3 + a_4 r_- - u^6 (a'_6 - b_0))}_{\in T^{-1} k[T^{-1}]} = 0.$$

Thus, we see $r_+^3 + a_4 r_+ - u^6 b_0 + a_6 = 0$, but this means that substituting $y = \overline{y}$, $x = \overline{x} + r_+$ in W yields a Weierstrass equation $\overline{W}$ for E with constant coefficients:

$$\overline{W} : \overline{y}^2 = \overline{x}^3 + a_4 \overline{x} + u^6 b_0$$

with $a_4 \in k$ and $u^6 b_0 \in k$, since $u, b_0 \in k$.

Since the argument in characteristic 2 is similar we will just sketch the proof. We first consider the case $j(E) \neq 0$. By Proposition 12 (b) we may choose a globally minimal Weierstrass equation W for E over $\operatorname{Spec} k[T] = \mathbf{P}^1_k \setminus \{\infty\}$ of the form

$$W : y^2 + xy = x^3 + a_2x^2 + a_6$$

with $a_2 \in k[T], a_6 \in k^\times$ (we have $a_6 = \Delta \in k^\times$), as well as for E over $\operatorname{Spec} k[T^{-1}] = \mathbf{P}^1_k \setminus \{0\}$:

$$W' : y'^2 + x'y' = x'3 + a'_2x'^2 + a'_6$$

with $a'_6 \in k^\times$, $a'_2 \in k[T^{-1}]$, say $a'_2 = \sum_{i=0}^{k} b_iT^{-i}$. A substitution between W and W' must be of the form $y = y' + sx'$, $x = x'$ for some $s \in k(T)$ satisfying $s^2 + s + a_2 + a'_2 = 0$. Applying Gauss's lemma as above we must have $s = \sum_{i=-m}^{n} s_iT^i \in k[T, T^{-1}]$. By setting $s_+ := \sum_{i=0}^{n} s_iT^i$, we get $s_+^2 + s_+ + b_0 + a_2 = 0$, but this means that substituting $y = \overline{y} + s_+\overline{x}$, $x = \overline{x}$ in W yields a Weierstrass equation $\overline{W}$ for E with constant coefficients:

$$\overline{W} : \overline{y}^2 + \overline{xy} = \overline{x}^3 + b_0\overline{x}^2 + a_6$$

$a_6 \in k^\times$ and $b_0 \in k$.

Now we consider the case characteristic 2 and $j(E) = 0$. Again we get globally minimal Weierstrass equations for E over $\operatorname{Spec} k[T]$:

$$W : y^2 + a_3y = x^3 + a_4x + a_6$$

with $a_3 \in k^\times$ and $a_4, a_6 \in k[T]$, as well as for E over $\operatorname{Spec} k[T^{-1}]$:

$$W' : y'^2 + a'_3y' = x'^3 + a'_4x' + a'_6$$

with $a'_3 \in k^\times$ and $a'_4, a'_6 \in k[T^{-1}]$. Comparing them over $\operatorname{Spec} k[T, T^{-1}]$, we get a substitution of the form $x = u^2x' + s^2$ and $y = u^3y' + u^2sx' + t$ with $u, s, t \in k(T)$ (see Proposition 12 (a) (iv)) satisfying:

$$u^3 = \frac{a_3}{a'_3} \tag{11}$$

$$s^4 + a_3s + a_4 + u^4a'_4 = 0 \tag{12}$$

$$t^2 + a_3t + s^6 + a_4s^2 + a_6 + u^6a'_6 = 0. \tag{13}$$

In particular (11) yields $u \in k^\times$ and, together with (12), we get $s \in k[T, T^{-1}]$. After substituting $y = \widetilde{y} + s_+\widetilde{x}$, $x = \widetilde{x} + s_+^2$ in W as well as $y' = \widetilde{y}' + u^{-1}s_-\widetilde{x}'$, $x' = \widetilde{x}' + u^{-2}s_-^2$ in W' we may assume a_4 and a_4' constant. Then (13) yields $t \in k[T, T^{-1}]$. One more substitution of the form $\widetilde{y} = \overline{y} + t_+$, $\widetilde{x} = \overline{x}$ yields a Weierstrass equation $\overline{W}$ for E with constant coefficients.

The following examples show that in both characteristics 2 and 3 there are non-constant elliptic curves, if we allow bad reduction in one point. ◻

Example 14. (a) Let k be an algebraically closed field of characteristic 2. Then the elliptic curve $E/k(T)$ defined by

$$W: \quad y^2 + xy = x^3 + Tx^2 + 1$$

has discriminant $\Delta_W = 1$ and j-invariant $j(E) = 1$. From $\Delta_W \in k^\times = k[T]^\times$, we see that the elliptic curve E has good reduction everywhere over $\mathbf{A}_k^1$ and a similar computation as done in Example 9 shows that E is non-constant.

(b) Let k be an algebraically closed field of characteristic 3. Then the elliptic curve $E/k(T)$ given by

$$W : y^2 = x^3 - x + T$$

with $\Delta_W = 1$ and $j(E) = 0$ has good reduction everywhere over $\mathbf{A}_k^1$ and is also non-constant.

Acknowledgments. I would like to thank N. Naumann for suggesting this work to me and for his mentoring advice as well as R. Schoof for his initiating idea towards an elementary solution of the problem.

REFERENCES

1. Arnaud Beauville, *Le nombre minimum de fibres singulières d'une courbe stable sur P1*, Astérisque **86** (1981), 97–108.

2. Jean-Marc Fontaine, *Il n'y a pas de variété abélienne sur Z*, Invent. Math. **81** (1985), 515–538.

3. Dale Husemöller, *Elliptic curves*, second ed., Grad. Texts Math. **111**, Springer-Verlag, New York, 2004; with appendices by Otto Forster, Ruth Lawrence and Stefan Theisen.

4. Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Grad. Texts Math. **6**, Oxford University Press, Oxford, 2002, Oxford Science Publications (translated from the French by Reinie Erné).

5. R.C. Mason, *Diophantine equations over function fields*, Lond. Math. Soc. Lect. Note Ser. **96**, Cambridge University Press, Cambridge, 1984.

6. René Schoof, *Abelian varieties over Q with bad reduction in one prime only*, Compos. Math. **141** (2005), 847–868.

7. Joseph H. Silverman, *The arithmetic of elliptic curves*, Grad. Texts Math. **106**, Springer-Verlag, New York, 1986.

Universität Regensburg, 93040 Regensburg, Germany
Email address: johannes.sprang@mathematik.uni-regensburg.de