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# ON THE PATHWISE UNIQUENESS OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH NON-LIPSCHITZ COEFFICIENTS

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ABSTRACT. In this paper, we prove a pathwise uniqueness result of a class of stochastic partial differential equations driven by space-time white noise whose coefficients satisfy non-Lipschitz conditions.

**1. Introduction.** Many mathematicians and physicists have investigated the uniqueness of the following stochastic partial differential equations (SPDE):

$$\frac{\partial}{\partial t}\nu_t(x) = \Delta\nu_t(x) + \sigma(\nu_t(x))\underline{\dot{W}}(t,x), \quad \nu_0 = \mu, \tag{1.1}$$

where $\underline{\dot{W}}$ is the space-time white noise. It is a very important model which was proposed by Dawson in 1972 as follows:

$$\frac{\partial}{\partial t}\nu_t(x) = \Delta\nu_t(x) + \sigma\sqrt{\nu_t(x)}\underline{\dot{W}}(t,x), \quad \nu_0 = \mu. \tag{1.2}$$

In this case, the uniqueness of the solution of the SPDE (1.2) is only proved in the weak sense using that of the martingale problem. The difficulty in proving pathwise uniqueness in (1.2) arises from the fact that $\sqrt{\nu(t,x)}$ is non-Lipschitz. For a more detailed description the

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reader is referred to [**1–7, 12–14**]. We will not resolve the pathwise uniqueness question for SPDE (1.2) but will succeed in solving the SPDE (1.1) with non-Lipschitz noise coefficients by using a new class of backward doubly stochastic differential equations (BDSDE) driven by space-time white noise with non-Lipschitz coefficients. Mytnik [**8**] and Mytnik, Perkins and Sturm [**9**] obtained the pathwise uniqueness for stochastic heat equations with non-Lipschitz coefficients, but we will investigate this problem by using backward method. BSDE and BDSDE were first introduced by Pardoux and Peng, respectively, in [**10, 11**]. We will establish the connection between BDSDE and SPDE driven by space-time white noise. Consider the SPDE as follows:

$$\nu_t(y) = \mu(y) + \int_0^t \int_R \sigma(u, \nu_s(y)) \underline{W}(ds du) + \int_0^t \triangle \nu_s(y)\, ds, \tag{1.3}$$

where $\sigma$ satisfies the following condition

$$\int_R |\sigma(u, y_1) - \sigma(u, y_2)|^2 du \le \rho(|y_1 - y_2|^2), \tag{1.4}$$

where $\rho$ is a concave and nondecreasing function from $R^+$ to $R^+$ such that $\rho(0) = 0$, $\rho(u) > 0$ for $u > 0$ and $\int_{0+} (du)/\rho(u) = \infty$.

Under assumption (1.4), we obtain the pathwise uniqueness for SPDE (1.3).

**2. Backward doubly stochastic differential equations driven by space-time white noise.** We consider a new form of BDSDE as follows:

$$y_t = \xi + \int_t^T \int_R g(u, y_s) W(\overleftarrow{ds} du) - \int_t^T z_s dB_s, \quad 0 \le t \le T, \tag{2.5}$$

where $\xi$ is an $\mathcal{F}_T^B$-measurable random variable, where $\mathcal{F}_T^B = \sigma(B_s : 0 \le s \le T)$, $B$ is a Brownian motion, $W$, independent of $B$, is a space-time white noise in $(0, \infty)^2$ and the notation $W(\overleftarrow{ds} du)$ stands for the backward Itô integral.

**Definition 2.1.** The pair of processes $(y_t, z_t)$ is a solution to the BDSDE (2.5) if they are $\mathcal{G}_t$-adapted and, for each $t \in [0, T]$, the identity

(2.5) holds almost surely, where $\mathcal{G}_t = \sigma(B_s, s \leq t; W([r,T] \times A), r \in [t,T], A \in \mathcal{B}(R))$.

We shall need the following extension of the well-known Itô formula.

**Lemma 2.1.** *Let $g : [0,T] \times R \times \Omega \to R$ be a $\mathcal{G}_t$-adapted random field, and let $x_t$ be given by*

$$x_t = x + \int_0^t \int_R g(s,u) W(\overleftarrow{ds} du) + \int_0^t z_s dB_s. \tag{2.6}$$

*Then, for any $f \in C_b^2(R)$, we have*

$$\begin{aligned} f(x_t) = f(x) &+ \int_0^t \int_R f'(x_s) g(s,u) W(\overleftarrow{ds} du) + \int_0^t f'(x_s) z_s dB_s \\ &- \frac{1}{2} \int_0^t \int_R f''(x_s) g^2(s,u)\, du\, ds + \frac{1}{2} \int_0^t f''(x_s) z_s^2 ds. \end{aligned} \tag{2.7}$$

The proof is similar to the proof of Lemma 1.3 in Pardoux and Peng [**11**].

With the help of the above lemma, we can now prove the following theorem.

**Theorem 2.1.** *Suppose function $g$ satisfies the assumption* (1.4). *Then the* BDSDE (2.5) *has at most one solution.*

*Proof.* Let $(y_\cdot, z_\cdot), (y'_\cdot, z'_\cdot)$ be two solutions of BDSDE (2.5) and $\beta > 0$. By virtue of Lemma 2.1, we have

$$\begin{aligned} E|y_t - y'_t|^2 e^{\beta t} + E\int_t^T \beta |y_s - y'_s|^2 e^{\beta s} ds + E\int_t^T |z_s - z'_s|^2 ds \\ = E\int_t^T \int_R e^{\beta s} |g(u,y_s) - g(u,y'_s)|^2 du\, ds. \end{aligned}$$

From the assumption (1.4), for all $t \in [0,T]$, we derive

$$E|y_t - y'_t|^2 + E\int_t^T |z_s - z'_s|^2 ds \leq e^{\beta T} \int_t^T \rho(E[|y_s - y'_s|^2])\, ds. \tag{2.8}$$

Therefore,

$$E|y_t - y'_t|^2 \leq e^{\beta T} \int_t^T \rho(E[|y_s - y'_s|^2])\, ds.$$

Then we can get $E|y_t - y'_t|^2 = 0$, $t \in [0,T]$; this means that $y_t = y'_t$, almost surely. It then follows from (2.8) that $z_t = z'_t$, almost surely, for any $t \in [0,T]$. □

## 3. Connection between SPDE and BDSDE.

We consider the SPDE as follows:

$$\nu_t(y) = \mu(y) + \int_0^t \int_R g(u, \nu_s(y)) \underline{W}(dsdu) + \int_0^t \triangle \nu_s(y)\, ds. \tag{3.9}$$

For $T$ fixed, we define the random field $u_t(y) = \nu_{T-t}(y)$, for all $t \in [0,T]$, $y \in R$. We also introduce the new noise $W$ by

$$W(t,x) = \underline{W}(T,x) - \underline{W}(T-t,x), \quad \text{for all } t \in [0,T], x \in R.$$

Then the SPDE (3.9) is converted to its backward version as follows:

$$u_t(y) = \mu(y) + \int_t^T \int_R g(v, u_s(y)) W(\overleftarrow{ds} dv) + \int_t^T \triangle u_s(y)\, ds. \tag{3.10}$$

It is clear that the SPDE (3.9) and (3.10) have the same uniqueness property.

Denote

$$X_s^{t,x} = x + B_s - B_t, \quad \text{for all } t \leq s \leq T.$$

Consider the following BDSDE:

$$Y_s^{t,x} = \mu(X_T^{t,x}) + \int_s^T \int_R g(v, Y_r^{t,x}) W(\overleftarrow{dr} dv) - \int_s^T Z_r^{t,x} dB_r, \quad t \leq s \leq T. \tag{3.11}$$

**Theorem 3.1.** *If* $\{u_t(x)\}$ *is a solution to* (3.10) *satisfying*

$$\sup_{(t,x)\in[0,T]\times R} E[|\partial_x u(t,x)|^2] < \infty,$$

*then* $u(t,x) = Y_t^{t,x}$*, where* $Y_t^{t,x}$ *is a solution to the* BDSDE (3.11).

*Proof.* First, we smooth the function $u_t(y)$ by using the Brownian semigroup. For any $\varepsilon > 0$, let $u_t^\varepsilon(y) = T_\varepsilon u_t(y)$, where $T_\varepsilon f(x) = \int_R P_\varepsilon(x-y)f(y)\,dy$ and $P_\varepsilon(x) = 1/\sqrt{2\pi\varepsilon}\exp(-x^2/2\varepsilon)$. Applying the Brownian semigroup to both sides of (3.10), we get

$$\begin{aligned} u_t^\varepsilon(y) = T_\varepsilon\mu(y) + \int_t^T \int_R \int_R P_\varepsilon(y-z)g(v,u_s(z))\,dz W(\overleftarrow{ds}dv) \\ + \int_t^T \triangle u_s^\varepsilon(y)\,ds. \end{aligned} \tag{3.12}$$

Let $Y_s^{t,x,\varepsilon} = u_s^\varepsilon(X_s^{t,x})$ and $Z_s^{t,x,\varepsilon} = \partial_x u_s^\varepsilon(X_s^{t,x})$. Let $s = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[s,T]$. We use Lemma 2.1 for $u_{t_i}^\varepsilon$ and the SPDE (3.12) with $y = X_{t_{i+1}}$; then

$$\begin{aligned} & u_s^\varepsilon(X_s^{t,x}) - T_\varepsilon\mu(y) \\ &\quad = \sum_{i=0}^{n-1}(u_{t_i}^\varepsilon(X_{t_i}^{t,x}) - u_{t_i}^\varepsilon(X_{t_{i+1}}^{t,x})) \\ &\qquad + \sum_{i=0}^{n-1}(u_{t_i}^\varepsilon(X_{t_{i+1}}^{t,x}) - u_{t_{i+1}}^\varepsilon(X_{t_{i+1}}^{t,x})) \\ &\quad = -\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \partial_x u_{t_i}^\varepsilon(X_r^{t,x})\,dB_r \\ &\qquad - \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \triangle u_{t_i}^\varepsilon(X_r^{t,x})\,dr \\ &\qquad + \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \triangle u_{t_i}^\varepsilon(X_r^{t,x})\,dr \\ &\qquad + \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \int_R\int_R P_\varepsilon(X_{t_{i+1}^{t,x}} - z)g(v,u_r(z))W(\overleftarrow{dr}dv). \end{aligned} \tag{3.13}$$

Taking the mesh size to 0, we have

$$\begin{aligned} & u_s^\varepsilon(X_s^{t,x}) - T_\varepsilon\mu(y) \\ &\quad = -\int_s^T \partial_x u_r^\varepsilon(X_r^{t,x})\,dB_r \\ &\qquad + \int_s^T \int_R\int_R P_\varepsilon(X_r^{t,x} - z)g(v,u_r(z))\,dz W(\overleftarrow{dr}dv). \end{aligned} \tag{3.14}$$

As $\varepsilon \to 0$, we note that

$$
\begin{aligned}
(3.15)\quad E\left|\int_s^T \partial_x u_r^\varepsilon(X_r^{t,x})\,dB_r - \int_s^T \partial_x u_r(X_r^{t,x})\,dB_r\right|^2 \\
&\le \int_s^T E|\partial_x u_r^\varepsilon(X_r^{t,x}) - \partial_x u_r(X_r^{t,x})|^2 dr \\
&\le \int_s^T \frac{1}{s-t}||T_\varepsilon \partial_x u_r - \partial_x u_r||^2_{L^2(R)} dr \to 0,
\end{aligned}
$$

and

$$
\begin{aligned}
(3.16)\quad E\Bigg|\int_s^T \int_R \int_R P_\varepsilon(X_r^{t,x} - z) g(v, u_r(z))\,dz W(\overleftarrow{dr}dv) \\
- \int_s^T \int_R g(v, u_r(X_r^{t,x}))W(\overleftarrow{dr}dv)\Bigg|^2 \\
\le \int_s^T \int_R E|T_\varepsilon g(v, u_r(X_r^{t,x})) - g(v, u_r(X_r^{t,x}))|^2 dr\,dv \\
\longrightarrow 0.
\end{aligned}
$$

Finally, we take $\varepsilon \to 0$ on both sides of (3.14). Then (3.11) follows from (3.14). $\square$

**Theorem 3.2.** *Let the function $g$ satisfy assumption* (1.4)*, and let $\mu$ be bounded in* (3.9)*. Then the pathwise uniqueness holds for the* SPDE (3.9)*.*

*Proof.* Because $\mu$ is bounded, this implies

$$\sup_{(t,x)\in[0,T]\times R} E[|\partial_x \nu(t,x)|^2] < \infty,$$

and the proof immediately follows from Theorems 2.1 and 3.1. $\square$

More generally, we can consider the following SPDE:

(3.17)
$$\nu_t(y) = \mu(y) + \int_0^t \int_R \sigma(s, u, \nu_s(y))\underline{W}(dsdu) + \int_0^t (\triangle \nu_s(y) + b(s, \nu_s(y)))\,ds,$$

where $b$ satisfies

$$|b(t, y_1) - b(t, y_2)|^2 \leq \rho(t, |y_1 - y_2|^2), \quad t \in [0, T]; \tag{3.18}$$

and $\sigma$ satisfies

$$\int_R |\sigma(t, u, y_1) - \sigma(t, u, y_2)|^2 du \leq \rho(t, |y_1 - y_2|^2), \quad t \in [0, T], \tag{3.19}$$

where $\rho : [0, T] \times R^+ \to R^+$ satisfies:

- for fixed $t \in [0, T]$, $\rho(t, \cdot)$ is a concave and nondecreasing function such that $\rho(0) = 0$.
- For fixed $v$, $\int_0^T \rho(t, v)\, dt < \infty$.
- For any $C > 0$, the following ODE

$$\begin{cases} v' = -C\rho(t, v), \\ v(T) = 0. \end{cases} \tag{3.20}$$

has a unique solution $v(t) = 0$, $t \in [0, T]$.

We further assume that $\mu$ is bounded. Then we have:

**Theorem 3.3.** *Under assumptions* (3.18) *and* (3.19), *the* SPDE (3.17) *has at most one solution.*

*Proof.* The proof of Theorem 3.3 is similar to that of Theorem 3.2; so, we omit it. □

*Remark* 3.1. Our method can be applied in the SPDE driven by colored noise; the proof is similar to that of Theorem 3.2, we omit it.

*Remark* 3.2. Under the suitable assumptions about generator $g$, we can prove the existence of BSDE to get the existence of SPDE.

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